

QUARK MASS EFFECTS AND ANOMALIES

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When one takes into account that different quarks have unequal masses, five-point functions associated with pentagon graphs with two external gluons ($2g$) are modified by terms involving quark mass ratios. In this context we prove an exact low energy theorem for the amplitudes $(2g) \rightarrow rj\ell$ with r, j, ℓ in the octet of pseudoscalar Goldstone bosons. A possible manifestation of these effects could be an enhancement of processes $(2g) \rightarrow K\bar{K}\pi$ over $(2g) \rightarrow \eta\pi$ ones as indeed are seen experimentally in τ decays. A contribution to $\eta' \rightarrow \eta\pi\pi$ is also found.

1. Introduction

It has been known for a long time [1] that axial currents have triangle, square and pentagon anomalies [1,2], which can be traced to the divergences of the triangle graph. For example, and concentrating on gluon anomalies [3]^{#1}, the current $A_{q\bar{q}}^\mu = \bar{q}\gamma^\mu\gamma_5 q$ has a triangle (gluon) anomaly, so that

$$\partial_\mu A_{q\bar{q}}^\mu = (T_F \alpha_s / 2\pi) \tilde{G}G + 2im_q \bar{q}\gamma_5 q, \quad (1)$$

where $T_F = \frac{1}{2}$ is defined by $\text{Tr } t^a t^b = T_F \delta_{ab}$, t^a the generators of colour $SU(3)$ and

$$G_a^{\mu\nu} = \partial^\mu B_a^\nu - \partial^\nu B_a^\mu + gf_{abc} B_b^\mu B_c^\nu, \quad \tilde{G}_a^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} G_{a\alpha\beta}.$$

Moreover, the anomaly does not get renormalized [4] which leads to low energy theorems, the best known of which is that giving the $\pi^0 \rightarrow 2\gamma$ decay amplitude via the (photon) anomaly and PCAC.

The analysis of low energy theorems related to anomalies is usually carried neglecting the mass differences of the (light, u, d, s) quarks. While for triangle and square anomalies this is likely to be a valid procedure (to corrections $O(m_q/\text{typical hadronic mass})$) the situation is different for five-point functions. Loosely speaking, we have the following: let

$$F(g_1^a(p_1)g_2^b(p_2)) \rightarrow r(k_1)j(k_2)\ell(k_3) \quad (2)$$

be the amplitude for the gluons $g_1 g_2$ into the pseudoscalar mesons r, j, ℓ ; p, k are the momenta, a, b , the color indices of the gluons. For the pentagon case, we have sufficiently many momenta that we can take two k 's to zero and still get a nonzero value for F . With standard current algebra techniques, each zero momentum is seen to give a commutator and we thus get double commutators which are nonzero for nonvanishing quark masses. The remainder is an anomalous triangle, which can be evaluated exactly in the limit of small momenta. The net

^{#1} The extension to photon anomalies, together with more details on the present work will be presented in ref. [3].

result is a low energy theorem for F expressed in terms of the anomaly times coefficients depending on quark mass ratios, m_q/m_d , hence nonvanishing in the chiral limit. For example, if $r=K^+$, $j=\bar{K}^0$, $\ell=\pi^-$ we obtain that the effective lagrangian

$$\mathcal{L}_{\text{int}}^{\text{eff}}(K^+\bar{K}^0\pi^-) = (T_F\alpha_s/8\pi f_\pi^3)\lambda(K^+\bar{K}^0\pi^-)\phi_{K^+}\phi_{\bar{K}^0}\phi_{\pi^-}\tilde{G}, \quad (3a)$$

$$\lambda(K^+\bar{K}^0\pi^-) = \frac{(m_s+m_d)(m_s+m_u)(m_d+m_u)}{2\sqrt{2}m_s m_d m_u} \frac{f_\pi^2}{f_{\bar{K}^0}^2} \quad (3b)$$

gives $F(g_1+g_2\rightarrow K^+\bar{K}^0\pi^-)$ correct to $O((p,k)^4)$; and neglecting higher order flavor $SU(3)$ corrections. (Eqs. (3) appear to be singular if one of the quark masses vanish. This is not so as will be discussed below.)

We have not been able to find an application of our results as clear and precise as the calculation of $\pi^0\rightarrow 2\gamma$ using the standard anomaly; but a possible phenomenological implication would be for a rough calculation of the widths and partial rates of the $\iota(1440)$ glueball, and the width of the $\eta'(960)$, in qualitative agreement with experiment. (Note that the two first are hitherto unexplained phenomena; for the first applying naive Zweig rule considerations [5]^{#2}, we would expect a width $\Gamma_\iota\sim 0.3\text{--}1$ MeV, two orders of magnitude smaller than the experimental figure).

2. Low energy theorem

Consider the axial currents $A_r^\mu, A_j^\nu, A_\ell^\lambda$ associated with particles of the octet of pseudoscalar Goldstone mesons; if, e.g., $r=K^+$, $j=\bar{K}^0$, $\ell=\pi^-$,

$$A_r^\mu = \bar{u}\gamma^\mu\gamma_5 s, \quad A_j^\nu = -\bar{s}\gamma^\nu\gamma_5 d, \quad A_\ell^\lambda = \bar{d}\gamma^\lambda\gamma_5 u.$$

The field operators themselves will be denoted by $\phi_r, \phi_j, \phi_\ell$. We will use the PCAC definitions $\phi_i = (1/\sqrt{2}f_i M_i^2)\partial A_i$, with $f_\pi \simeq 92$ MeV. Since none of the A_i have anomalies we may also write, if $A_i^\alpha = \bar{q}_1\gamma^\alpha\gamma_5 q_2$, $\partial A_i = i(m_1+m_2)\bar{q}_1\gamma_5 q_2$, with m the quark masses. If $e_i(p_i)$ are the polarization vectors of the gluons, we can use gauge invariance to write the amplitude (2) as

$$F = (F_2^{\alpha\beta} + F_4^{\alpha\beta})e_{1\alpha}(p_1)e_{2\beta}(p_2)\delta_{ab}, \quad (4a)$$

$$F_2^{\alpha\beta} = \Phi(p_1, \dots, k_3)\epsilon^{\alpha\beta\mu\nu}p_{1\mu}p_{2\nu}, \quad F_4^{\alpha\beta} = \Psi(p_1, \dots, k_3)\epsilon^{\alpha\lambda\mu\nu}p_{1\lambda}k_{i\mu}(g_\nu^\beta p_1 p_2 - p_1^\beta p_{2\nu}) + \text{permutations}. \quad (4b,c)$$

Note that F_2 is $O((p,k)^2)$, and F_4 is $O((p,k)^4)$. By using the reduction formulas, we may consider the F to be defined for arbitrary values of the momenta. The existence of Adler's zeros [6] implies that Φ must vanish when one of the k vanishes, and the others are on the mass shell, so if we write

$$\Phi(p_1, \dots, k_3) = (2 - k_r^2/M_r^2 - k_j^2/M_j^2 - k_\ell^2/M_\ell^2)\tilde{\Phi}(p_1, \dots, k_3), \quad (4d)$$

then $\tilde{\Phi}$ is free from zeros. It is also free from poles to leading order in non-chiral $SU(3)$ breaking, so in this limit it will be a smooth function of the momenta. The physical value of Φ is for all particles on their mass shells, i.e., $\Phi_{\text{phys}} = -\tilde{\Phi}$. Expanding in powers of the momenta, $\tilde{\Phi} = \tilde{\Phi}(0) + O((p,k)^2)$; $\Phi(0) = 2\tilde{\Phi}(0)$. We thus find that, to corrections $O((p,k)^4)$, the physical scattering amplitude is given by

$$F_{\text{phys}} \simeq F_{2\text{phys}}^{\alpha\beta}e_{1\alpha}e_{2\beta}\delta_{ab}, \quad F_{2\text{phys}}^{\alpha\beta} = -\frac{1}{2}\Phi(0)\epsilon^{\alpha\beta\mu\nu}p_{1\mu}p_{2\nu}. \quad (5)$$

Our task is to evaluate $\Phi(0)$. Using the PCAC definitions and reduction formulas, and neglecting terms $O((p,k)^4)$,

^{#2} The result is obtained by comparing with the decay of η_c , $\Gamma(\eta_c) \simeq 11$ MeV. The only difference is the replacement of $C_F\alpha_s(M_{\eta_c}^2)$ by $T_F\alpha_s(M_\iota^2)$, and phase space. The spread from 0.3 to 1 MeV for $\Gamma(\iota)$ depends upon the relative weight of the various decay channels for ι .

$$F = \frac{i^3}{2^{3/2} f_r f_j f_\ell} \frac{(2\pi)^4}{(2\pi)^{9/2}} \int dx dy \exp[i(k_1 \cdot x + k_2 \cdot y)] \langle 0 | T \partial A_r(x) \partial A_j(y) \partial A_\ell(0) | g_1 g_2 \rangle. \quad (6)$$

To calculate the RHS we consider

$$R_{ij\ell}'' = \int dx dy \exp[i(k_1 \cdot x + k_2 \cdot y)] \langle 0 | T A_r''(x) \partial A_j(y) \partial A_\ell(0) | g_1 g_2 \rangle,$$

contracting with $-ik_{1\mu}$ we find

$$\begin{aligned} -ik_{1\mu} R_{ij\ell}'' &= \int dx dy \exp[i(k_1 \cdot x + k_2 \cdot y)] \langle 0 | T \partial A_r(x) \partial A_j(y) \partial A_\ell(0) | g_1 g_2 \rangle \\ &+ \int dy \exp(ik_2 \cdot y) \langle 0 | T S_{r\ell}(0) \partial A_j(y) | g_1 g_2 \rangle + \int dz \exp(ik_3 \cdot z) \langle 0 | T S_{rj}(0) \partial A_\ell(z) | g_1 g_2 \rangle. \end{aligned} \quad (7)$$

Using gauge invariance it is easy to verify that $k_{1\mu} R_{ij\ell}''$ is actually of *fourth* order in the momenta so eq. (7) tells us that, to the order we are working, the $\int \dots \langle 0 | T \partial A \partial A \partial A | g_1 g_2 \rangle$ term on the RHS of (7) is just minus the sum of the $\int \dots \langle 0 | T S \partial A | g_1 g_2 \rangle$ ones there. To evaluate the last, consider again

$$\begin{aligned} \frac{1}{i} k_{1\mu} \int dx \exp(ik \cdot x) \langle 0 | T S_{ri}(0) A_n''(x) | g_1 g_2 \rangle \\ = \int dx \exp(ik \cdot x) \langle 0 | T S_{ri}(0) \partial A_n(x) | g_1 g_2 \rangle + \int dx \exp(ik \cdot x) \langle 0 | \delta(x^0) [A_n^0(x), S_{ri}(0)] | g_1 g_2 \rangle. \end{aligned}$$

The LHS is of $O((p, k)^4)$; we may then substitute the RHS into (7) to obtain finally

$$\int dx dy \exp[i(k_1 \cdot x + k_2 \cdot y)] \langle 0 | T \partial A_r(x) \partial A_j(y) \partial A_\ell(0) | g_1 g_2 \rangle = \langle 0 | P_{jr\ell}(0) + P_{\ell rj}(0) | g_1 g_2 \rangle + O((p, k)^4), \quad (8a)$$

where the P are pseudoscalar operators defined by

$$\delta(x) \delta(y) P_{jr\ell}(0) = \delta(x^0 - y^0) [A_j^0(x), \delta(y^0) [A_r^0(y), \partial A_\ell(0)]]. \quad (8b)$$

Evaluating the commutators we can see that if the quark masses are nonvanishing (and only in this case) the P are expressible as

$$P_{jr\ell} = \sum_q c(jr\ell, q) 2im_q \bar{q} \gamma_5 q, \quad (8c)$$

where the coefficients c depend on quark mass ratios, m_q/m_q . Thus the RHS of (8a) is a combination of terms $\langle 0 | 2im_q \bar{q}(0) \gamma_5 q(0) | g_1 g_2 \rangle$.

These terms contain, in principle poles due to intermediate π^0 , η , η' , ι particles. Of these, the η' , ι poles are suppressed in the chiral limit; the π^0 , η ones only enter through mixing with η' , ι which is of the order of flavor SU(3) breaking. Therefore, these expressions can be obtained, to the order we are interested in, in terms of the standard anomaly, eq. (1). Indeed, $\langle 0 | 2im_q \bar{q} \gamma_5 q | g_1 g_2 \rangle = (-T_F \alpha_s / 2\pi) \langle 0 | G\tilde{G} | g_1 g_2 \rangle$. (It is to be noted that in spite of superficial appearances to the contrary, the result is symmetric under permutations of r, j, ℓ .) We have succeeded in extracting the RHS of (6), hence $\Phi(0)$, and, in view of (5), the physical scattering amplitude to $O((p, k)^4)$ corrections. The results (which we have checked by an explicit calculation of the pentagon graph) can best be expressed in the form of effective lagrangians for two gluons with color indices a, b , and mesons r, j, ℓ .

$$\mathcal{L}_{\text{int}}^{\text{eff}}(r, j, \ell) = (T_F \alpha_s / 8\pi f_\pi^3) \lambda(rj\ell) \phi_r \phi_j \phi_\ell \tilde{G} G \delta_{ab}, \quad (9a)$$

where $\lambda(K^+ \bar{K}^0 \pi^-) = \lambda(K^- K^0 \pi^+)$ was given in eq. (3b) and

$$\lambda(K^+K^-\pi^0) = \frac{(m_s + m_u)^2 f_\pi^2}{2m_u m_s f_K^2}, \quad \lambda(K^0\bar{K}^0\pi^0) = \frac{-(m_s + m_d)^2 f_\pi^2}{2m_s m_d f_K^2},$$

$$\lambda(\eta\pi^+\pi^-) = \frac{(m_d + m_u)^2 f_\pi}{\sqrt{3}m_d m_u f_\eta}, \quad \lambda(\eta\pi^0\pi^0) = -\frac{1}{2!} \frac{4 f_\pi}{\sqrt{3} f_\eta}, \quad (9b)$$

$\lambda(\pi^+\pi^-\pi^0) = \lambda(\pi^0\pi^0\pi^0) = 0$, etc. These lagrangians give the exact mass-shell scattering amplitudes, to corrections $O((p, k)^4)$. These corrections are of two types: the terms F_4 in eq. (4), and corrections to F_2 . The first may be seen to be $O(|k|, M/(\text{typical hadronic mass}))$; in all our applications this will be small. In fact, it will vanish in the chiral limit. The corrections to F_2 may be estimated by writing a dispersion relation for Φ_{phys} in the variable $s = (p_a + p_b)^2$, subtracted at $s=0$ where (9) give the exact result. One finds that the only intermediate states that contribute appreciably are the $\eta', \iota(1440)$ poles. The expected corrections are of some 50% for $s = M_\iota^2$ and smaller for $s = M_{\eta'}^2$, so our calculation should give at least the correct order of magnitude.

Before plunging into the applications a few words may be said regarding two important questions. First, because of π^0, η mixing with η', Φ (for example) will have poles at $(p_a + p_b)^2 = M_{\pi^0}^2, M_{\eta'}^2$. As stated before, these poles give corrections of higher order in SU(3) breaking; in $\eta\pi\pi$ they are small (20% level) in the $K\bar{K}\pi$ cases much smaller. Details about these corrections will be found in ref. [3]; here we neglect them. Second, a formula like, say, eq. (3) is surprising: a divergent result would seem to follow if one of the quark masses vanishes. It is not true. Our derivation assumes implicitly and explicitly, and that several times, that quarks are *not* massless particles. To see what happens, assume that one of the quark masses is small, μ , and that the two others are kept finite, m (and equal to simplify the calculation). A Feynman diagram evaluation gives an amplitude for $rj\ell \rightarrow g\bar{g}$ of the form $\tilde{G}G\mu m \int_0^1 dx x \int_0^1 dy [\mu^2 - 2x(1-x)yp_a \cdot p_b]^{-2}$. Thus, if $p_a \cdot p_b$ were exactly zero, we would obtain $\tilde{G}Gm/\mu$ as in eq. (3); if, however, $\mu=0$ to begin with, we get a zero result: the singularity at $\mu=0$ is only apparent. (The same situation arises indeed in the ordinary triangle anomaly.)

A full discussion of these two points will be given in ref. [3].

3. Phenomenological implications

We will consider the three meson decays of the ι and η' particles. For the $\iota(1440)$, presumably a pseudoscalar glueball [7] we perform the calculation in the bag model, i.e., we consider the iota to be two free gluons inside a bag of radius R . Then,

$$\Gamma_i(rj\ell) = \frac{1 - \delta(rj\ell)}{8\sqrt{3}} \frac{M_\iota^2 (M_\iota - M_r - M_j - M_\ell)^2}{(2Rf_\pi)^3 (4\pi f_\pi)^3} \left(\frac{T_F \alpha_s}{\pi}\right)^2 \lambda^2(rj\ell), \quad (10)$$

where δ are the relativistic corrections to phase space (10–30%). For numerical evaluations we take $m_s/m_d = 20$, $m_d/m_u = 2$, $f_K = 1.18f_\pi$, $f_\eta = 0.73f_\pi$, $A = 140$ MeV for the QCD parameter; and let $(2Rf_\pi)^3 = \rho$. Then,

$$\Gamma_i(\bar{K}^0K^+\pi^-) = \Gamma_i(K^0K^-\pi^+) = 4.6 \text{ MeV}/\rho, \quad \Gamma_i(K^+K^-\pi^0) = 4.0 \text{ MeV}/\rho, \quad \Gamma_i(K^0\bar{K}^0\pi^0) = 1.1 \text{ MeV}/\rho,$$

$$\Gamma_i(\eta\pi^+\pi^-) = 0.74 \text{ MeV}/\rho, \quad \Gamma_i(\eta\pi^0\pi^0) = 0.3 \text{ MeV}/\rho, \quad \Gamma_i(\pi^+\pi^-\pi^0) = \Gamma_i(\eta\eta\pi^0) = \Gamma_i(\pi^0\pi^0\pi^0) = 0. \quad (11)$$

This explains correctly the dominance of the decays into kaons, compared to those involving η . For the total width we have to decide on a value for R , and this is what causes the largest uncertainties in our calculation. Allowing R^{-1} to vary [8] between 145 MeV and 313 MeV, we have

$$\Gamma_i(\text{all}) = 7.3\text{--}76 \text{ MeV} \quad (\text{experimentally, } \Gamma_i = 76 \text{ MeV}). \quad (12)$$

For the η' decay we find a formula like (10) substituting M_ι by $M_{\eta'}$ and multiplying by $\sin^2\theta_{\eta'g}$ for the gluonic component of the η' . With $\sin^2\theta_{\eta'g} \sim \frac{1}{2}$,

$$\Gamma_{\eta}^{\text{gluc}}(\eta\pi\pi) = 0.02 - 0.16 \text{ MeV} \quad (\text{experimentally } \Gamma_{\eta}(\eta\pi\pi) = 0.18 \text{ MeV}). \quad (13)$$

(The η' has a contribution from its quark component, easily evaluated with the techniques of ref. [9] to be $\Gamma_{\eta'}^{\text{quark}}(\eta\pi\pi) \simeq 0.04 \text{ MeV}$, so $\Gamma_{\eta}^{\text{gluc}}$ should really be compared with $\Gamma^{\text{exp}} - \Gamma^{\text{quark}} \simeq 0.14 \text{ MeV}$).

We finish this note with a few comments on our phenomenological results. It is clear that we could fit both total widths with reasonable values for R but this we deem a profitless occupation at the present state of the art. We merely prefer to note *qualitative* agreement in a very wide range of widths, due largely to the values of the $\lambda(\tau j \ell)$:

$$\sum_{\text{K}} \lambda^2(\text{KK}\pi) : \sum_{\pi} \lambda^2(\eta\pi\pi) \sim 790:16.$$

Particular attention should be drawn to the natural explanation of the large ratio $\Gamma_{\text{K}}(\text{KK}\pi)/\Gamma_{\eta}(\eta\pi\pi)$, contrary to phase space expectations and, as far as we know, not given by any mechanism other than ours.

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Note added

The results of this paper cannot be obtained with the method of phenomenological lagrangians. This will be discussed in ref. [3].

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