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AN ORIENTATION PRESERVING FIXED POINT FREE HOMEOMORPHISM OF THE PLANE WHICH ADMITS NO CLOSED INVARIANT LINE

Morton BROWN
Mathematics Department, University of Michigan, Ann Arbor, MI 48109, USA

Edward E. SLAMINKA* and William TRANSUE
Department of Mathematics, F.A.T., Auburn University, AL36849-3501, USA

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Though fixed point free homeomorphisms of the plane would appear to exhibit the simplest dynamical behavior, we show that the minimal sets can be quite complex. Every homeomorphism which is conjugate to a translation must have a closed invariant line. However we construct an orientation preserving fixed point free homeomorphism of the plane which admits no closed invariant line. We verify that no such line exists by considering the 'fundamental regions' of our example. Fundamental regions, studied first by Stephen Andrea, are equivalence classes of points in the plane associated with a given homeomorphism. Two points are said to be in the same equivalence class if they can be connected by an arc which diverges to infinity under both the forward and backward iterates of the homeomorphism. Our example contains no invariant fundamental regions.

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fundamental region closed invariant line

1. Introduction

In [2], the first author constructed a Brouwer homeomorphism (that is, an orientation preserving, fixed point free homeomorphism of the euclidean plane) which is non-invariant on each closed topological line. This negatively settled a venerable conjecture. The purpose of this paper is to present a simpler and more accessible example.

2. Fundamental regions

Andrea, in his important study of Brouwer homeomorphisms [1], introduced the following definition. Two points are defined to be in the same fundamental region of a Brouwer homeomorphism h provided that there is a simple arc A containing

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them with the property that for each compact subset C of the plane, there is a sufficiently large integer N such that $h^n(A) \cap C = \emptyset$, whenever $|n| > N$. Obviously, the fundamental regions of a homeomorphism partition the plane, and the homeomorphism permutes them. The principal result of [1] is that a Brouwer homeomorphism is conjugate to a translation of the plane if and only if the plane itself is the only fundamental region of the homeomorphism.

We shall construct a Brouwer homeomorphism that leaves no fundamental region invariant. Such a homeomorphism h could not be invariant on l, any closed topological line for, suppose h is invariant on l, that is, $h(l) = l$. Since h is fixed point free, $h|l$ is conjugate to a translation of l. Let x and $h(x)$ be two points of l and let A be the subarc of l joining them. Since l is closed in the plane, it is easily seen that A satisfies the requirement of the previous paragraph. Thus x and $h(x)$ are in the same fundamental region which contradicts the assumption that h leaves no fundamental region invariant.

3. Construction of the example

Refer to Fig. 1. The homeomorphism will commute with both the 2 unit vertical shift and the unit horizontal shift. Horizontal strips, bounded above and below by 'plungers' two units apart, will remain invariant.

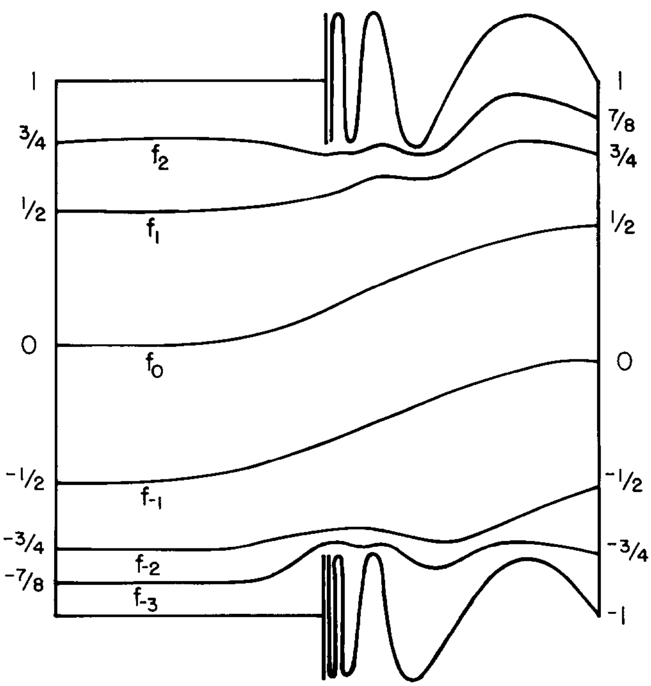

Fig. 1.

Our basic building block will be

$$\left\{(x, y) \mid 0 \leq x \leq 1;\ -1 \leq y \leq 1 \text{ if } x < \tfrac{1}{2},\ -1.1 \leq y \leq 1.1 \text{ if } x = \tfrac{1}{2},\right.$$

$$\left.\text{and } -1 + 0.1 \sin \frac{\pi}{x - \frac{1}{2}} \leq y \leq 1 + 0.1 \sin \frac{\pi}{x - \frac{1}{2}} \text{ if } x > \tfrac{1}{2}\right\}.$$

We foliate the interior of this block with a collection f_λ, λ real, of functions from $[0, 1]$ into $(-1.1, 1.1)$ with the following properties:

(1) $f_\lambda(0) = (\lambda/|\lambda|)(1 - 2^{-|\lambda|})$ or 0 if $\lambda = 0$,

(2) $f_\lambda(1) = f_{\lambda+1}(0)$,

(3) if $0 < \lambda - \mu \leq 1$ and $x \in [0, 1]$ then $0 < f_\lambda(x) - f_\mu(x) < 8/(|\lambda| + 1)$.

To construct such a collection one may first construct $f_{-8}, f_{-4}, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots, f_{2n}, \ldots$, keeping the order correct making the endpoints satisfy (1) and (2), and limiting top and bottom on the appropriate $\sin(1/x)$ curves. The intermediate f_λ, for integral λ, can then be obtained from those already chosen by linear interpolation suitably modified near 0 and 1 so as to keep (1) and (2) satisfied. With this, since $|f_{2n} - f_{2n+1}| < 2$, we have that $|f_k - f_{k+1}| < 4/k$. If we then interpolate logarithmically [to keep (1) and (2) satisfied] to get the f_λ for nonintegral λ, (3) will be satisfied.

These blocks are now joined end to end, with the sequence of $\sin(1/x)$ lines forming a 'plunger' so as to form a foliation g_λ, λ real, of the infinite strip by continuous functions from R into $(-1.1, 1.1)$ where $g_\lambda(x) = f_{\lambda + \text{int}(x)}(x - \text{int}(x))$. (See Fig. 2).

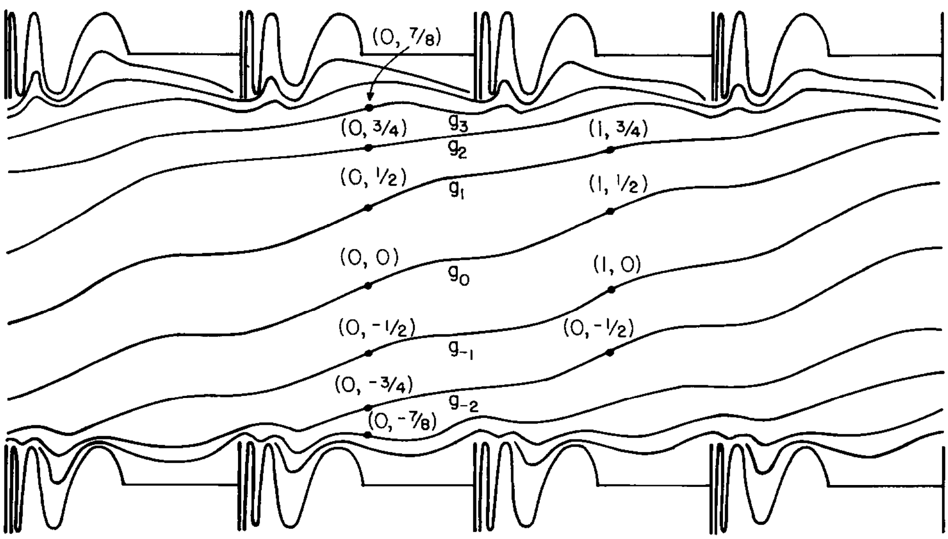

Fig. 2.

Our homeomorphism h is now defined on the strip containing the x-axis as the composite of two (commuting) homeomorphisms, the unit left shift $l(x, y) = (x-1, y)$, and a motion, k, along the graphs g_λ, leaving the points $(n, 0)$ (n an integer) fixed, and moving all other points to the right, with a rightward motion of two units in the vicinity of each of the plungers which bound the strip. More precisely, let m_λ be the function which is linear on $[-4-\lambda, -\lambda]$ and $[-\lambda, -\lambda+2]$ so that $m_\lambda(-\lambda) = \min(\lambda - \text{int}(\lambda), \text{int}(\lambda+1) - \lambda)$ and $m_\lambda(x) = x+2$ if $x \leqslant -4-\lambda$ or $x \geqslant 2-\lambda$. We now let $k(x, g_\lambda(x)) = (m_\lambda(x), g_\lambda(m_\lambda(x)))$. Condition (3) above assures us that k extends continuously to a right shift of two units on the plungers. (See Fig. 3.) We let $h = k \circ l$, and extend h to the whole plane by requiring that $h(x, y+2n) = h(x, y) + (0, 2n)$ for each integer n.

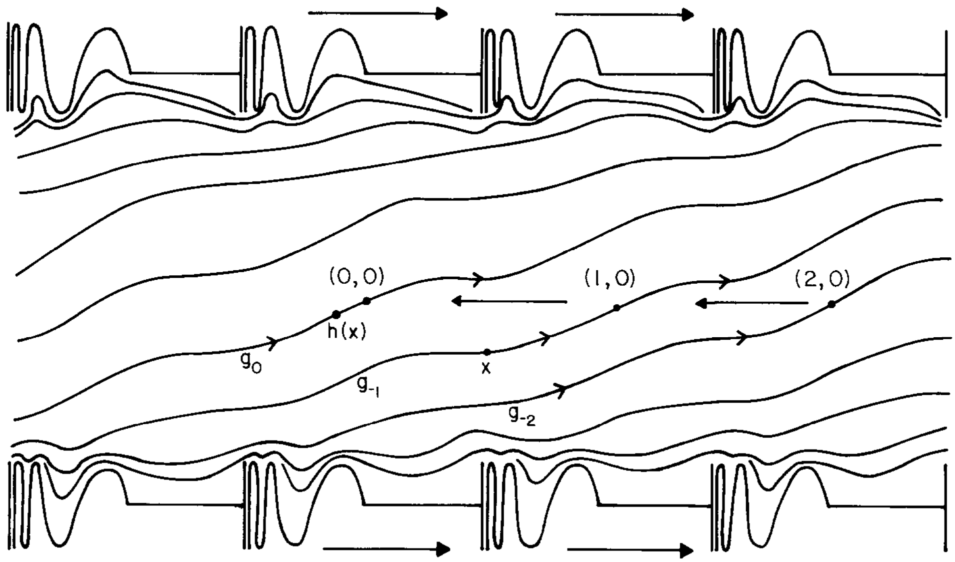

Fig. 3. $(n, 0) \mapsto (n-1, 0)$, $g_i \mapsto g_{i+1}$.

We now examine the orbits of the points in the canonical strip containing the x-axis. Points are classified (l, l), (l, r), (r, l) or (r, r) according as their (backwards, forwards) orbits move ultimately to the left or right. The classification is as follows:

(r, l): points $(n, 0)$, n an integer,
(l, l): points of the form $(x, g_n(x))$, n an integer, $x < n$,
(r, r): points of the form $(x, g_n(x))$, n an integer, $x > n$,
(l, r): all other points (including the plungers).

The fundamental regions are as follows (see Fig. 4):

– each point of type (r, l),

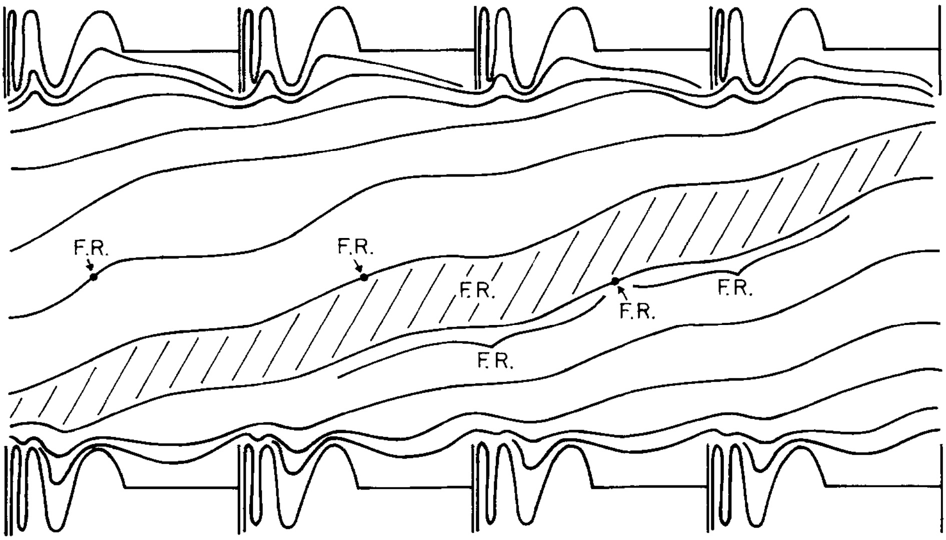

Fig. 4. F.R. = Fundamental Region.

- each ray with points of type (l, l), for each n,
- each ray with points of type (r, r), for each n,
- each path connected component of the plungers,
- each region bounded by the lines $(x, g_n(x)) \cup (n, 0)$ where n is an integer.

Since no point may be joined to its image by an arc missing another fundamental region, there does not exist an invariant fundamental region. Thus we have constructed an example which admits no closed invariant line.

By altering properties of f, adding suitable differentiability conditions, match-up conditions and requiring that the graphs f_n as $n \to \pm\infty$ converge to the plungers sufficiently slowly (so that $\lim_{n+\infty} |(f_n(x) - f_{n+1}(x))/(f_n(x) - \lim f_k(x))| = 0$), the homeomorphism could be made C^1, and, with somewhat more effort, C^n. On the other hand, the authors attempted without success to construct an example not using the local pathology of the plungers, so that a question remaining is: is there a Brouwer homeomorphism of $\mathbb{R}^2$ which leaves no proper closed connected set invariant?

References

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[2] M. Brown, Fundamental regions of Brouwer homeomorphisms, to appear.