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NOTE

# A CORRECTION TO A RESULT IN LINEAR PROGRAMMING

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We show that Theorem 1 in [1, p. 236] about linear programs is incorrect as stated. We provide the correct version of this theorem and an elementary proof of it.

## The result

In a recent paper [1] Chandrasekaran stated the following theorem about linear programs without proof (this is Theorem 1 in [1, p. 236]; the wording of our statement is different, but the content is the same).

**Theorem 1.** *Consider the convex polyhedron defined by the system of constraints*

$$Ax = b, \quad x \geq 0.$$

*Let $cx$, $c'x$ be two linear objective functions defined on this polyhedron. The difference between these two linear functions, $(c-c')x$, is a constant over this polyhedron iff there is some $u$ such that $c-c'=uA$.*

This theorem is incorrect as stated. We provide the following counterexample. Let

$$x = (x_1, x_2, x_3, x_4, x_5, x_6)^{\mathrm{T}},$$

$$c = (3, 9, -7, 8, 1, 2), \qquad c' = (19, 0, 18, 4, 1, 2);$$

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \qquad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Let $P=\{x\colon Ax=b,\ x\geq 0\}$, with the above data. It can be verified that $x_1=x_2=x_3=x_4=0$ at all points $x\in P$. Hence $(c-c')x=0$ for all $x\in P$, and yet there exists no $u$ such that $(c-c')=uA$, in this case.

The purpose of this note is to give the correct version of Chandrasekaran's result, together with an elementary proof of it.

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If $E$ is any matrix, we will denote its $i$th row by $E_{i.}$. If $S$ and $T$ are arbitrary sets, we will denote the set of all elements in $S$ which are not in $T$ by $S \setminus T$.

## Correct version of Theorem 1 for general systems of linear constraints

**Theorem 1′.** *Let $A, B, b, d, c, c'$ be given real matrices of orders $m \times n$, $p \times n$, $m \times 1$, $p \times 1$, $1 \times n$, $1 \times n$, respectively. Let the convex polyhedron $K$ be the set of feasible solutions of*

$$Ax = b, \qquad Bx \geq d. \tag{1}$$

*We assume that $K \neq \emptyset$. An inequality constraint in* (1), *say*

$$B_{r.}x \geq d_r,$$

*is said to be a 'binding inequality constraint' in* (1) *iff $B_{r.}x = d_r$ for all $x \in K$. Let*

$$J = \{r\colon 1 \leq r \leq p \text{ and } B_{r.}x \geq d_r \text{ is a binding inequality constraint in (1)}\}.$$

*Then $(c-c')x$ is a constant over $K$ iff $(c-c')$ is in the linear hull of*

$$\Gamma = \{A_{i.}\colon i = 1, \ldots, m\} \cup \{B_{r.}\colon r \in J\}.$$

**Proof.** If $(c-c')$ is in the linear hull of $\Gamma$, there exists $v_i$ for $1 \leq i \leq m$, and $u_r$ for $r \in J$ such that

$$c - c' = \sum_{i=1}^{m} v_i A_{i.} + \sum_{r \in J} u_r B_{r.}. \tag{2}$$

From the definition of $J$, $B_{r.}x = d_r$ for all $r \in J$ and $x \in K$. So, (2) implies that

$$(c - c')x = \sum_{i=1}^{m} v_i b_i + \sum_{r \in J} u_r d_r$$

a constant, for all $x \in K$, establishing the 'if' part of the theorem.

To prove the 'only if' part, we now assume that $(c-c')$ is not in the linear hull of $\Gamma$. We will show that this implies that $(c-c')x$ is not a constant over $K$.

From the hypothesis in the theorem, $K$ is the set of feasible solutions of the system

$$A_{i.}x = b_i, \quad i = 1, \ldots, m,$$
$$B_{r.}x \begin{cases} = d_r, & r \in J, \\ \geq d_r, & r \in \bar{J} = \{1, \ldots, p\} \setminus J, \end{cases} \tag{3}$$

and for each $r \in \bar{J}$, there exists an $x \in K$ satisfying $B_{r.}x > d_r$.

Let $q$ = rank of the set $\Gamma$ of vectors. Since we assumed that the vector $c - c'$ is not in the linear hull of $\Gamma$, $q < n$. From linear algebra, we know that the system of linear equations,

$$A_{i.}x = b_i, \quad i = 1, \ldots, m,$$

$$B_{r.}x = d_r, \quad r \in J,$$

can be transformed by the Gauss–Jordan elimination method into an equivalent system which expresses $q$ of the variables among $x_1, \ldots, x_n$ as affine functions of the remaining $n-q$ variables. Without any loss of generality, assume that this equivalent system is

$$x_i = a''_{0i} + \sum_{j=q+1}^{n} a''_{ij}x_j, \quad i = 1, \ldots, q. \tag{4}$$

Substitute the expressions for $x_1, \ldots, x_q$ from (4) in $(c-c')x$. Suppose this leads to the affine function

$$c''_0 + \sum_{j=q+1}^{n} c''_j x_j.$$

Since $(c-c')$ is not in the linear hull of $\Gamma$, $(c''_{q+1}, \ldots, c''_n) \neq 0$.

Substitute the expressions for $x_1, \ldots, x_q$ from (4) in the inequality $B_{r.}x \geq d_r$, for $r \in \bar{J}$. Suppose this transforms it into

$$\sum_{j=q+1}^{n} b''_{rj}x_j \geq d''_r.$$

So, the system (3) is equivalent to

$$\begin{aligned} x_i &= a''_{0i} + \sum_{j=q+1}^{n} a''_{ij}x_j, \quad i = 1, \ldots, q \\ \sum_{j=q+1}^{n} & b''_{rj}x_j \geq d''_r, \quad r \in \bar{J}. \end{aligned} \tag{5}$$

So, $K$ is the set of feasible solutions of (5). $(c-c')x$ is a constant for all $x \in K$ iff

$$\sum_{j=q+1}^{n} c''_j x_j$$

is a constant over the set of feasible solutions of

$$\sum_{j=q+1}^{n} b''_{rj}x_j \geq d''_r, \quad r \in \bar{J} \tag{6}$$

in the space of the variables $X = (x_{q+1}, \ldots, x_n)^{\mathrm{T}}$. Let $\Delta$ denote the set of feasible solutions of (6). From our hypothesis, there exists a point $x \in K$ which satisfies each of the inequality constraints in (5) as a strict inequality, this implies that there exists an $X \in \Delta$ which satisfies each of the inequalities in (6) as a strict inequality, that is $\Delta$ has an interior point in the space of $X$. Hence, in the space of $X$, $\Delta$ is of full dimension (i.e., dimension of $\Delta$ is $n-q$).

Let

$$l(X) = \sum_{j=q+1}^{n} c''_j x_j.$$

If $l(X)$ is a constant over $\Delta$, say $\alpha$, then $\Delta$ is a subset of the hyperplane defined by $l(X)=\alpha$ in the space of $X$, contradicting the fact that $\Delta$ has full dimension in the space of $X$. So $l(X)$ is not a constant over $\Delta$, this implies that $(c-c')X$ is not a constant over $K$, completing the 'only if' part of the theorem. □

**Corollary 1.** *Let cx be a linear objective function defined on $K$, which is the set of feasible solutions of (1). cx is a constant over $K$ iff c is in the linear hull of $\Gamma=\{A_{i.}: 1\le i\le m\}\cup\{B_{r.}: r\in J\}$.*

**Corollary 2.** *Let $P$ be the set of feasible solutions of*

$$Ax=b,\quad x\ge 0,$$

*where $A$ is a given real matrix of order $m\times n$ and rank $s$. If $P\ne\emptyset$, and $P$ has dimension $n-s$, the linear objective function $cx$ is a constant over $P$ iff $c$ is in the linear hull of rows of $A$.*

We should point out that even though Theorem 1 in [1, p. 236] is incorrect as stated, all the other results about the assignment and traveling salesman problems derived there using Theorem 1, are correct, since the stronger conditions in Corollary 2 here, hold for those problems.

## References

[1] R. Chandrasekaran, Recognition of the Gilmore-Gomory traveling salesman problem, Discrete Appl. Math. 14 (1986) 231-238.