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SELECTIVE ULTRAFILTERS AND HOMOGENEITY

Andreas BLASS
University of Michigan, Ann Arbor, MI 48109 and Pennsylvania State University, University Park, PA 16802, USA

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We develop a game-theoretic approach to partition theorems, like those of Mathias, Taylor, and Louveau, involving ultrafilters. Using this approach, we extend these theorems to contexts involving several ultrafilters. We also develop an analog of Mathias forcing for such contexts and use it to show that the proposition (considered by Laver and Prikry) "every non-trivial c.c.c. forcing adjoins Cohen-generic reals or random reals" implies the non-existence of P-points. We show that, in the model obtained by Lévy collapsing to ω all cardinals below a Mahlo cardinal κ, any countably many selective ultrafilters are mutually generic over the Solovay (Lebesgue measure) submodel. Finally, we show that a certain natural group of self-homeomorphisms of $\beta\omega - \omega$, chosen so as to preserve selectivity of ultrafilters, in fact preserves isomorphism types.

Introduction

Three different meanings of 'homogeneity' are relevant to this paper. The first is the homogeneity of the sets whose existence is asserted by various partition relations. Our central theorems are partition relations extending Mathias's version [19, Theorem 13], involving selective ultrafilters, of Silver's theorem [28] on analytic partitions. Among the consequences of these theorems are several that shed light on a second sort of homogeneity, the informal idea that selective ultrafilters look alike, that they cannot be distinguished from each other by reasonable combinatorial properties. Finally, in a section that is independent of the rest of the paper except for the preliminaries, we consider topological homogeneity, viewing ultrafilters as points in the Stone–Čech compactification of a discrete space, as in [25].

After a section in which we explain our terminology (including the terminology used in this introduction) and record some preliminary facts, we devote Sections 2 through 4 to the proofs of our partition theorems. Mathias's theorem, which asserts that every selective ultrafilter contains homogeneous sets for all partitions of $[\omega]^\omega$ into an analytic and a coanalytic piece, as well as theorems of Taylor [31] and Louveau [17] dealing with non-selective ultrafilters, will be generalized to deal with several ultrafilters. For example, our results imply that, if $[\omega]^\omega$ is partitioned into an analytic and a coanalytic piece, and if $\mathcal{U}_0$ and $\mathcal{U}_1$ are P-point ultrafilters, then there exist sets $A_0 \in \mathcal{U}_0$, $A_1 \in \mathcal{U}_1$, and a function $g : \omega \to \omega$ such

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that one piece of the partition contains every set

$$X = \{x_0 < x_1 < \cdots < x_n < x_{n+1} < \cdots\} \subseteq \omega$$

such that, for all n, $x_{2n} \in A_0$, $x_{2n+1} \in A_1$, and $g(x_n) < x_{n+1}$. Our approach to these partition theorems uses game-theoretic ideas to simplify the unwieldly combinatorics that would otherwise arise in generalizing Mathias's techniques; a significant part of the complexity of the argument can be isolated in the intuitively natural principle that "infinite strings of generalized quantifiers are unambiguous", which we present in Section 2. As a by-product of this approach, we find a connection between partition relations and games, a connection that makes the Galvin–Prikry theorem [11] on Borel partitions (strengthened to assert that the homogeneous set is in a specified selective ultrafilter) an immediate consequence of Borel determinacy [18]. In proving our main results, in Sections 3 and 4, we do not invoke Borel determinacy, but a crucial part of the argument is closely related to Wolfe's proof of F_σ-determinacy [33].

In Section 5 we recall, in a form suitable for our purposes, information about various forcing constructions, primarily Mathias forcing [19] with respect to a selective ultrafilter, forcing to add a generic ultrafilter on ω, and Lévy forcing [29] to collapse to ω all the cardinals below an inaccessible κ. We discuss analogs of Mathias forcing for non-selective ultrafilters, and we recall from [19] some properties of the Lévy model when κ is a Mahlo cardinal.

Section 6 is devoted to studying selective ultrafilters in the Lévy model obtained from a Mahlo cardinal κ. We show that any such ultrafilter is generic, and any countably many non-isomorphic such ultrafilters are mutually generic, over the submodel HDVR of sets hereditarily ordinal definable with real parameters and parameters from the ground model. This result implies that, in this Lévy model, all selective ultrafilters look alike in certain senses. (Similar techniques were used in [6] to show that, if two non-isomorphic selective ultrafilters are viewed as type two objects in the sense of higher-type recursion theory, then neither is recursive in the other.)

Finally, in Section 7, we obtain a negative result about an attempt to modify, so as to apply to selective ultrafilters, the topological homogeneity result of W. Rudin [25]. Rudin showed, assuming the continuum hypothesis, that the P-point ultrafilters constitute a single orbit of the self-homeomorphism group of the space $\beta\omega - \omega$ of non-principal ultrafilters on ω. This group has a subgroup, with a simple topological definition, whose members preserve selectivity, and it was reasonable to hope that the selective ultrafilters would constitute a single orbit of this subgroup. We shall dash this hope by showing that the orbits of this subgroup are in fact just the isomorphism classes of ultrafilters.

1. Preliminaries

Throughout this paper, 'ultrafilter' will mean 'non-principal ultrafilter on a denumerable set'. We shall often formulate definitions as though all ultrafilters were on the set ω of natural numbers. Concepts so defined are to be transferred to ultrafilters on other enumerable sets via bijections with ω; the choice of bijection will never matter.

If $\mathcal{U}$ is an ultrafilter, we shall say that a statement $\varphi(n)$ is true for *$\mathcal{U}$-almost all n*, and we shall write $(\mathcal{U}n)\,\varphi(n)$, to mean that $\{n \mid \varphi(n)\} \in \mathcal{U}$. The definition of 'ultrafilter' implies that the quantifier $(\mathcal{U}n)$ commutes with negation and conjunction and therefore with all propositional connectives.

An ultrafilter $\mathcal{U}$ is *selective* (resp. a *P-point*) if every function on ω becomes one-to-one (resp. finite-to-one) or constant, when restricted to a suitable set in $\mathcal{U}$. Although the existence of ultrafilters, in fact of $2^{2^{\aleph_0}}$ ultrafilters, on ω can be proved without any unusual hypothesis (unless one considers the axiom of choice unusual), the existence of selective ultrafilters, or even of P-points, cannot [15, 26, 32]. The continuum hypothesis (CH), however, is more than enough to ensure the existence of selective ultrafilters and non-selective P-points in great profusion [8, 24, 4].

Ramsey's theorem [22] asserts that, for any $n \in \omega$, if the set $[\omega]^n$ of n-element subsets of ω is partitioned into finitely many pieces, then there is an infinite set $H \subseteq \omega$ that is homogeneous in the sense that $[H]^n$ is included in one of the pieces. Kunen showed [8] that an ultrafilter $\mathcal{U}$ is selective if and only if the homogeneous set H in Ramsey's theorem can always be taken to be an element of $\mathcal{U}$. For this reason, selective ultrafilters are often called Ramsey ultrafilters.

The natural generalization of Ramsey's theorem for the family $[\omega]^\omega$ of infinite subsets of ω is false. It is not difficult (with the axiom of choice) to partition $[\omega]^\omega$ into two pieces so that any two sets that differ by a single element lie in different pieces; clearly such a partition has no homogeneous set. The generalization becomes true, however, if the partition is required to be well-behaved. More precisely, Silver [28] showed that homogeneous sets exist for any partition of $[\omega]^\omega$ into an analytic set and a coanalytic set, and Mathias [19] showed that a homogeneous set for such a partition can be found in any prescribed selective ultrafilter. The topological notion of 'analytic' (= continuous image of a Borel set $= \Sigma^1_1$) refers to the topology on $[\omega]^\omega$ given by the following metric: If the longest common initial segment of X and Y has size n, then the distance between X and Y is 2^{-n}. Thus, a typical basic open set consists of all the $X \in [\omega]^\omega$ that have a fixed finite set as an initial segment; note that such a basic open set is also closed.

Ultrafilters on ω are the points of the so-called Stone–Čech remainder, $\beta\omega - \omega$. (Had we admitted principal ultrafilters, we would have obtained the whole Stone–Čech compactification $\beta\omega$.) A basis for the topology of $\beta\omega - \omega$ consists of the sets

$$\hat{A} = \{\mathcal{U} \in \beta\omega - \omega \mid A \in \mathcal{U}\}$$

for all $A \subseteq \omega$. It is easy to verify that the operation $\hat{}$ commutes with finitary Boolean operations, that $\hat{A} \subseteq \hat{B}$ if and only if $A - B$ is finite (which is sometimes expressed by saying that A is almost included in B), and that $\beta\omega - \omega$ is a compact Hausdorff space. $\mathcal{U}$ is a P-point if and only if every countable intersection of neighborhoods of $\mathcal{U}$ is again a (not necessarily open) neighborhood of $\mathcal{U}$. It follows that self-homeomorphisms of $\beta\omega - \omega$ send P-points to P-points, and Rudin [25] showed that CH implies the converse: any P-point can be mapped to any other by a self-homeomorphism of $\beta\omega - \omega$. (By contrast, every self-homeomorphism of $\beta\omega$ is induced by a permutation of ω, so there are only $2^{\aleph_0}$ of these, whereas CH yields $2^{2^{\aleph_0}}$ P-points.) Since selective ultrafilters are P-points, one can view Rudin's result as a homogeneity result for selective ultrafilters, but such a view is distorted because self-homeomorphisms of $\beta\omega - \omega$ fail to preserve selectivity. In Section 7, we shall consider a natural restricted class of self-homeomorphisms that can be shown to preserve selectivity. The result we obtain (assuming CH) is, however, just the opposite of homogeneity.

If U is any ultrafilter on ω and $f:\omega \to \omega$ is not constant on any set in $\mathcal{U}$, then

$$\hat{f}(\mathcal{U}) = \{A \subseteq \omega \mid f^{-1}(A) \in \mathcal{U}\}$$

is an ultrafilter. It is the image of $\mathcal{U}$ under the unique continuous extension of f to $\hat{f}:\beta\omega \to \beta\omega$, and it satisfies the equivalence

$$(\hat{f}(\mathcal{U})n)\, \varphi(n) \;\leftrightarrow\; (\mathcal{U}n)\, \varphi(f(n)).$$

The Rudin–Keisler (RK) order of ultrafilters [8, 24] is defined by putting $\mathcal{U} \leq \mathcal{V}$ if and only if $\mathcal{U} = \hat{f}(\mathcal{V})$ for some f. If $\mathcal{U} = \hat{f}(\mathcal{V})$ for some f that is one-to-one on a set in $\mathcal{V}$, then in fact $\mathcal{U} = \hat{f}(\mathcal{V})$ for some permutation f of ω, and we say that $\mathcal{U}$ and $\mathcal{V}$ are isomorphic ($\mathcal{U} \cong \mathcal{V}$). A basic, but not entirely trivial, result about mappings of ultrafilters is that $\hat{f}(\mathcal{U}) = \mathcal{U}$ only if $f(n) = n$ for $\mathcal{U}$-almost all n. This implies that $\mathcal{U} \leq \mathcal{V} \leq \mathcal{U}$ is equivalent to $\mathcal{U} \cong \mathcal{V}$, so the RK order induces a partial order of isomorphism classes. The selective ultrafilters are precisely the RK-minimal ultrafilters, and the set of P-points is closed downward in the RK order.

A stronger relation, the Rudin–Frolík (RF) order, is defined by putting $\mathcal{U} <_{\mathrm{RF}} \mathcal{V}$ if and only if there exist ultrafilters $\mathcal{W}_n$, for $n \in \omega$, such that $\mathcal{V}$ is isomorphic to the ultrafilter

$$\mathcal{U}\text{-}\textstyle\sum_n \mathcal{W}_n = \{X \subseteq \omega \times \omega \mid (\mathcal{U}n)\,(\mathcal{W}_n k)\,(n, k) \in X\}$$

on $\omega \times \omega$, called the $\mathcal{U}$-sum of the $\mathcal{W}_n$. An equivalent statement is that $\mathcal{V}$ is the limit, with respect to $\mathcal{U}$, of some discrete sequence in $\beta\omega - \omega$. (Had we allowed the ultrafilters $\mathcal{W}_n$ to be principal, the definition would have yielded $\leq_{\mathrm{RF}}$ instead of $<_{\mathrm{RF}}$.) Since the projection from $\omega \times \omega$ to the first factor ω sends $\mathcal{U}\text{-}\sum_n \mathcal{W}_n$ to $\mathcal{U}$, we see that the RF order is stronger than the RK order. Furthermore, this projection is neither constant nor finite-to-one on any set in $\mathcal{U}\text{-}\sum_n \mathcal{W}_n$, so this ultrafilter is not a P-point; all P-points are therefore RF-minimal. We write $\mathcal{U} \otimes \mathcal{W}$ for $\mathcal{U}\text{-}\sum_n \mathcal{W}_n$ when all of the $\mathcal{W}_n$ are the same ultrafilter $\mathcal{W}$.

In Section 8, we shall need two facts about sums of ultrafilters. The first is part of a theorem of M.E. Rudin [23]; it asserts that, if

$$\mathcal{U}\text{-}\textstyle\sum_n \mathcal{W}_n \cong \mathcal{U}'\text{-}\sum_n \mathcal{W}'_n,$$

then either

(a) for $\mathcal{U}$-almost all n, $\mathcal{W}_n \cong \mathcal{Y}_n\text{-}\Sigma_n \mathcal{W}'_n$ for some ultrafilters $\mathcal{Y}_n$, or

(b) for $\mathcal{U}'$-almost all n, $\mathcal{W}'_n \cong \mathcal{Y}_n\text{-}\Sigma_m \mathcal{W}_m$ for some ultrafilters $\mathcal{Y}_n$, or

(c) for $\mathcal{U}$-almost all n, $\mathcal{W}_n$ is isomorphic to some $\mathcal{W}'_m$.

The second is a characterization, given in [3], of the RK-predecessors of $\mathcal{U} \otimes \mathcal{V}$ when $\mathcal{V}$ is a P-poi. These predecessors are of three sorts: the RK-predecessors of $\mathcal{U}$, the RK-predecessors of $\mathcal{V}$, and ultrafilters isomorphic to $\mathcal{U}'\text{-}\Sigma_n \mathcal{V}'_n$, where $\mathcal{U}' \leq \mathcal{U}$ and $\mathcal{V}'_n \leq \mathcal{V}$ for all n.

2. Quantifiers and games

We shall need some facts about the semantics of infinite strings of the 'almost all' quantifiers $(\mathcal{U}n)$ associated to ultrafilters $\mathcal{U}$. Some of these facts seem to be of sufficient independent interest to warrant a presentation in somewhat greater generality than we shall actually need. We therefore devote the first part of this section to a discussion of infinite strings of generalized quantifiers. Afterward, we relate this discussion to partition relations and specialize to quantifiers of the form $(\mathcal{U}n)$.

Consider an infinite expression of the form

$$(\mathcal{Q}_\emptyset x_0)(\mathcal{Q}_{(x_0)} x_1) \cdots (\mathcal{Q}_{(x_0,\ldots,x_{n-1})} x_n) \cdots (x_0, x_1, \ldots, x_n, \ldots) \in \mathcal{X}, \tag{1}$$

where the variables x_n range over some set A, where $\mathcal{X} \subseteq A^\omega$, and where $\mathcal{Q}_s$ is, for each finite sequence $s \in A^{<\omega}$, a (generalized) quantifier on A, that is, a family of subsets of A closed under supersets. To simplify notation, we shall write $\boldsymbol{\mathcal{Q}}$ for the entire system of quantifiers $\mathcal{Q}_s$ indexed by $A^{<\omega}$, and we shall abbreviate (1) as $(\boldsymbol{\mathcal{Q}}\boldsymbol{x})\, \boldsymbol{x} \in \mathcal{X}$, where $\boldsymbol{x}$ means $(x_0, x_1, \ldots, x_n, \ldots)$. We shall also use the symbol $^\frown$ for concatenation of sequences.

The semantics of a single generalized quantifier $\mathcal{Q}$ is defined by

$$\begin{aligned}(\mathcal{Q}x)\, \varphi(x) &\leftrightarrow (\exists X \in \mathcal{Q})(\forall x \in X)\, \varphi(x) \\ &\leftrightarrow \{x \in A \mid \varphi(x)\} \in \mathcal{Q}.\end{aligned} \tag{2}$$

The semantics of an infinite string (1) of *ordinary* quantifiers $\forall$ and $\exists$ (which are identified with the generalized quantifiers $\{A\}$ and $\{X \subseteq A \mid X \neq \emptyset\}$ respectively) is usually given in terms of a game between two players, $\forall$ and $\exists$, who consecutively choose values in A for $x_0, x_1, \ldots, x_n, \ldots$. At move n, when the values of $x_0, \ldots, x_{n-1}$ have already been chosen and are known to both players, player $\mathcal{Q}_{(x_0,\ldots,x_{n-1})}$ chooses a value for x_n. The outcome of a play is that $\exists$ (resp. $\forall$) wins if the sequence $\boldsymbol{x}$ of chosen values is in (resp. out of) $\mathcal{X}$. If $\exists$ has a winning strategy in this game, then we say that (1) is true. We shall refer to the game just described as the *game for* (1). Note that the quantifier system $\boldsymbol{\mathcal{Q}}$ alone determines what counts as a play of the game; $\mathcal{X}$ is involved only in deciding the outcome. Thus, it makes sense to speak of 'playing the game $\boldsymbol{\mathcal{Q}}$' in the sense of having $\forall$

and $\exists$ select values for $\mathbf{x}$ in accordance with the rules, but it does not make sense to speak of the outcome of the game unless $\mathscr{X}$ is also specified. Again, we can speak of 'strategies for the game $\mathscr{Q}$', but the concept of a winning strategy depends on $\mathscr{X}$.

When the quantifiers in (1) are not ordinary ones, it is natural to replace them with pairs of ordinary quantifiers by formally invoking the equivalence (2) and then to interpret the resulting formula by means of games as just described. Thus, we arrive at the notion of the *canonical game* for (1). The game consists of stages of two moves each. At stage n, values of $x_0, \ldots, x_{n-1}$ have already been chosen and are known to both players. Player $\exists$ begins stage n by choosing a set $X_n \in \mathscr{Q}_{(x_0,\ldots,x_{n-1})}$, and player $\forall$ responds by choosing a value for $x_n \in X_n$. (This part of the description covers playing the canonical game $\mathscr{Q}$.) Player $\exists$ wins a play if the chosen values satisfy $\mathbf{x} \in \mathscr{X}$, and (1) is, by definition, true if and only if $\exists$ has a winning strategy. This approach to interpreting infinite strings of generalized quantifiers was used (with all the $\mathscr{Q}_s$ equal) in [1].

The reduction (2) of generalized quantifiers to ordinary ones is not the only such reduction. For example, the quantifier 'for infinitely many natural numbers', given by

$$\infty = \{X \subseteq \omega \mid X \text{ is infinite}\},$$

is usually reduced to ordinary quantifiers by means of the equivalence

$$(\infty x)\, \varphi(x) \;\leftrightarrow\; (\forall k \in \omega)(\exists x \geq k)\, \varphi(x). \tag{3}$$

More generally, if $\mathscr{Q}$ is any quantifier on a set A, we define its dual by

$$\begin{aligned}\check{\mathscr{Q}} &= \{Y \subseteq A \mid A - Y \notin \mathscr{Q}\}\\ &= \{Y \subseteq A \mid (\forall X \in \mathscr{Q})\, X \cap Y \neq \emptyset\},\end{aligned}$$

and we have the reduction

$$(\mathscr{Q}x)\, \varphi(x) \;\leftrightarrow\; (\forall Y \in \check{\mathscr{Q}})(\exists x \in Y)\, \varphi(x) \tag{4}$$

of $\mathscr{Q}$ to ordinary quantifiers.

The definition of truth for $(\mathscr{Q}\mathbf{x})\, \mathbf{x} \in \mathscr{X}$ depended on singling out a particular 'canonical' reduction (2) which leads to the canonical game for this formula. It is certainly debatable whether the adjective 'canonical' is appropriate, since (2) doesn't seem intrinsically preferable to (3), say. But we shall show that the debate is unnecessary: all reductions lead to the same truth condition for (1). Before reading further, the reader may find it instructive to compare the canonical reduction

$$(\exists X_0 \in \infty)(\forall x_0 \in X_0)(\exists X_1 \in \infty)(\forall x_1 \in X_1) \cdots \mathbf{x} \in \mathscr{X} \tag{5}$$

of

$$(\infty x_0)(\infty x_1) \cdots \mathbf{x} \in \mathscr{X}$$

with the alternative reduction obtained from (3)

$$(\forall k_0 \in \omega)(\exists x_0 \geqslant k_0)(\forall k_1 \in \omega)(\exists x_1 \geqslant k_1) \cdots \boldsymbol{x} \in \mathscr{X}. \tag{6}$$

In particular, suppose that you are player $\forall$ and you have a winning strategy for the game for the canonical reduction; how do you go about winning the game for the alternative reduction?

We shall prove two versions of the result that different reductions yield equivalent truth conditions. The first version covers only the canonical reduction (2) and its dual (4). This version is indirectly proved (when all the $\mathscr{Q}_s$ are the same) in [1], and it is all we shall need for our study of ultrafilters. The second version, covering arbitrary reductions, is included to avoid leaving an obvious gap. It is somewhat more complicated than the first version, but the extra complexity arises mostly from the need to say what an arbitrary reduction is.

If we replace each generalized quantifier in (1) by its reduction via (4), we obtain the formula

$$(\forall Y_0 \in \breve{\mathscr{Q}}_\emptyset)(\exists x_0 \in Y_0) \cdots (\forall Y_n \in \breve{\mathscr{Q}}_{(x_0,\ldots,x_{n-1})})(\exists x_n \in Y) \cdots \boldsymbol{x} \in \mathscr{X},$$

whose game, which we call the *dual game* for (1), also consists of stages of two moves each. At stage n, $x_0, \ldots, x_{n-1}$ are already known, $\forall$ begins by choosing $Y_n \in \breve{\mathscr{Q}}_{(x_0,\ldots,x_{n-1})}$, and $\exists$ replies by choosing $x_n \in Y_n$. As before, $\exists$ wins if $\boldsymbol{x} \in \mathscr{X}$.

Theorem 1 (preliminary version). *The canonical and dual games for* $(\mathscr{Q}\boldsymbol{x})\,\boldsymbol{x} \in \mathscr{X}$ *are equivalent in the sense that a player who has a winning strategy in one of the games also has a winning strategy in the other.*

Proof. We prove the theorem for the case where $\exists$ wins one of the games; the proof for $\forall$ is the same. So we assume that $\exists$ has a winning strategy σ for one of the games, and we give instructions for how $\exists$ should play the other game. These instructions will define a strategy, and we shall see that it is a winning strategy. The instructions will always involve having $\exists$ pretend that he is playing, in addition to the actual game, an auxiliary play of the other game. $\exists$'s moves in the auxiliary game will be in accordance with σ, so he will win the auxiliary play. Furthermore, the instructions will tell $\exists$ what moves he should pretend that $\forall$ is making in the auxiliary game. These moves will be chosen so that the x's selected in the actual and auxiliary games are identical. Since the outcome of each game is determined by whether $\boldsymbol{x} \in \mathscr{X}$, and since $\exists$ wins the auxiliary game, he also wins the actual game.

Suppose first that σ is a winning strategy for $\exists$ in the canonical game. Here are the instructions $\exists$ should follow when playing the dual game and pretending to play an auxiliary canonical game. At stage n, you have already arranged that the sequence s of previous choices $x_0, \ldots, x_{n-1}$ is the same in both games. If your actual opponent now selects $Y_n \in \breve{\mathscr{Q}}_s$, use σ in the auxiliary game to select $X_n \in \mathscr{Q}_s$. By definition of $\breve{\mathscr{Q}}_s$, there is an element $x_n \in X_n \cap Y_n$. Play such an x_n in the actual game, and pretend that $\forall$ played the same x_n in the auxiliary game. Thus, the sequence of choices is still the same, namely $s^\frown\langle x_n\rangle$, in both games, as required.

Now suppose σ is a winning strategy for $\exists$ in the dual game. Here are the instructions $\exists$ should follow when playing the canonical game and pretending to play an auxiliary dual game. At stage n, you have already arranged that $s=\langle x_0,\ldots,x_{n-1}\rangle$ is the same in both games. You are required now to choose $X_n\in\mathcal{Q}_s$. Consider all possible moves $Y_n\in\check{\mathcal{Q}}_s$ that $\forall$ could make in the auxiliary game, and let X_n be the set of all σ's responses to these moves;

$$X_n=\{\sigma(W,Y_n)\mid Y_n\in\check{\mathcal{Q}}_s\}$$

where W stands for all the moves already made in the auxiliary game. By definition, X_n contains at least one element from each $Y_n\in\check{\mathcal{Q}}_s$, so $A-X_n\notin\check{\mathcal{Q}}_s$, which means that $X_n\in\mathcal{Q}_s$. Thus, X_n is a legal move in the actual game; play it. If $\forall$ responds with $x_n\in X_n$, then, by definition of X_n, there is a $Y_n\in\check{\mathcal{Q}}_s$ to which σ would respond by choosing x_n. Pretend that $\forall$ played such a Y_n in the auxiliary game and that you responded, using σ, with x_n. Again, the sequence of choices $s^\frown\langle x_n\rangle$ is the same in both games, so the proof is complete. $\square$

We return to the general situation. By a *reduction* of a quantifier $\mathcal{Q}$ on A we mean a finite string of ordinary quantifiers plus a function,

$$(Q^0y^0)(Q^1y^1)\cdots(Q^{k-1}y^{k-1}),f, \tag{7}$$

where each Q^i is $\forall$ or $\exists$, where each y^i ranges over some set (that may depend on the values of $y^0,\ldots,y^{i-1}$), where $f(y^0,y^1,\ldots,y^{k-1})\in A$ for y's in the appropriate sets, and where

$$\mathcal{Q}=\{X\subseteq A\mid (Q^0y^0)(Q^1y^1)\cdots(Q^{k-1}y^{k-1})f(y^0,y^1,\ldots,y^{k-1})\in X\}.$$

For any such reduction, we have the schema

$$(\mathcal{Q}x)\,\varphi(x)\;\leftrightarrow\;(Q^0y^0)(Q^1y^1)\cdots(Q^{k-1}y^{k-1})\,\varphi(f(y^0,y^1,\ldots,y^{k-1})), \tag{8}$$

and we refer to (7) as the reduction described by the schema (8). Thus, for example, the reduction described by (2) has $k=2$, $Q^0=\exists$, $Q^1=\forall$, y^0 ranging over $\mathcal{Q}$, y^1 ranging over y^0, and $f(y^0,y^1)=y^1$.

A reduction (7) defines a 'pseudo-game' between two players $\forall$ and $\exists$. At move i, where $0\leqslant i<k$, player Q^i, knowing the values of y^0, $y^1,\ldots,y^{i-1}$, chooses a value for y^i in the appropriate set. The outcome of a play is not a win or loss for either player (hence the terminology 'pseudo') but rather the element $f(y^0,y^1,\ldots,y^{k-1})$ of A.

Suppose we are given a formula of the form (1) and reductions (7_s) of all the quantifiers $\mathcal{Q}_s$ occurring in it. We refrain from writing out (7_s) explicitly; the reader can obtain it by attaching a subscript s to every occurrence of Q, y, k, f in (7). The schemas (8_s) describing these reductions can be used to formally replace the generalized quantifiers in (1) with ordinary quantifiers. The game associated to the resulting formula can be described as follows. Players $\forall$ and $\exists$ begin by playing the pseudo-game defined by ($7_\emptyset$). If the outcome is $x_0 \in A$, then they play the pseudo-game defined by ($7_{(x_0)}$). If the outcome of this stage is $x_1 \in A$, then they play the pseudo-game defined by ($7_{(x_0,x_1)}$), and so forth. Finally $\exists$ (resp. $\forall$) wins the play if $(x_0, x_1, \ldots)$ is in (resp. out of) $\mathscr{X}$. We call this game the *game for* (1) *reduced by* (7_s).

Theorem 1 (general version). *Suppose two reductions are given for each of the quantifiers $\mathscr{Q}_s$. Then the games for $(\mathscr{Q}x)\, x \in \mathscr{X}$, reduced by these two reductions, are equivalent in the sense that, if a player has a winning strategy for one of the games, then he also has a winning strategy for the other.*

Proof. We refer to the two reductions as (7_s) and ($7'_s$) and to the corresponding games as G and G'. Suppose σ is a winning strategy for $\exists$ in G. We shall give instructions whereby $\exists$ can play G' and pretend to play an auxiliary play of G, in such a way that, first, he plays G in accordance with σ and therefore wins the auxiliary game and, second, the outcome x_n of each stage is the same in the actual game and the auxiliary game. Since the condition for winning, $x \in \mathscr{X}$, depends only on the sequence x of these outcomes, it follows that $\exists$ will also win the actual game G'. We shall therefore have, in these instructions, a winning strategy for $\exists$ in G'. This will complete the proof of the theorem for $\exists$; the proof for $\forall$ is the same.

Here are the instructions for $\exists$ to follow at stage n. You have already arranged that $s = (x_0, \ldots, x_{n-1})$ is the same in both games. Let S be the set of all outcomes x_n that could arise in stage n of the auxiliary game G if you continue to follow σ while $\forall$ plays arbitrarily. Then

$$(Q^0y^0)(Q^1y^1)\cdots(Q^{k-1}y^{k-1})f(y^0, y^1, \ldots, y^{k-1}) \in S \tag{9}$$

holds, where subscripts s (on Q, y, k, f) have been omitted. Indeed, the game interpretation of this sentence is that you have a strategy forcing the outcome of stage n to be in S, and, by definition of S, σ is such a strategy. (Technically, it is not σ but the part of σ that deals with stage n, given the chosen moves at previous stages.) Thus, by (8_s), we have $(\mathscr{Q}_s x_n)\, x_n \in S$, and, by ($8'_s$), it follows that the formula that is like (9) but has a prime (in addition to the s already suppressed) on every Q, y, k, f to refer to ($7'_s$) also holds. In other words, there is a strategy whereby you can play stage n of the actual game G' so as to ensure that its outcome x_n is in S. Use this strategy in the actual game. When the actual stage n is finished, let x_n be its outcome. So $x_n \in S$. This means, by definition of S, that there is a play of stage n of the auxiliary game in which you use σ and the outcome is the same x_n. Pretend that this play occurs in the auxiliary game. The sequence of outcomes, $s^\frown(x_n)$, is still the same in both games, so the instructions work as desired. $\square$

Theorem 1 assures us that the preference given to the canonical reduction in the truth definition for (1) is only apparent; any reduction could be used.

One can generalize Theorem 1 a bit by allowing the quantifier Q^i in a reduction to depend on the values of $y^0, \ldots, y^{i-1}$; the proof is unchanged.

It is convenient to describe strategies for $\exists$ in the canonical game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ in terms of certain trees. A $\mathcal{Q}$-tree is a nonempty subset Y of $A^{<\omega}$ (the set of finite sequences from A) satisfying

$$T \text{ is closed under initial segments, and} \tag{10}$$

$$\text{for each } s \in T, \text{ the 'branching set'} \tag{11}$$

$$T(s) = \{x \in A \mid s^\frown(x) \in T\}$$

belongs to $\mathcal{Q}_s$, i.e., $(\mathcal{Q}_s x)\, s^\frown(x) \in T$.

If σ is a strategy for $\exists$ in the canonical game for (1), then the finite initial segments of infinite sequences $(x_0, x_1, \ldots)$ produced when $\exists$ uses σ and $\forall$ plays arbitrarily constitute a $\mathcal{Q}$-tree T_σ. Conversely, every $\mathcal{Q}$-tree T is T_σ for a unique σ, namely the strategy: after your opponent has chosen the terms of a sequence s, reply by choosing $T(s)$. Thus, we may identify strategies for $\exists$ with $\mathcal{Q}$-trees. The infinite sequences $(x_0, x_1, \ldots)$ that can arise when $\exists$ uses σ are precisely the (unions of) paths through T_σ. Thus, σ is a winning strategy if and only if all paths through T_σ lie in $\mathcal{X}$, and (1) holds if and only if there is a $\mathcal{Q}$-tree all of whose paths lie in $\mathcal{X}$. If σ and τ are strategies such that $T_\sigma \subseteq T_\tau$, then we call σ an *improvement* of τ; the terminology is justified by the observation that, if τ is a winning strategy for $(\mathcal{Q}x)\, x \in \mathcal{X}$, then so is σ.

Observe that the canonical game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ and the dual game for $(\check{\mathcal{Q}}x)\, x \notin \mathcal{X}$ differ only in that the names of the players have been interchanged. Thus, $\forall$ has a winning strategy in the canonical game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ if and only if $(\check{\mathcal{Q}}x)\, x \notin \mathcal{X}$ is true. The formula $(\check{\mathcal{Q}}x)\, x \notin \mathcal{X}$ can be formally obtained from the negation of $(\mathcal{Q}x)\, x \in \mathcal{X}$ by applying infinitely often the 'prenex operation' indicated by

$$\neg(\mathcal{Q}x)\, \varphi(x) \;\leftrightarrow\; (\check{\mathcal{Q}}x)\neg\varphi(x).$$

This equivalence is always correct, by definition of $\check{\mathcal{Q}}$, but its infinite iteration is justified only when the (canonical) game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ is determined. (We have parenthesized 'canonical', and we shall omit it altogether in such contexts in the future, since Theorem 1 assures us that the game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ reduced in any other way is determined if and only if the canonical game is determined.) The following theorem summarizes the preceding discussion.

Theorem 2. *The following are equivalent.*

(a) *Either* $(\mathcal{Q}x)\, x \in \mathcal{X}$ *or its 'formally prenexed negation'* $(\check{\mathcal{Q}}x)\, x \notin \mathcal{X}$ *is true.*

(b) *The game for* $(\mathcal{Q}x)\, x \in \mathcal{X}$ *is determined.*

(c) *Either there is a $\mathcal{Q}$-tree all of whose paths are in $\mathcal{X}$ or there is a $\check{\mathcal{Q}}$-tree none of whose paths are in $\mathcal{X}$.*

In the situation of particular interest to us where all the quantifiers $\mathcal{Q}_s$ are ultrafilters $\mathcal{U}_s$ on ω, several simplifications occur. The first is that, since ultrafilters satisfy $\check{\mathcal{U}} = \mathcal{U}$ by definition, we can omit the dualizations in Theorem 2. The second is that, since ultrafilters are closed under finite intersection, the intersection of finitely many $\mathcal{U}$-trees is a $\mathcal{U}$-tree. The third is that, since all our ultrafilters are assumed to be non-principal, the tree of strictly increasing finite sequences from ω is a $\mathcal{U}$-tree, which we identify with the set $[\omega]^{<\omega}$ of finite subsets of ω (ordered by 'initial segment') by identifying a subset of ω with its increasing enumeration. Similarly, we identify the set of paths through $[\omega]^{<\omega}$ with $[\omega]^{\omega}$. In virtue of the second and third of the preceding observations we can, when discussing $\mathcal{U}$-trees, usually confine our attention to subtrees of $[\omega]^{<\omega}$, and we shall do so without further comment.

Combining our observations, we immediately obtain the following corollary of Theorem 2.

Corollary 2.1. *If the game for $(\mathcal{U}x)\, x \in \mathcal{X}$ is determined and if all the $\mathcal{U}_s$ are ultrafilters, then there is a $\mathcal{U}$-subtree of $[\omega]^{<\omega}$ all or none of whose paths are in $\mathcal{X}$.*

If $\mathcal{X}$ is a Borel subset of $[\omega]^{\omega}$, then the game mentioned in Corollary 2.1 is a Borel game over $\omega \cup$ (Power set of ω), because the auxiliary conditions $X_n \in \mathcal{U}_{(x_0,\ldots,x_{n-1})}$ and $x_n \in X_n$ are all clopen. Since Martin's proof [18] of Borel determinacy works over arbitrary sets, the hypothesis of the corollary is automatically satisfied.

Corollary 2.2. *If $\mathcal{U}_s$ is an ultrafilter for each $s \in [\omega]^{<\omega}$ and if $\mathcal{X}$ is a Borel subset of $[\omega]^{\omega}$, then there is a $\mathcal{U}$-tree all or none of whose paths lie in $\mathcal{X}$.*

From this corollary, we easily obtain a proof of the Galvin–Prikry partition theorem [11] for Borel sets. Since a direct proof of the Galvin–Prikry theorem is simpler than any known proof of Borel determinacy, this result should be viewed, not as an alternate method for establishing the Galvin–Prikry theorem, but rather as a clarification of a connection between determinacy and partition properties.

Corollary 2.3 (Galvin–Prikry). *If $\mathcal{X}$ is a Borel subset of $[\omega]^{\omega}$, then there is an infinite $H \subseteq \omega$ all or none of whose infinite subsets are in $\mathcal{X}$.*

Proof. Let $\mathcal{U}$ be an ultrafilter, and take all $\mathcal{U}_s = \mathcal{U}$ in the preceding corollary. Let T be a $\mathcal{U}$-tree (i.e., a $\mathcal{U}$-tree with all $\mathcal{U}_s = \mathcal{U}$) all or none of whose paths lie in $\mathcal{X}$. We shall complete the proof by finding an infinite $H \subseteq \omega$ such that every infinite subset of H is a path through T. Equivalently, we find H such that every finite subset of H is a node of T. But this is easily accomplished by choosing the elements of H inductively. After the set H_n of the first n elements of H has been determined, let x be any element of the intersection of the sets $T(s)$ for all $s \subseteq H_n$; such an x exists as all of the sets $T(s)$ belong to the ultrafilter $\mathcal{U}$. Then set $H_{n+1} = H_n \cup \{x\}$; it is clear by induction on n that every subset of H_n is a node of T, so $H = \bigcup_{n \in \omega} H_n$ has the desired properties. $\square$

In the next section, we shall need some notation for dealing with games 'started in the middle'. To be more precise, consider a formula $(\mathcal{Q}x)\, x \in \mathcal{X}$ as before, and consider a finite sequence $s \in A^{<\omega}$. Then the *canonical game for* $(\mathcal{Q}x)\, x \in \mathcal{X}$ *starting at position* s is defined to be the canonical game for $(\mathcal{Q}'x)\, s^\frown x \in \mathcal{X}$ where $\mathcal{Q}'_t = \mathcal{Q}_{s^\frown t}$. To play this game, one essentially plays the canonical game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ beginning at stage length(s) and pretending that the previous (omitted) stages resulted in the sequence of outcomes s. A strategy σ for $\exists$ in this game can, of course, be described by a $\mathcal{Q}'$-tree T_σ, but we prefer to append s at the beginning and set

$$T_{\sigma,s} = \{s^\frown t \mid t \in T_\sigma\} \cup \{\text{initial segments of } s\}.$$

Then σ is a winning strategy for the canonical game for $(\mathcal{Q}x)\, x \in \mathcal{X}$ starting at s if and only if all paths through $T_{\sigma,s}$ lie in $\mathcal{X}$. Also, when writing the formula $(\mathcal{Q}'x)\, s^\frown x \in \mathcal{X}$ out explicitly, we prefer to start the indexing of the variables x at $k = \text{length}(s)$ rather than at 0, and we may then write $(x_0, \ldots, x_{k-1})$ for s.

3. Analytic determinacy for ultrafilter games

Our proof of Corollary 2.3, the Galvin–Prikry theorem, required the hypothesis that $\mathcal{X}$ is a Borel set in order to use Martin's general theorem on Borel determinacy. The same technique can be used to establish Silver's partition theorem [28] for analytic sets, but we can no longer rely on general determinacy results. Rather, we prove a determinacy theorem for games of the specific sort needed for partition properties. This section is devoted to this determinacy theorem, from which Silver's theorem follows. In the next section, the same determinacy theorem will be used to establish partition relations involving ultrafilters, generalizations of the theorem of Mathias quoted in the introduction.

We begin by fixing some notation. Let $\mathcal{U}$ be a system of ultrafilters $\mathcal{U}_s$ on ω, indexed by the finite subsets s of ω. If $\mathcal{X} \subseteq [\omega]^\omega$, we refer to the canonical game for $(\mathcal{U}x)\, \{x\} \in \mathcal{X}$ as the $\mathcal{X}$-game, and we define the $\mathcal{X}$-game starting at position $s \in [\omega]^{<\omega}$ similarly. Here $\{x\}$ means the set whose members are the terms of the infinite sequence x. Recall from Section 2 that, since the $\mathcal{U}_s$ are ultrafilters, we can confine our attention to plays whose outcomes x_n are in increasing order, so that x and $\{x\}$ are essentially equivalent. We write $s \to \mathcal{X}$ to mean that $\exists$ has a winning strategy in the $\mathcal{X}$-game starting at s, in other words, that

$(\mathcal{U}'x)\, s \cup \{x\} \in \mathcal{X}$ in the notation of Section 2. Our primary interest will be in the relation $\emptyset \to \mathcal{X}$, but the more general relation $s \to \mathcal{X}$ is needed in the analysis of it. Note that $\emptyset \to \mathcal{X}$ simply means that $\exists$ has a winning strategy in the $\mathcal{X}$-game; he can choose sets $X_n \in \mathcal{U}_{\langle x_0,\dots,x_{n-1}\rangle}$ so that, no matter what $x_n \in X_n$ his opponent chooses, the set of all the selected x's is in $\mathcal{X}$. And, by Theorem 1 and the discussion preceding Theorem 2, $\emptyset \to [\omega]^\omega - \mathcal{X}$ is equivalent to the statement that $\forall$ wins the $\mathcal{X}$-game.

Let C be the collection of those $\mathcal{X} \subseteq [\omega]^\omega$ such that, for every $s \in [\omega]^{<\omega}$, either $s \to \mathcal{X}$ or $s \to [\omega]^\omega \to \mathcal{X}$. Thus, $\mathcal{X}$ belongs to C if and only if the $\mathcal{X}$-game, and also the $\mathcal{X}$-game starting at any position, are determined. Let I be the subclass of C consisting of those $\mathcal{X}$ such that, for every $s \in [\omega]^{<\omega}$, $s \to [\omega]^\omega - \mathcal{X}$; for such an $\mathcal{X}$, $\forall$ can win the $\mathcal{X}$-game starting at any position.

Theorem 3. (a) *C is a Boolean σ-algebra of subsets of $[\omega]^\omega$, and it contains all basic open sets.*

(b) *I is a σ-ideal in C and is closed under arbitrary subsets.*

(c) *Any family of disjoint sets in $C - I$ is countable.*

(d) *For any set $\mathcal{X} \subseteq [\omega]^\omega$, there exist an F_σ-set $\mathcal{X}^-$ and a G_δ-set $\mathcal{X}^+$ such that $\mathcal{X}^- \subseteq \mathcal{X} \subseteq \mathcal{X}^+$ and no subset of $\mathcal{X} - \mathcal{X}^-$ or of $\mathcal{X}^+ - \mathcal{X}$ lies in $C - I$.*

Before proving the theorem, we point out some of its consequences.

Corollary 3.1. *C is closed under Souslin's operation $\mathcal{A}$. In particular, if $\mathcal{X}$ is analytic then the $\mathcal{X}$-game is determined.*

Proof. Either parts (a), (b), and (c) or parts (a), (b), and (d) of the theorem yield the first sentence of the corollary, by classical results of Szpilrajn [30]. (The proof using (a), (b), (d) is given in [16] and [20], while the one using (a), (b), (c) is given in [19].) The last part of the corollary follows, since $\mathcal{X} \in C$, so $\emptyset \to \mathcal{X}$ or $\emptyset \to [\omega]^\omega - \mathcal{X}$. □

Corollary 3.2. *If $\mathcal{X}$ is an analytic subset of $[\omega]^\omega$, then there is a $\mathcal{U}$-tree all or none of whose paths are in $\mathcal{X}$.*

Proof. Combine Corollaries 2.1 and 3.1. □

Corollary 3.3 (Silver [28]). *If $\mathcal{X}$ is an analytic subset of $[\omega]^\omega$, then there is an infinite $H \subseteq \omega$ all or none of whose infinite subsets are in $\mathcal{X}$.*

Proof. Proceed as in the proof of Corollary 2.3, using 3.2 in place of 2.2. □

If all the ultrafilters $\mathcal{U}_s$ are equal to a single ultrafilter $\mathcal{U}$, then the class C coincides with the class $\mathcal{C}_{\mathcal{U}}$ introduced (with a different definition) by Louveau

[17]. It is shown in [17] that this class is the class of subsets of $[\omega]^\omega$ having the Baire property in a certain topology, similar to the topology introduced by Ellentuck [10]. Since Louveau's topology satisfies the countable antichain condition for open sets, his result gives an alternate proof of parts (a) through (c) of Theorem 3 as well as the corollaries above, for the case that all $\mathscr{U}_s$ are equal.

Proof of Theorem 3. It is clear from the definitions that C is closed under complementation, that I is closed under subsets, and that $I \subseteq C$. That C contains all basic open sets is a consequence of open determinacy. A simpler proof is to verify directly that, if

$$\mathscr{X} = \{X \in [\omega]^\omega \mid t \text{ is an initial segment of } X\},$$

then $s \to \mathscr{X}$ if t is an initial segment of s, and $s \to [\omega]^\omega - \mathscr{X}$ otherwise.

We shall complete the proof of part (a) of the theorem by showing that C is closed under countable unions. Let $\mathscr{X}_n \in C$ for every $n \in \omega$, let $\mathscr{X} = \bigcup_{n\in\omega} \mathscr{X}_n$, and let $s = \{x_0, \dots, x_{k-1}\} \in [\omega]^{<\omega}$. We must prove that $s \to \mathscr{X}$ or $s \to [\omega]^\omega - \mathscr{X}$. We distinguish two cases, according to whether the formula

$$(\mathscr{U}_s x_k)(\mathscr{U}_{s\cup\{x_k\}} x_{k+1}) \cdots [\exists m\, \exists n\, \{x_0, x_1, \dots, x_{k+m-1}\} \to \mathscr{X}_n], \tag{12}$$

which we abbreviate as $(\mathscr{U}'x)\,[\exists m\, \exists n\, s \cup \{x \mid m\} \to \mathscr{X}_n]$, holds.

Case 1: The formula (12) *holds.* This case hypothesis means that, starting at position s in the $\mathscr{X}$-game, player $\exists$ can play so as to ensure that, at some finite stage of the game, he will be able to win the $\mathscr{X}_n$-game for some n. If he plays in this manner until he reaches a position from which he can win an $\mathscr{X}_n$-game and then plays so as to win that $\mathscr{X}_n$-game, then he wins the $\mathscr{X}$-game, since $\mathscr{X}_n \subseteq \mathscr{X}$. Therefore, $s \to \mathscr{X}$.

Case 2: The formula (12) *does not hold.* The part of (12) in square brackets defines an open subset of $[\omega]^\omega$, so, by open determinacy and Theorem 1, falsity of (12) means

$$(\mathscr{U}'x)\,[\forall m\, \forall n\, s \cup \{x \mid m\} \not\to \mathscr{X}_n].$$

Since $\mathscr{X}_n \in C$, we can infer $(\mathscr{U}'x)\,[\forall m\, \forall n\, s \cup \{x \mid m\} \to [\omega]^\omega - \mathscr{X}_n]$. Thus, $\exists$ has a strategy τ for the canonical game for $\mathscr{U}'$ whereby he can ensure that, at any stage of the game, he can play, from that point on, so as to force the final result $s \cup \{x\}$ out of any prescribed $\mathscr{X}_n$. We define an improvement σ of τ that will force $s \cup \{x\}$ out of $\mathscr{X}$. Recall from Section 2 that an improvement is simply a σ which, in every situation, chooses a subset (in the appropriate $\mathscr{U}_s$ of course) of the set that τ chooses; recall also that any improvement σ of τ wins every game that τ wins, so our description of τ above applies equally to σ. The instructions for σ will tell $\exists$ not only what to choose at each stage but also how to construct a sequence of auxiliary strategies ρ_n. Here are the instructions for stage m.

You have already constructed $\rho_0, \dots, \rho_{m-1}$, and the outcomes $x_k, \dots, x_{k+m-1}$ of the previous stages are (because σ improves τ) such that you can win any prescribed $[\omega]^\omega - \mathscr{X}_n$ starting at $t_m = \{x_0, x_1, \ldots, x_{k+m-1}\}$. In particular, you can win $[\omega]^\omega - \mathscr{X}_m$ starting at t_m by means of some strategy ρ_m. Play, as X_m, a set in $\mathscr{U}_{t_m}$ that is a subset of each of the sets chosen by $\tau, \rho_0, \ldots, \rho_m$ at position t_m. Such an X_m exists as $\mathscr{U}_{t_m}$ is an ultrafilter.

If $\exists$ follows these instructions then, from stage m on, he is using an improvement of ρ_m which guarantees that the final result $s \cup \{x\}$ will be outside $\mathscr{X}_m$. This holds for all m, so the result is outside $\mathscr{X}$. Therefore, $s \to [\omega]^\omega - \mathscr{X}$. This completes the proof of part (a).

To prove part (b) it remains only to check that $\boldsymbol{I}$ is closed under countable unions. The proof of this is like the proof for $\boldsymbol{C}$ just completed, except for the simplification that Case 1 never arises and in Case 2 τ is not needed.

Part (c) is easy if we recall, from Section 2, that the intersection of two $\boldsymbol{\mathscr{U}}$-trees is a $\boldsymbol{\mathscr{U}}$-tree, so that any two strategies have a common improvement. It follows that, if $s \to \mathscr{X}$ and $s \to \mathscr{Y}$, then $s \to \mathscr{X} \cap \mathscr{Y}$. In particular, if $\mathscr{X}$ and $\mathscr{Y}$ are disjoint, then no s can simultaneously satisfy $s \to \mathscr{X}$ and $s \to \mathscr{Y}$. But, by definition of $\boldsymbol{C}$ and $\boldsymbol{I}$, for each $\mathscr{X} \in \boldsymbol{C} - \boldsymbol{I}$, there is an s with $s \to \mathscr{X}$. There are only countably many $s \in [\omega]^{<\omega}$ and disjoint $\mathscr{X}$'s require distinct s's, so there cannot be uncountably many disjoint $\mathscr{X} \in \boldsymbol{C} - \boldsymbol{I}$.

To prove (d), it suffices to prove the part about $\mathscr{X}^-$, as the rest follows by complementation. Let $\mathscr{X}$ be given. For each $s \in [\omega]^{<\omega}$ such that $s \to \mathscr{X}$, fix a winning strategy σ_s for $\exists$ in the $\mathscr{X}$-game starting at s, and let $\mathscr{F}_s$ be the set of all possible outcomes $s \cup Y \in [\omega]^\omega$, where Y is the set of $\forall$'s choices in some play of this game in which $\exists$ uses strategy σ_s. As σ_s wins for $\exists$, $\mathscr{F}_s \subseteq \mathscr{X}$. $\mathscr{F}_s$ is a closed subset of $[\omega]^\omega$, namely the set of all paths through the tree $T_{\sigma_s, s}$. We define $\mathscr{X}^-$ to be the union of the $\mathscr{F}_s$ for all s such that $s \to \mathscr{X}$, so $\mathscr{X}^-$ is clearly an F_σ subset of $\mathscr{X}$. To complete the proof, we suppose that $\mathscr{Y} \in \boldsymbol{C} - \boldsymbol{I}$ and $\mathscr{Y} \subseteq \mathscr{X} - \mathscr{X}^-$, and we derive a contradiction. Since $\mathscr{Y} \in \boldsymbol{C} - \boldsymbol{I}$, there is an s such that $s \to \mathscr{Y}$; fix such an s. As $\mathscr{Y} \subseteq \mathscr{X}$, we have $s \to \mathscr{X}$, so σ_s and $\mathscr{F}_s$ are defined. Note that $s \to \mathscr{F}_s$ by definition of $\mathscr{F}_s$. Therefore $s \to \mathscr{F}_s \cap \mathscr{Y}$. But this is absurd, as $\mathscr{F}_s \cap \mathscr{Y} \subseteq \mathscr{X}^- \cap \mathscr{Y} = \emptyset$. $\square$

4. Ultrafilter partition theorems

This section is devoted to partition theorems, for analytic sets, in which homogeneous sets of various sorts are found in prescribed ultrafilters. All these results are based on Corollary 3.2, which asserts, for partitions of $[\omega]^\omega$ into an analytic piece and a coanalytic piece, the existence of a $\boldsymbol{\mathscr{U}}$-tree that is homogeneous in the sense that all its paths lie in the same piece of the partition. We shall show that, in various situations, the concepts of $\boldsymbol{\mathscr{U}}$-trees and paths through them can be replaced by simpler concepts.

For our first result, we specialize to the case that all the ultrafilters $\mathscr{U}_s$ in the system $\boldsymbol{\mathscr{U}}$ are the same ultrafilter $\mathscr{U}$; in this case we refer to $\boldsymbol{\mathscr{U}}$-trees as $\mathscr{U}$-trees.

Theorem 4. *Let $\mathcal{U}$ be an ultrafilter on ω and let $\mathcal{X}$ be an analytic subset of $[\omega]^\omega$.*

(a) *There are sets $H \supseteq H_0 \supseteq H_1 \supseteq \cdots \supseteq H_n \supseteq \cdots$ in $\mathcal{U}$ such that $\mathcal{X}$ contains all or none of the infinite sets $X \subseteq H$ that satisfy*

$$\forall x, y \in X\, [x < y \rightarrow y \in H_x].$$

(b) *There is a set $H \in \mathcal{U}$ and there is a function $f : \omega \rightarrow \omega$ such that f is not bounded on any set in $\mathcal{U}$ and $\mathcal{X}$ contains all or none of the infinite sets $X \subseteq H$ that satisfy*

$$\forall x, y \in X\, [x < y \rightarrow x < f(y)].$$

(c) (Taylor [31], see also [2, Theorem 2.3]). *If $\mathcal{U}$ is a P-point, then there is a set $H \in \mathcal{U}$ and there is a function $g : \omega \rightarrow \omega$ such that $\mathcal{X}$ contains all or none of the infinite sets $X \subseteq H$ that satisfy*

$$\forall x, y \in X\, [x < y \rightarrow g(x) < y].$$

(d) (Mathias [19]). *If $\mathcal{U}$ is selective, then there is a set $H \in \mathcal{U}$ such that $\mathcal{X}$ contains all or none of the infinite subsets of H.*

Remark. Consider the following very special case of the theorem. Let $[\omega]^k$ be partitioned into two pieces for some finite k. Let $\mathcal{X}$ consist of those infinite subsets of ω whose k smallest elements form a set in the first piece of $[\omega]^k$. Then the theorem applies to $\mathcal{X}$ since $\mathcal{X}$ is clopen. Part (d) gives us Kunen's theorem that every selective ultrafilter contains a homogeneous set for the given partition of $[\omega]^k$. Part (c) gives us a theorem of A. Taylor (see [2, Theorem 2.3]) that every P-point contains a set H that is homogeneous in the weak sense that all sufficiently spread-out (i.e., $x < y \rightarrow g(x) < y$) k-element subsets of H lie in the same piece of the given partition. It is easy to see that the spread-out condition is needed for non-selective P-points and that it does not suffice for ultrafilters that are not P-points. Indeed, if $f : \omega \rightarrow \omega$, then the partition of $[\omega]^2$ into $\{\{x, y\} \mid f(x) = f(y)\}$ and its complement has the property that on any homogeneous set f is one-to-one or constant, and on any set that is homogeneous in the weaker spread-out sense f is finite-to-one or constant. Part (b) of Theorem 4 gives, for arbitrary ultrafilters on ω, a partition theorem similar to the one for P-points but with an even weaker sort of homogeneity. This theorem does not seem to be explicit in the literature, but it is (for $k = 2$) essentially equivalent to a theorem of P[illegible] describing generating sets for ultrafilters of the form $\mathcal{U} \otimes \mathcal{U}$.

The proof of Theorem 4 reduces, by virtue of Corollary 3.2, to the following result (a special case of Theorem 4).

Theorem 4'. *Let $\mathcal{U}$ be an ultrafilter on ω and let T be a $\mathcal{U}$-tree.*

(a) *There are sets $H \supseteq H_0 \supseteq H_1 \supseteq \cdots \supseteq H_n \supseteq \cdots$ in $\mathcal{U}$ such that every infinite*

$X \subseteq H$ satisfying

$$\forall x, y \in X [x < y \rightarrow y \in H_x]$$

is a path through T.

(b) *There is a set $H \in \mathcal{U}$ and there is a function $f : \omega \rightarrow \omega$ such that f is not bounded on any set in $\mathcal{U}$ and such that every infinite $X \subseteq H$ satisfying*

$$\forall x, y \in X [x < y \rightarrow x < f(y)]$$

is a path through T.

(c) *If $\mathcal{U}$ is a P-point, then there is a set $H \in \mathcal{U}$ and there is a function $g : \omega \rightarrow \omega$ such that every infinite $X \subseteq H$ satisfying*

$$\forall x, y \in X [x < y \rightarrow g(x) < y]$$

is a path through T.

(d) (Grigorieff [12]). *If $\mathcal{U}$ is selective, then there is a set $H \in \mathcal{U}$ such that every infinite $X \subseteq H$ is a path through T.*

Proof. Since T is a $\mathcal{U}$-tree, we have, for each $s \in T$,

$$T(s) = \{x \in \omega \mid s \cup \{x\} \in T\} \in \mathcal{U}.$$

(a) Let $H = T(\emptyset)$ and, for each $n \in \omega$, let

$$H_n = \bigcap_{\substack{s \in T \\ \max(s) \leq n}} T(s);$$

this is a finite intersection, so $H_n \in \mathcal{U}$. If $X \subseteq H$ satisfies the condition in (a) and if $x_0, x_1, \ldots$ are the elements of X in increasing order, then $x_0 \in T(\emptyset)$ and, for each n, $x_{n+1} \in H_{x_n} \subseteq T(\{x_0, x_1, \ldots, x_n\})$. Therefore X is a path through T.

(b) Let $H \supseteq H_0 \supseteq H_1 \supseteq \cdots$ be as in (a) and assume, by removing all numbers less than n from H_n, that $\bigcap_{n \in \omega} H_n = \emptyset$. Define $f : \omega \rightarrow \omega$ by

$$f(n) = \text{the smallest } k \text{ such that } n \notin H_k.$$

Then f is not bounded by any $k \in \omega$ on any $A \in \mathcal{U}$, because otherwise H_k and A would be disjoint whereas they are both in $\mathcal{U}$. The definition of f implies that any X satisfying the condition in (b) also satisfies the condition in (a).

(c) Let $H' \in \mathcal{U}$ and f have the properties specified in (b). Since $\mathcal{U}$ is a P-point and f is not constant on any set in $\mathcal{U}$, f must be finite-to-one on some set $H \in \mathcal{U}$, which we can take to be a subset of H'. As f is finite-to-one on H, there exists, for each $x \in \omega$, an upper bound $g(x)$ for the elements of H that f maps to values $\leq x$. Thus, for any $y \in H$,

$$g(x) < y \rightarrow x < f(y),$$

so the condition on X in (c) implies the one in (b).

(d) Having mentioned Kunen's theorem (selective ultrafilters are Ramsey) as a consequence of Theorem 4(d), we should avoid using the former in the proof of the latter. What follows is therefore not the quickest proof of (d); that would involve applying Kunen's theorem to the partition of pairs $\{x<y\}$ according to whether $g(x)<y$.

Since selective ultrafilters are P-points, find $H'\in\mathcal{U}$ and $g:\omega\to\omega$ with the properties specified in (c). Increasing g if necessary, we may assume that g is strictly increasing and that $g(x)>x$ for all x. Consider the sequence $0<g(0)<g^2(0)<\cdots$ obtained by starting with 0 and repeatedly applying g. Let $h:\omega\to\omega$ be the function which, on each interval $[g^k(0), g^{k+1}(0))$, is constant with value k. (Here $g^0(0)$ means 0.) Since h is not constant on any infinite set, selectivity of $\mathcal{U}$ requires h to be one-to-one on some $A\in\mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, it contains one of the sets $\{x\in\omega \mid h(x) \text{ is even}\}$ and $\{x\in\omega \mid h(x) \text{ is odd}\}$; let B be the intersection of this set with A. If $x<y$ are two elements of B, and $k=h(x)$, then $k<h(y)$ as h is one-to-one on B, and $k+2\leq h(y)$ as the parity of $h(y)$ matches that of $h(x)=k$. By definition of h and monotonicity of g, we have $x<g^{k+1}(0)$ and $g(x)<g^{k+2}(0)\leq y$. Thus, the condition in (c) is satisfied whenever x and y are in B, so $H=H'\cap B$ is as required in (d). □

We turn now to applications of Corollary 3.2 in which the ultrafilters $\mathcal{U}_s$ are not all the same. Since our main results about this situation, Theorems 6 and 7 below, are notationally complicated, we give first a special case whose proof is more transparent. (This special case was used in [6].) If A and B are two infinite subsets of ω (disjoint in all the interesting cases), we say that an infinite $X\subseteq\omega$ is *chosen alternately from A and B* if, when the elements of X are listed in increasing order as $x_0<x_1<\cdots$, x_n is in A for all even n and in B for all odd n.

Theorem 5. *Let $\mathcal{U}$ and $\mathcal{V}$ be non-isomorphic selective ultrafilters on ω, and let $\mathcal{X}$ be an analytic subset of $[\omega]^\omega$. There exist sets $A\in\mathcal{U}$ and $B\in\mathcal{V}$ such that $\mathcal{X}$ contains all or none of the infinite sets chosen alternately from A and B.*

Proof. Define $\mathcal{U}$ by letting $\mathcal{U}_s$ be $\mathcal{U}$ or $\mathcal{V}$ according to whether the cardinality $|s|$ of s is even or odd. As before, Corollary 3.2 reduces the proof of Theorem 5 to the following special case.

Theorem 5′. *Let $\mathcal{U}$ be as above and let T be a $\mathcal{U}$-tree. There exist sets $A\in\mathcal{U}$ and $B\in\mathcal{V}$ such that all infinite sets X chosen alternately from A and B are paths through T.*

Proof. As T is a $\mathcal{U}$-tree, the sets $T(s)$ are in $\mathcal{U}$ or $\mathcal{V}$ according to whether $|s|$ is even or odd. Thus, for every $m\in\omega$,

$$A_m=\bigcap_{\substack{s\in T,\,|s|\text{even}\\ \max(s)\leq m}} T(s)\in\mathcal{U},\quad\text{and}\quad B_m=\bigcap_{\substack{s\in T,\,|s|\text{odd}\\ \max(s)\leq m}} T(s)\in\mathcal{V}.$$

There is an $A' \in \mathcal{U}$ such that

$$\forall x, y \in A' [x < y \rightarrow y \in A_x];$$

this can be seen either by following the proof of Theorem 4′ or by applying the Ramsey property of $\mathcal{U}$ to the partition of the pairs $\{x < y\}$ according to whether or not $y \in A_x$. (Any homogeneous set A' for this partition either is as required or satisfies $\forall x, y \in A' [x < y \rightarrow y \notin A_x]$, but the latter is absurd since it makes A' disjoint from $A_a - \{a\} \in \mathcal{U}$, where a is the smallest member of A'.) Similarly, there is a $B' \in \mathcal{V}$ such that

$$\forall x, y \in B' [x < y \rightarrow y \in B_x].$$

Since $\mathcal{U}$ and $\mathcal{V}$ are non-isomorphic and are minimal in the Rudin–Keisler ordering, no function $f : \omega \rightarrow \omega$ can map one of them to the other. We apply this first with

$$f(x) = \text{the first element of } B' \text{ larger than } x.$$

Since $f(\mathcal{U}) \neq \mathcal{V}$, there is a set $V \in \mathcal{V}$ with $f^{-1}(V) \notin \mathcal{U}$ and therefore $f^{-1}(\omega - V) \in \mathcal{U}$. Similarly, taking

$$g(x) = \text{the first element of } A' \text{ larger than } x,$$

we obtain a set $U \in \mathcal{U}$ such that $g^{-1}(\omega - U) \in \mathcal{V}$. Set

$$A = A' \cap f^{-1}(\omega - V) \cap U \cap T(\emptyset) \quad \text{and} \quad B = B' \cap V \cap g^{-1}(\omega - U).$$

So $A \in \mathcal{U}$ and $B \in \mathcal{V}$. We intend to show that, if X is chosen alternately from A and B, then X is a path through T. Let the members of X be $x_0 < x_1 < \cdots$, so $x_n \in A$ for even n and $x_n \in B$ for odd n. We show by induction that every initial segment of X is in T. Assume $\{x_0, \ldots, x_{n-1}\} \in T$; we must show that $x_n \in T(\{x_0, \ldots, x_{n-1}\})$. Suppose n is even. (The other case is analogous.) If $n = 0$, then $x_0 \in T(\emptyset)$ because $A \subseteq T(\emptyset)$. So assume $n > 0$. Since $x_n \in A' \cap U$ and $x_{n-1} \in g^{-1}(\omega - U)$, we see that $g(x_{n-1}) \neq x_n$ and therefore $g(x_{n-1})$ is an element $z \in A'$ strictly between x_{n-1} and x_n. By our choice of A', we have $x_n \in A_z$, which means that $x_n \in T(s)$ for all $s \in T$ with $|s|$ even and $\max(s) \leq z$. But $\{x_0, \ldots, x_{n-1}\}$ is such an s, being in T by induction hypothesis. So $x_n \in T(\{x_0, \ldots, x_{n-1}\})$ as desired. $\square$

Remark. In Theorems 5 and 5′, the assumption that $\mathcal{U}$ and $\mathcal{V}$ are not isomorphic is essential. Consider, for example, what happens when $\mathcal{U}$ and $\mathcal{V}$ are distinct but isomorphic selective ultrafilters on ω. Let f be an isomorphism from $\mathcal{U}$ to $\mathcal{V}$. We may assume (by interchanging $\mathcal{U}$ with $\mathcal{V}$ and f with f^{-1}, and by altering f on a set not in $\mathcal{U}$, if necessary) that $f(x) > x$ for all x. Then the clopen set $\mathcal{X}$ consisting of those $X = \{x_0 < x_1 < \cdots\} \in [\omega]^\omega$ such that $f(x_0) = x_1$ does not have the partition property asserted in Theorem 5. (Thus, even the partition relation for pairs fails.) There is also a Borel function $F : [\omega]^\omega \rightarrow \mathcal{P}(\omega)$ such that, if $A \in \mathcal{U}$ and $B \in \mathcal{V}$, then every $Z \subseteq \omega$ occurs as $F(X)$ for some X chosen alternately from A and B, namely

$$F(\{x_0 < x_1 < \cdots\}) = \{n \in \omega \mid f(x_{2n}) = x_{2n+1}\}.$$

This F is in some sense the 'most general' counterexample to Theorem 5 when $\mathcal{U}$, $\mathcal{V}$, and f are as above. In particular, if $\mathcal{X}$ is an analytic subset of $[\omega]^\omega$, then there are $A \in \mathcal{U}$ and $B \in \mathcal{V}$ such that $\mathcal{X}$ contains all or none of the sets X chosen alternately from A and B and satisfying $F(X) = \emptyset$. This can be proved similarly to Theorem 5, and it can also be deduced without much difficulty from Mathias's Theorem 4(d).

Theorem 6. *Let $\mathcal{U}_s$ be a P-point ultrafilter on ω for each $s \in [\omega]^{<\omega}$, and let $\mathcal{X}$ be an analytic subset of $[\omega]^\omega$. There exists a function $g: \omega \to \omega$ and there exists a function Z, assigning to each ultrafilter $\mathcal{U}$ that occurs as a $\mathcal{U}_s$ some element $Z(\mathcal{U}) \in \mathcal{U}$, such that $\mathcal{X}$ contains all or none of the infinite subsets $\{x_0 < x_1 < \cdots\}$ of ω that satisfy, for all $n \in \omega$,*

$$x_n \in Z(\mathcal{U}_{\{x_0,\dots,x_{n-1}\}}) \quad \text{and} \quad g(x_n) < x_{n+1}.$$

As in the previous situations, this follows from a special case via Corollary 3.2.

Theorem 6'. *Let $\mathcal{U}_s$ be a P-point ultrafilter on ω for each $s \in [\omega]^\omega$, and let T be a $\mathcal{U}$-tree. There exist g and Z as in Theorem 6 such that every infinite subset $\{x_0 < x_1 < \cdots\}$ satisfying for all $n \in \omega$*

$$x_n \in Z(\mathcal{U}_{\{x_0,\dots,x_{n-1}\}}) \quad \text{and} \quad g(x_n) < x_{n+1}$$

is a path through T.

Remark. If all of the $\mathcal{U}_s$ were distinct, we could just set $Z(\mathcal{U}_s) = T(s)$ and dispense with g. The point of the theorem is that, if the same $\mathcal{U}$ occurs as $\mathcal{U}_s$ for many different s (e.g., if $\mathcal{U}_s$ depends only on $|s|$ as in Theorem 5), then the same $Z(\mathcal{U})$ works for all these s simultaneously. Here g and the assumption that $\mathcal{U}$ is a P-point will be essential.

The special case of Theorem 6 where $\mathcal{U}_s$ depends only on the parity of $|s|$ (so there are only two $\mathcal{U}$'s) is the partition theorem quoted in the introduction.

Proof of Theorem 6'. For each $\mathcal{U}$ that occurs as $\mathcal{U}_s$ for some s, let $I(\mathcal{U})$ be the set of such indices s. Define, for each such $\mathcal{U}$ and each $m \in \omega$,

$$B_m(\mathcal{U}) = \bigcap_{\substack{s \in I(\mathcal{U}) \\ \max(s) \leq m}} T(s),$$

so $B_m(\mathcal{U}) \in \mathcal{U}$. As $\mathcal{U}$ is a P-point, it contains a set that is almost included in $B_m(\mathcal{U})$ for every m. Let $Z(\mathcal{U})$ be such a set, chosen to be $\subseteq T(\emptyset)$ if $\mathcal{U} = \mathcal{U}_\emptyset$, and let $b_m(\mathcal{U}) \in \omega$ be larger than all the (finitely many) members of $Z(\mathcal{U}) - B_m(\mathcal{U})$.

Define $g(m)$ to be the maximum value of $b_m(\mathcal{U})$, where $\mathcal{U}$ ranges over the (finitely many) ultrafilters that occur as $\mathcal{U}_s$ with $\max(s) \leq m$. We shall show that the functions Z and g so defined satisfy the conclusion of Theorem 6′. Let $\{x_0 < x_1 < \cdots\}$ satisfy the requirements in that conclusion. In particular, $x_0 \in Z(\mathcal{U}_\emptyset) \subseteq T(\emptyset)$, so $\{x_0\} \in T$. Assume as an induction hypothesis that $s = \{x_0, \ldots, x_{n-1}\} \in T$ for a certain $n \geq 1$. Then $x_n \in Z(\mathcal{U}_s)$ and $x_n > g(x_{n-1}) \geq b_{x_{n-1}}(\mathcal{U}_s)$. By definition of $b_m(\mathcal{U})$ and $B_m(\mathcal{U})$, it follows that $x_n \in B_m(\mathcal{U}_s) \subseteq T(s)$, which means that $\{x_0, \ldots, x_{n-1}, x_n\} \in T$. So $\{x_0, x_1, \ldots\}$ is a path through T, as desired. $\square$

Our next theorem will allow us, when the $\mathcal{U}_s$ are selective and distinct $\mathcal{U}_s$ are non-isomorphic, to remove all references to g from Theorem 6. The proof uses the following lemma, which may be of some independent interest.

Lemma 7.1. *Let $\mathcal{U}_n$ $(n \in \omega)$ be RK-incomparable P-points, and let $A_n \in \mathcal{U}_n$. Then there exist sets $B_n \in \mathcal{U}_n$ such that $B_n \subseteq A_n$ and*

$$\forall u, v, m, n\, [u \in B_m,\ v \in B_n,\ u < v,\ m \neq n \rightarrow (\exists w \in A_m)\, u < w < v].$$

Proof. Temporarily fix m and n with $m \neq n$. The function f, defined on all but a finite subset of ω by

$$f(v) = \text{the largest element of } A_m \text{ that is } < v,$$

does not map $\mathcal{U}_n$ to $\mathcal{U}_m$ as these are RK-incomparable ultrafilters. So there exist $C^n_m \in \mathcal{U}_m$ and $D^m_n \in \mathcal{U}_n$ such that f is defined at all points of D^m_n and maps them to points outside C^n_m. (The notation for the C's and D's was chosen so that the subscript of a set indicates the ultrafilter in which that set lies.) Thus,

$$u \in C^n_m,\ v \in D^m_n,\ u < v \ \rightarrow\ (\exists w \in A_m)\, u < w < v. \tag{13}$$

Now un-fix m and n. Each $\mathcal{U}_m$ is a P-point, so let $E_m \in \mathcal{U}_m$ be almost included in C^n_m and in D^n_m for every n. If E_m were actually included, rather than almost included, in every C^n_m and D^n_m, then we would have

$$u \in E_m,\ v \in E_n,\ u < v,\ m \neq n \ \rightarrow\ u \in C^n_m,\ v \in D^m_n,\ u < v$$

and so

$$u \in E_m,\ v \in E_n,\ u < v,\ m \neq n \ \rightarrow\ (\exists w \in A_m)\, u < w < v, \tag{14}$$

and the lemma would be proved. But, because of the almost-inclusion, there may be counterexamples to (14).

Temporarily fix m and n again, with $m \neq n$. The counterexamples to (14) are of two (possibly overlapping) sorts; either

$$u \in E_m - C^n_m, \quad v \in E_n, \quad \text{and} \quad u < v \leq \min\{w \in A_m \mid w > u\} \tag{15}$$

or

$$u \in E_m, \quad v \in E_n - D^m_n, \quad u < v. \tag{16}$$

By choice of E_m, there are only finitely many u as in (15), and for each such u there are only finitely many v as in (15) because $v \leq \min\{w \in E_m \mid w > u\}$. Similarly, by choice of E_n, there are only finitely many v as in (16) and for each such v only finitely many u, as $u < v$. Thus, for our fixed m and n, there are only finitely many counterexamples (u, v) to (14).

Again un-fix m and n, and refer to the counterexamples discussed in the preceding paragraph as (m, n)-counterexamples. For each n, obtain B_n by removing from $A_n \cap E_n$ any element that is a component of an (m, n)-counterexample or an (n, m)-counterexample for any $m < n$. As A_n and E_n are in $\mathcal{U}_n$ and only finitely many elements have been removed from $A_n \cap E_n$, we have $B_n \in \mathcal{U}_n$. But every counterexample (for any (m, n)) has had at least one of its components removed (the component corresponding to the larger of m and n), so (14) becomes true if the E's are replaced by B's. $\square$

Corollary 7.2. *Let $\mathcal{U}_n$ $(n \in \omega)$ be pairwise non-isomorphic selective ultrafilters on ω and let $g : \omega \to \omega$. There are sets $B_n \in \mathcal{U}_n$ such that*

$$(\forall x, y \in \bigcup_n B_n)\,[x < y \to g(x) < y].$$

Proof. By selectivity, find sets $A_n \in \mathcal{U}_n$ satisfying

$$\forall x, y \in A_n\,[x < y \to g(x) < y]$$

(as in the proof of Theorem 4′(d)). Then let $B_n \in \mathcal{U}_n$ be as in Lemma 7.1, and suppose $x < y$ are in $\bigcup_n B_n$. If x and y are in the same B_n, then as $B_n \subseteq A_n$ we have $g(x) < y$ by choice of A_n. If they are in different B's, say $x \in B_m$ and $y \in B_n$, then the choice of the B's ensures that $x < w < y$ for some $w \in A_m$. So $g(x) < w < y$. $\square$

The following is an immediate consequence of Theorems 6 and 6′ and Corollary 7.2.

Theorem 7 (and 7′). *Assume that $\mathcal{U}_s$ is a selective ultrafilter on ω for every $s \in [\omega]^{\omega}$ and that, for any $s, t \in [\omega]^{<\omega}$ the ultrafilters $\mathcal{U}_s$ and $\mathcal{U}_t$ are either equal or non-isomorphic. Then the function g in Theorem 6 (and 6′) can be taken to the identity function.*

Of course, taking g to be the identity function amounts to deleting the requirement that $g(x_n) < x_{n+1}$ and therefore amounts to removing g altogether from the statement of Theorem 6 (and 6′).

5. Some notions of forcing

Mathias [19] associated to every selective ultrafilter $\mathcal{U}$ on ω a notion of forcing, $P_{\mathcal{U}}$, which adjoins to the universe an infinite subset R of ω such that each set in $\mathcal{U}$

contains all but finitely many elements of R. We shall discuss in this section a similar notion of forcing $Q(\mathcal{U})$ associated to a family $\mathcal{U} = (\mathcal{U}_s : s \in [\omega]^{<\omega})$ of ultrafilters on ω; we obtain analogs in this context of many of Mathias's results for the selective case, and we use this forcing construction to make some progress on a question of Prikry and Laver. At the end of the section, we also review some well-known facts about adjoining an ultrafilter on ω by forcing and about Solovay's Lebesgue measure model [29].

The terminology 'Mathias forcing' is used with somewhat different meanings by different authors. Mathias [19] defined a notion of forcing $P_{\mathcal{A}}$, where $\mathcal{A}$ is the complement of a free ideal on ω. (In particular, $\mathcal{A}$ could be an ultrafilter.) $P_{\mathcal{A}}$ consists of pairs $\langle s, A\rangle$ where $s \in [\omega]^{<\omega}$, $A \in \mathcal{A}$, and all elements of s are smaller than $\min(A)$; $\langle s, A\rangle$ is an extension of $\langle t, B\rangle$ if $s \supseteq t$, $A \subseteq B$, and $s - t \subseteq B$. The forcing is viewed as adjoining an infinite subset R of ω, about which $\langle s, A\rangle$ gives the information that $s \subseteq R \subseteq s \cup A$. Two special cases received much attention in [19], the case that $\mathcal{A}$ consists of all infinite subsets of ω and the case that $\mathcal{A}$ is a selective ultrafilter on ω. By 'Mathias forcing', some people mean one of these special cases, some mean the other, some mean either one, some mean the case where $\mathcal{A}$ is an arbitrary ultrafilter on ω, and some mean the most general case considered by Mathias, where $\mathcal{A}$ is merely the complement of a free ideal. It will be convenient in this paper to use the name 'Mathias forcing' in a sense broad enough to include $P_{\mathcal{A}}$ for all ultrafilters $\mathcal{A}$ on ω; we shall not have occasion to consider more general $\mathcal{A}$. The reader should be warned, however, that Mathias forcing for non-selective ultrafilters does not share all the familiar properties of Mathias forcing for selective ultrafilters; for example, it need not adjoin any functions $\omega \to \omega$ that eventually dominate all ground model functions [7]. It seems that, in some respects, the appropriate analog for non-selective ultrafilters of Mathias forcing for selective ultrafilters is not Mathias forcing but rather the forcing $Q(\mathcal{U})$ (or its special case with all $\mathcal{U}_s$ equal) that we are about to define.

Let $\mathcal{U} = (\mathcal{U}_s : s \in [\omega]^{<\omega})$ be a family of ultrafilters on ω indexed by the finite subsets s of ω. As in Section 3, if $\mathcal{X} \subseteq [\omega]^{\omega}$ and $s \in [\omega]^{<\omega}$, we write $s \to \mathcal{X}$ to mean that player $\exists$ has a winning strategy in the $\mathcal{X}$-game (the canonical game for $(\mathcal{U}x)\ \{x\} \in \mathcal{X}$) starting at position s. Such a strategy is represented, as in Section 2, by a tree T (there called $T_{\sigma,s}$), a subtree of $[\omega]^{<\omega}$ ordered by 'initial segment of', in which every node is comparable with s and every node $t \supseteq s$ has branching set

$$T(t) = \{n > \max(t) \mid t \cup \{n\} \in T\} \in \mathcal{U}_t.$$

Thus, T consists of a trunk, ending at s, beyond which every node has many immediate successors, 'many' being in the sense of the appropriate ultrafilter. We call such a tree a *$\mathcal{U}$-tree with stem s.*

As in Section 3, let $\boldsymbol{I}$ (resp. $\boldsymbol{C}$) be the collection of all $\mathcal{X} \subseteq [\omega]^{\omega}$ such that, for every $s \in [\omega]^{<\omega}$, $s \to [\omega]^{\omega} - \mathcal{X}$ (resp. $s \to \mathcal{X}$ or $s \to [\omega]^{\omega} - \mathcal{X}$). By Theorem 3(a, b), $\boldsymbol{C}$ is a Boolean σ-algebra and $\boldsymbol{I}$ is a σ-ideal, so the quotient algebra $\boldsymbol{B} = \boldsymbol{C}/\boldsymbol{I}$ is also a Boolean σ-algebra. By Theorem 3(c), $\boldsymbol{B}$ satisfies the countable chain condition, i.e., every collection of pairwise disjoint non-zero elements of $\boldsymbol{B}$ is countable; it is well known [27 (20.5)] that, in the presence of the countable chain condition, countable completeness implies completeness, so $\boldsymbol{B}$ is in fact a complete Boolean algebra. $\boldsymbol{B}$ is the complete Boolean algebra associated to the forcing $Q(\mathcal{U})$ (or just Q, as $\mathcal{U}$ is fixed) that we shall study. We could simply define Q to be $\boldsymbol{B}-\{0\}$, but it will often be more convenient to use a certain dense subset instead. Before defining this Q, we record for future reference that we use $[\mathscr{X}]$ for the equivalence class in $\boldsymbol{B}$ of a set $\mathscr{X}\in\boldsymbol{C}$ and that Theorem 3(d) implies that every element of $\boldsymbol{B}$ is $[\mathscr{X}]$ for some Borel set $\mathscr{X}$, for if $\mathscr{X}\in\boldsymbol{C}$ then $\mathscr{X}$, $\mathscr{X}^+$, and $\mathscr{X}^-$ (as in Theorem 3(d)) all differ by sets in $\boldsymbol{I}$. Thus, $\boldsymbol{B}$ can also be viewed as the quotient of the algebra of Borel sets by its intersection with $\boldsymbol{I}$.

Consider any non-zero $[\mathscr{X}]\in\boldsymbol{B}$. So $\mathscr{X}\in\boldsymbol{C}-\boldsymbol{I}$, which implies that $s\twoheadrightarrow\mathscr{X}$ for at least one s. Fix such an s, and let T be a $\mathcal{U}$-tree with stem s representing a winning strategy for $\exists$ in the $\mathscr{X}$-game starting at position s. As the strategy is a winning one, all paths through T are in $\mathscr{X}$. Writing Paths(T) for the set of all paths through T, we have

$$\begin{aligned}&\mathrm{Paths}(T)\subseteq\mathscr{X},\\ &\mathrm{Paths}(T)\in\boldsymbol{C}\qquad\text{(as Paths}(T)\text{ is closed), and}\\ &\mathrm{Paths}(T)\notin\boldsymbol{I}\qquad\text{(as } s\twoheadrightarrow\mathrm{Paths}(T)\text{, thanks to the strategy represented by } T).\end{aligned}$$

Thus, the elements of $\boldsymbol{B}$ of the form [Paths(T)], where T ranges over $\mathcal{U}$-trees with arbitrary stems, constitute a dense subset of $\boldsymbol{B}-\{0\}$. We observe that the trivial implications (where T, T' are $\mathcal{U}$-trees with arbitrary stems)

$$T\subseteq T'\;\Rightarrow\;\mathrm{Paths}(T)\subseteq\mathrm{Paths}(T')\;\Rightarrow\;[\mathrm{Paths}(T)]\leqslant[\mathrm{Paths}(T')]$$

are reversible, for if T has a node t not in T', then the subtree T'' of T consisting of the nodes of T comparable with t has Paths(T'') included in Paths(T) but disjoint from Paths(T'), so $[\mathrm{Paths}(T)]\nleqslant[\mathrm{Paths}(T')]$.

Thus the partial ordering $Q=Q(\mathcal{U})$ of all $\mathcal{U}$-trees with arbitrary stems, ordered by inclusion, is embedded as a dense subset of $\boldsymbol{B}$ by $T\mapsto[\mathrm{Paths}(T)]$. The Boolean-valued universe $V^{\boldsymbol{B}}$ may therefore be viewed as arising from V by adjoining either a V-generic ultrafilter G in $\boldsymbol{B}$ or the V-generic subset $G\restriction Q=\{T\in Q\mid[\mathrm{Paths}(T)]\in G\}$ of Q.

Each of the primed theorems of Section 4 (Theorems 4′ to 7′) describes, under certain hypotheses on $\mathcal{U}$, a dense subset of $Q(\mathcal{U})$, which may provide a more convenient, though of course equivalent, way to view Q-forcing. (Actually, they describe $\mathcal{U}$-trees without stems, but it is trivial to take stems into account.) In particular, Theorem 4′(d) says that, when all of the $\mathcal{U}_s$ are the same selective ultrafilter $\mathcal{U}$, then the trees of the form

$$\{t\in[\omega]^{<\omega}\mid s\text{ is an initial segment of }t\text{ and }t-s\subseteq H\},$$

where $s\in[\omega]^{<\omega}$ and $H\in\mathcal{U}$, are dense in Q. Since the inclusion relation on such trees corresponds exactly to the ordering of Mathias conditions $\langle s, H\rangle$, we see that Q-forcing is in this case equivalent to Mathias forcing.

Similarly, Theorem 4′(c) says that, when all $\mathcal{U}_s$ are the same P-point $\mathcal{U}$ (not necessarily selective), then Q has a dense subset consisting of trees of the form

$$\{t \in [\omega]^{<\omega} \mid s \text{ is an initial segment of } t \text{ and } t - s \subseteq H, \text{ and}$$
$$\forall x, y \in t - s\, [x < y \to g(x) < y]\},$$

where $s \in [\omega]^{<\omega}$, $H \in \mathcal{U}$, and $g: \omega \to \omega$. Thus, Q-forcing is equivalent in this case to forcing with triples $\langle s, H, g\rangle$ of this sort, the ordering of these triples being that $\langle s', H', g'\rangle$ extends $\langle s, H, g\rangle$ if $s' \supseteq s$, $H' \subseteq H$, $g' \geqslant g$ on H', $s' - s \subseteq H$, and $\forall x, y \in s' - s[x < y \to g(x) < y]$.

The remaining primed theorems of Section 4 clearly have similar, though more complicated, consequences in terms of dense subsets of Q.

Any V-generic ultrafilter G in $\boldsymbol{B}$ determines an infinite subset $R = R_G$ of ω by

$$n \in R_G \Leftrightarrow [\{X \in [\omega]^{\omega} \mid n \in X\}] \in G. \tag{17}$$

It follows, by an easy induction over Borel sets, that if $\mathcal{B}$ is any Borel subset of $[\omega]^{\omega}$ in V and $\bar{\mathcal{B}}$ is its canonical extension in some extension of V containing R_G (i.e., $\bar{\mathcal{B}}$ is coded by the same real as $\mathcal{B}$; see [29] for details), then

$$R_G \in \bar{\mathcal{B}} \Leftrightarrow [\mathcal{B}] \in G. \tag{18}$$

In particular, since every element of $\boldsymbol{B}$ is represented by a Borel set, R_G completely determines G; $V[G] = V[R_G]$. We may therefore view forcing with $\boldsymbol{B}$ (or Q) as simply adjoining a subset R_G of ω, and we call the R_G's that arise in this way Q-generic (over V). Notice that the generic subset $G \restriction Q$ of Q is definable from R_G by

$$\begin{aligned} T \in G\restriction Q &\Leftrightarrow R_G \in \overline{\mathrm{Paths}(T)} \\ &\Leftrightarrow R_G \text{ is a path through } T. \end{aligned} \tag{19}$$

It follows from (18) that, if X is Q-generic then $X \notin \bar{\mathcal{B}}$ for any Borel set (of V) $\mathcal{B} \in \boldsymbol{I}$. The converse holds also, for if X is not in the extension of any $\mathcal{B} \in \boldsymbol{I}$, then, for arbitrary Borel sets $\mathcal{B}$ of V, whether X belongs to $\bar{\mathcal{B}}$ depends only on $[\mathcal{B}]$, so we can define $G \subseteq \mathcal{B}$ by

$$[\mathcal{B}] \in G \Leftrightarrow X \in \bar{\mathcal{B}}.$$

Then clearly $X = R_G$, and it is not hard to verify that G is a V-generic ultrafilter in $\boldsymbol{B}$. (The proof uses the fact that every maximal antichain of $\boldsymbol{B}$ in V is countable and can be represented by Borel sets $\mathcal{B}_n$ whose union is all of $[\omega]^{\omega}$ in V. Then the $\bar{\mathcal{B}}_n$ cover $[\omega]^{\omega}$ in the extended universe, so, for some n, $X \in \bar{\mathcal{B}}_n$ and therefore $[\mathcal{B}_n] \in G$.) All this is entirely analogous to the discussion of random reals in [29]; under this analogy, $\boldsymbol{C}$ and $\boldsymbol{I}$ correspond to the algebra of Lebesgue measurable sets and the ideal of sets of measure zero.

The following definitions lead to a somewhat more concrete characterization of Q-generic subsets of ω, Theorem 8(a, c) below, analogous to Mathias's characterization [19] of the $P_{\mathcal{U}}$-generic sets for selective $\mathcal{U}$ as those infinite subsets of ω that are almost included in every set of $\mathcal{U}$ (in the ground model). By a *$\mathcal{U}$-regulation* (or just a regulation, as long as $\mathcal{U}$ is fixed), we mean a function Z assigning to each $s \in [\omega]^{<\omega}$ a set $Z(s) \in \mathcal{U}_s$. We say that an infinite subset $\{x_0 < x_1 < \cdots\}$ of ω *obeys* the regulation Z if, for every $n \in \omega$, $x_n \in Z(\{x_0, \ldots, x_{n-1}\})$. If this holds for all sufficiently large n (rather than for all n), then we say that X *eventually obeys* Z.

Theorem 8. *Let X be an infinite subset of ω in some extension of V. Then the following are equivalent.*

(a) *X is $Q(\mathcal{U})$-generic over V.*

(b) *$X \notin \bar{\mathcal{B}}$ for any Borel set $\mathcal{B} \in I$ in V.*

(c) *X eventually obeys each $\mathcal{U}$-regulation $Z \in V$.*

Proof. The equivalence of (a) and (b) was proved above. To show that (a) implies (c), it suffices to show, for each given regulation A, that the trees T, all of whose paths eventually obey Z, are dense in Q. But this is easy, since any tree $T_0 \in Q$ can be extended to such a T by replacing the branching sets $T_0(t)$ by $T_0(t) \cap Z(t)$. More precisely, if T_0 is a $\mathcal{U}$-tree with stem s, then

$$T = \{t \in T_0 \mid (\forall x \in t - s)\, x \in Z(t \cap x)\}$$

(where $t \cap x$ consists of the predecessors of x in t) is an extension of T_0 whose paths eventually obey Z (the obedience beginning as soon as the path gets beyond s).

It remains to show that (c) implies (b), so assume (c) and let $\mathcal{B}$ be a Borel set (of V) in I. That $\mathcal{B} \in I$ means that, for every $s \in [\omega]^{<\omega}$, there is a $\mathcal{U}$-tree T_s with stem s, none of whose paths are in $\mathcal{B}$. Define

$$Z(t) = \bigcap_{s \text{ initial segment of } t} T_s(t).$$

As each T_s is a $\mathcal{U}$-tree beyond s and as the intersection here is a finite one, we have $Z(t) \in \mathcal{U}_t$, so Z is a $\mathcal{U}$-regulation. Consider an arbitrary set $Y = \{y_0 < y_1 < \cdots\}$ in V that eventually obeys Z; let k be such that $y_n \in Z(\{y_0, \ldots, y_{n-1}\})$ for all $n \geqslant k$, and let $s = \{y_0, \ldots, y_{k-1}\}$. Then, for all $n \geqslant k$, we have $y_n \in T_s(\{y_0, \ldots, y_{n-1}\})$ by definition of Z, so Y is a path through T_s and therefore $Y \notin \mathcal{B}$. We have shown that no $Y \in \mathcal{B}$ eventually obeys Z (in V). This fact is Π^1_1, hence absolute. So the assumption (c) implies $X \notin \bar{\mathcal{B}}$. $\square$

Corollary 8.1. *If all the ultrafilters $\mathcal{U}_s$ are the same $\mathcal{U}$ and if X is Q-generic over V, then every infinite subset of X is also Q-generic over V.*

Proof. Let Z be any $\mathcal{U}$-regulation. Since all $\mathcal{U}_s$ are equal to $\mathcal{U}$,

$$Z'(t) = \bigcap_{s \subseteq t} Z(s)$$

defines a $\mathcal{U}$-regulation Z'. Clearly, if X eventually obeys Z', then every infinite subset eventually obeys Z. $\square$

When $\mathcal{U}$ is selective, Corollary 8.1 reduces to Mathias's theorem [19, Theorem 2.5].

If enough (but not necessarily all) of the $\mathcal{U}_s$ are equal, one obtains results analogous to Corollary 8.1 but more complicated. We confine our attention to the simplest such case, namely that $\mathcal{U}_s$ depends only on the parity of $|s|$. Let us write $\mathcal{U}_0$ (resp. $\mathcal{U}_1$) for the common value of $\mathcal{U}_s$ for all s of even (resp. odd) size. Suppose A_0 and A_1 are disjoint sets, with $A_i \in \mathcal{U}_i$. If X is Q-generic, then, with finitely many exceptions, the elements of X in increasing order are chosen alternately from A_0 and A_1. It is clear that this will not hold for arbitrary infinite subsets Y of X, so the conclusion of Corollary 8.1 fails in this situation. Indeed, it is clear that Y will have the necessary alternation property if and only if the elements removed from X in forming Y are, with an even finite number of exceptions, taken in consecutive (in X) pairs. The following corollary says that this restriction, obviously necessary if Y is to be Q-generic, is sufficient.

Corollary 8.2. *If $\mathcal{U}_s$ depends only on the parity of $|s|$, if X is Q-generic over V, and if Y is an infinite subset of X such that every sufficiently large $y \in Y$ has an even number of predecessors in $X - Y$, then Y is also Q-generic over V.*

Proof. Proceed as in the proof of Corollary 8.1, except that

$$Z'(t) = \bigcap_{s \subseteq t,\ |t-s| \text{ even}} Z(s). \qquad \square$$

Another easy consequence of Theorem 8 is that the enumeration of a Q-generic set dominates all ground model reals.

Corollary 8.3. *Let $\{x_0 < x_1 < \cdots\}$ be Q-generic over V, and let $f: \omega \to \omega$ be in V. Then, for all sufficiently large n, $f(n) < x_n$.*

Proof. Apply Theorem 8(c) to the regulation $Z(t) = \{x \in \omega \mid x > f(|t|)\}$. $\square$

By putting $\max(t)$ in place of $|t|$ in the proof, we could put $f(x_{n-1})$ in place of $f(n)$ in the corollary.

The next theorem extends to Q-forcing another result proved by Mathias [19, (2.9)] for the selective case. It will be convenient to say that a finite subset s of ω *favors* a sentence φ in the Q-forcing language if there is a $\mathcal{U}$-tree with stem

s that forces φ. Observe that two $\mathcal{U}$-trees with the same stem are always compatible in Q, for their intersection is another $\mathcal{U}$-tree with the same stem. Therefore, no s can favor both φ and $\neg\varphi$.

Theorem 9. *Let $s \in [\omega]^{<\omega}$ and let φ be a sentence of the Q-forcing language.*

(a) *s favors φ if and only if $(\mathcal{U}_s n)\, s \cup \{n\}$ favors φ.*

(b) *Either s favors φ or s favors $\neg\varphi$.*

Proof. (a) If a $\mathcal{U}$-tree T with stem s forces φ, then, for all $n \in T(s)$, the subtree $\{t \in T \mid t \text{ comparable with } s \cup \{n\}\}$ is a $\mathcal{U}$-tree with stem $s \cup \{n\}$ and forces φ because it is an extension of T. Conversely, given $\mathcal{U}$-trees T_n with stems $s \cup \{n\}$ forcing φ for all $n \in A$ where $A \in \mathcal{U}_s$, then the union T of the T_n's is a $\mathcal{U}$-tree with stem s, and it forces φ because each of its extensions in Q is compatible with some T_n and therefore cannot force $\neg\varphi$.

(b) Suppose s favors neither φ nor $\neg\varphi$. Define a tree T to consist of s, all initial segments of s, and those $t \in [\omega]^{<\omega}$ such that s is an initial segment of t and all initial segments of t longer than s (including t itself) favor neither φ nor $\neg\varphi$. Thus, for $t \in T$ and t longer than s, $T(t) = \{n > \max(t) \mid t \cup \{n\}$ favors neither φ nor $\neg\varphi\}$. By part (a) of the theorem (and the fact that each $\mathcal{U}_t$ is an ultrafilter), T is a $\mathcal{U}$-tree with stem s. Let T' be an extension of T deciding φ, and let t' be the stem of T'. So t' favors one of φ and $\neg\varphi$, yet is a node of T beyond s. This contradicts the definition of T. $\square$

To conclude our discussion of Q-forcing, we give an application to the following question raised by Laver and Prikry. Is it consistent with ZFC that

> if P is any non-trivial notion of forcing satisfying the countable chain condition, then P forces that there exists a real that is either Cohen generic or random over the ground model? (20)

Of course, (20) implies Souslin's hypothesis, for a Souslin tree is a c.c.c. notion of forcing that adjoins no reals at all. It is also known, and more relevant to the present discussion, that Mathias forcing $P_{\mathcal{U}}$ adds no Cohen or random reals if $\mathcal{U}$ is a selective ultrafilter but it adds Cohen reals if $\mathcal{U}$ is a non-selective ultrafilter. Thus, (20) implies that there are no selective ultrafilters. We shall extend this result by showing that, if all $\mathcal{U}_s$ are the same P-point $\mathcal{U}$, then Q-forcing adds no Cohen or random reals. Since Q satisfies the c.c.c. this means that (20) implies that there are no P-points. In fact, we shall prove a bit more, namely that, if all $\mathcal{U}_s$ are the same P-point and if A is any real in the $Q(\mathcal{U})$-forcing extension of V but not in V, then the submodel $V[A]$ contains a function $\omega \to \omega$ that eventually dominates all functions $\omega \to \omega$ in V. It is well-known that Cohen and random forcing do not produce any such dominating functions, so it follows that A cannot be Cohen-generic or random over V.

The proof of the existence of a dominating function in $V[A]$ begins with some general facts about reals in Q-forcing extensions. These facts, which generalize results of Mathias [19, Section 6] for the selective case, do not depend on special assumptions about the $\mathcal{U}_s$, so for the time being $\mathcal{U}$ can be any $[\omega]^{<\omega}$-indexed family of ultrafilters on ω.

Let $\mathring{A}$ be a name in the Q-forcing language such that "$\mathring{A} \subseteq \omega$" is forced (by every condition). For each $s \in [\omega]^{<\omega}$, we define

$$A(s) = \{k \in \omega \mid s \text{ favors "} k \in \mathring{A} \text{"}\},$$

which may be thought of as s's opinion of what A is. Notice that each $A(s)$ and indeed the whole function $s \mapsto A(s)$ are defined in the ground model V. By definition of 'favors' and by the observation that the intersection of finitely many $\mathcal{U}$-trees with stem s is again such a tree, we can find, for each s, a $\mathcal{U}$-tree T_s with stem s that forces "$k \in \mathring{A}$" (resp. "$k \notin \mathring{A}$") for all $k \leqslant \max(s)$ such that $k \in A(s)$ (resp. "$k \notin \mathring{A}(s)$"). Thus,

$$T_s \text{ forces "} \mathring{A} \text{ agrees with } A(s) \text{ up to and including } \max(s) \text{".}$$

Define a $\mathcal{U}$-regulation Z by

$$Z(t) = \bigcap_{s \subseteq t \text{ with } t \in T_s} T_s(t).$$

Now consider a set $R = \{r_0 < r_1 < \cdots\}$ Q-generic over V; we work in $V[R]$ for the time being. We write $\bar{r}(n)$ for the initial segment $\{r_0, \ldots, r_{n-1}\}$ of R, and we write G and $G \restriction Q$ for the V-generic subsets of $\boldsymbol{B}$ and Q associated with R via (18) and (19). By Theorem 8, R eventually obeys the regulation Z, so fix an n_0 such that, for all $n > n_0$, $r_n \in Z(\bar{r}(n))$. It follows that, if $n \geqslant k > n_0$, then $\bar{r}(n) \in T_{\bar{r}(k)}$ (by induction on n using the definition of Z), so R is a path through $T_{\bar{r}(k)}$. By (19), this means that $T_{\bar{r}(k)} \in G \restriction Q$ for all $k \geqslant n_0$, and therefore, by definition of T_s,

$$\mathring{A}_R \text{ agrees with } A(\bar{r}(k)) \text{ up to and including } r_{k-1}, \tag{21}$$

where $\mathring{A}_R$ is the denotation of the name $\mathring{A}$ with respect to the generic set $G \restriction Q$ corresponding to R. Thus, $\mathring{A}_R$ is obtained from R in a very simple fashion,

$$\mathring{A}_R = \bigcup_{k=n_0}^{\infty} A(\bar{r}(k+1)) \cap (r_k + 1),$$

via the function $s \mapsto A(s)$ which is in V.

Theorem 10. *Let $\mathcal{U}$ be a P-point on ω and let $\mathcal{U}_s = \mathcal{U}$ for all $s \in [\omega]^{<\omega}$. Let R be a Q-generic subset of ω, and let A be any subset of ω in $V[R]$ but not in V. Then $V[A]$ contains a function $\omega \to \omega$ eventually dominating all functions $\omega \to \omega$ from V.*

Proof. Fix a name $\mathring{A}$ such that $\mathring{A}_R = A$ and such that every condition forces

"$\mathring{A} \subseteq \omega$". We continue to use the notation introduced in the discussion preceding the theorem.

For each $s \in [\omega]^{<\omega}$, $A(s) \in V$ but $A \notin V$, so we can define $\delta(s)$ to be the smallest member of the symmetric difference $A \Delta A(s)$. We also define

$$D(s) = \{n > \max(s) \mid \text{either } n < \delta(s) \text{ or } \delta(s) \in A(s \cup \{n\}) \Delta A(s)\}.$$

For each fixed s, $D(s)$ is a set in V, and $D(s) \notin \mathscr{U}$ by Theorem 9(a) and the definitiou of $A(s)$. The functions δ and D are in the model $V[A]$.

Consider any $k > n_0$. By definition of δ, $\delta(\bar{r}(k)) \in A \Delta A(\bar{r}(k))$. On the other hand, if $r_k \geqslant \delta(\bar{r}(k))$, then, by (21), $\delta(\bar{r}(k)) \notin A \Delta A(\bar{r}(k+1))$. Therefore, either $r_k < \delta(\bar{r}(k))$ or $\delta(\bar{r}(k)) \in A(\bar{r}(k+1)) \Delta A(\bar{r}(k))$. As $\bar{r}(k+1) = \bar{r}(k) \cup \{r_k\}$, we have shown that

$$r_k \in D(\bar{r}(k)). \tag{22}$$

The sets $D(s)$, for $s \in [\omega]^{<\omega}$ constitute a countable family in $V[R]$ (in fact in $V[A]$) of sets in V. As $V[R]$ is a c.c.c. forcing extension of V, there is a countable family $\mathscr{D}$ in V that contains every $D(s)$. ($\mathscr{D}$ consists of all the sets in V that are forced by some condition in Q to be a $D(s)$.) As $\mathscr{U}$ is a P-point, it contains a set E that is almost disjoint from (i.e., has finite intersection with) every set in $\mathscr{D}$ that is not in $\mathscr{U}$. In particular, $E \cap D(s)$ is finite for every s. Define $g : \omega \to \omega$ in $V[A]$ by taking $g(n)$ large enough to be an upper bound for $E \cap D(s)$ for all s with $\max(s) \leqslant n$. We can arrange that g is a strictly increasing function.

As R eventually obeys the regulation that maps every $s \in [\omega]^{<\omega}$ to E, there is n_1 such that $r_k \in E$ for all $k > n_1$. Combining this with (22), and setting $n = \max\{n_0, n_1\}$, we have, for all $k > n$, $r_k \in E \cap D(\bar{r}(k))$ and therefore $r_k \leqslant g(r_{k-1})$. It follows, by induction on k, that $r_k \leqslant g^{k-n}(r_n)$ for all $k \geqslant n$.

The function $k \mapsto g^{k-n}(r_n)$ (for $k \geqslant n$, and extended arbitrarily for $k < n$) is in $V[A]$ because g is. It eventually dominates $k \mapsto r_k$ (from n on), which in turn eventually dominates every function $\omega \to \omega$ in V by Corollary 8.3. □

It is tempting to try to prove Theorem 20 without the hypothesis that $\mathscr{U}$ is a P-point and thereby refute (20). Unfortunately, there exist ultrafilters $\mathscr{U}$ such that $Q(\mathscr{U})$-forcing with all $\mathscr{U}_s = \mathscr{U}$ adds Cohen reals. Specifically, let P be the set of finite sequences of zeros and ones, ordered by inclusion, so P is a notion of forcing adjoining a Cohen real, and let $\mathscr{F}$ be the filter on P generated by the family $\mathscr{D}$ of dense open subsets of P. (Note that $\mathscr{D}$ is closed under finite intersections.) Let $\mathscr{U}$ be an ultrafilter on ω isomorphic to some ultrafilter on P that includes $\mathscr{F}$; or, more generally, let $\mathscr{U}$ be an ultrafilter on ω and let $f : \omega \to P$ be a function such that $f(\mathscr{U}) \supseteq \mathscr{F}$. If $R = \{r_0 < r_1 < \cdots\}$ is a Q-generic subset of ω (where $\mathscr{U}_s = \mathscr{U}$ for all s), then the infinite string of zeros and ones obtained by concatenating $f(r_0)^\frown f(r_1)^\frown \cdots$, which is clearly in $V[R]$, is Cohen-generic over V. To see this, let $D \in V$ be a dense open subset of P. For each $s = \{x_0 < \cdots <$

$x_{n-1}\} \in [\omega]^{<\omega}$, the set

$$D_s = \{p \in P \mid f(x_0)^\frown f(x_1)^\frown \cdots ^\frown f(x_{n-1})^\frown p \in D\}$$

is also dense and open, so $f^{-1}(D_s) \in \mathcal{U}$. Being Q-generic over V, R eventually obeys the regulation Z defined (in V) by $Z(s) = f^{-1}(D_s)$. So there is an n with $f(r_n) \in D_{\{r_0,\ldots,r_{n-1}\}}$. This means that $f(r_0)^\frown f(r_1)^\frown \cdots$ has an initial segment in D, namely $f(r_0)^\frown f(r_1)^\frown \cdots ^\frown f(r_n)$.

It follows from the result just proved and Theorem 10, that no ultrafilter that includes $\mathcal{F}$ is a P-point.

We conclude this section by summarizing some facts that we shall need later about two familiar forcing constructions, collapsing below an inaccessible cardinal and adjoining a generic ultrafilter on ω.

Let κ be an inaccessible cardinal in V. The notion of forcing for Lévy collapsing below κ, Lévy(κ), is the set of finite partial functions p from $(\kappa - \{0\}) \times \omega$ to κ satisfying $p(\alpha, k) < \alpha$ whenever $p(\alpha, k)$ is defined. A Lévy(κ)-generic set G codes functions g_α from ω onto α for all $\alpha \in \kappa - \{0\}$; $g_\alpha(k) = \beta$ if and only if some $p \in G$ has $p(\alpha, k) = \beta$. In $V[G]$, κ is $\aleph_1$. If $\lambda < \kappa$, then $\{p \in G \mid \text{domain}(p) \subseteq (\lambda - \{0\}) \times \omega\} = G_\lambda$ codes all the g_α for $\alpha < \lambda$ and can itself be coded by a single real in $V[G]$ as λ is countable in $V[G]$. Every real in $V[G]$ is in $V[G_\lambda]$ for all sufficiently large $\lambda < \kappa$. We write HDV$\mathbb{R}$ for the class of elements of $V[G]$ hereditarily (ordinal) definable in $V[G]$ from parameters in the ground model V and parameters that are reals (in $V[G]$). (The word 'ordinal' in the preceding sentence is redundant, as arbitrary parameters from V are allowed, but its inclusion makes it evident that general facts about ordinal definability with parameters are applicable. In particular, HDV$\mathbb{R}$ is definable, in $V[G]$, in the language of set theory augmented by a predicate symbol designating the ground model V. If one assumes $V = L$, then parameters from V can be eliminated in favor of ordinals, so HDV$\mathbb{R}$ is the class of sets hereditarily ordinal definable from reals in $V[G]$.) If $\mathcal{X} \in$ HDV$\mathbb{R}$ is a set of reals, then there is a formula $\psi(x, y, z)$, there is a real a in $V[G]$, and there is a parameter $p \in V$ such that, for all sufficiently large $\lambda < \kappa$ and for all reals $x \in V[G]$,

$$x \in \mathcal{X} \Leftrightarrow V[G_\lambda, x] \models \psi(x, a, p). \tag{23}$$

(Actually, λ need only be large enough to ensure $a \in V[G_\lambda]$.) All the preceding facts about Lévy (κ) forcing can be found in [19] or [29]. The following is in [19, Theorem 5.8]. If κ is a Mahlo cardinal in V and if $\mathcal{X} \in V[G]$ is a set of reals, then there exist arbitrarily large inaccessible (in V) cardinals $\lambda < \kappa$ such that $\mathcal{X} \cap V[G_\lambda] \in V[G_\lambda]$ and $(\mathbb{R}^{(V[G_\lambda])}, \mathcal{X} \cap V[G_\lambda])$ is an elementary submodel of $(\mathbb{R}^{(V[G])}, \mathcal{X})$, as models of second-order arithmetic with an additional unary predicate for $\mathcal{X}$. For example, if $\mathcal{U}$ is an ultrafilter on ω in $V[G]$, then $\mathcal{U} \cap V[G_\lambda]$ is an ultrafilter on ω in $V[G_\lambda]$ for many $\lambda < \kappa$.

The simplest notion of forcing to adjoin a new ultrafilter on ω is the set $[\omega]^\omega$ of infinite subsets of ω, ordered so that 'extension' means subset. A forcing condition $X \in [\omega]^\omega$ is thought of as saying about the ultrafilter being adjoined that it contains X. As defined here, the ordering is not separative; the separative quotient, which produces the same forcing extension, is obtained by identifying two conditions X and Y if the symmetric difference $X \Delta Y$ is finite, and the ordering corresponds to inclusion modulo finite sets. The separative quotient is countably closed because, given a sequence $X_0, X_1, \ldots$ of infinite sets that is decreasing modulo finite sets, we can form an infinite Y almost included in every X_n simply by choosing the nth element of Y from $X_0 \cap \cdots \cap X_n$. Thus, this notion of forcing adds no new ω-sequences of ordinals. This fact makes it easy to check that a generic subset G of $[\omega]^\omega$ is an ultrafilter on ω. The least trivial part of the verification is that G contains X or $\omega - X$ for each $X \subseteq \omega$; but, by the preceding observation, it suffices to consider $X \in V$, and then $\{Y \in [\omega]^\omega \mid Y \subseteq X$ or $Y \subseteq \omega - X\}$ meets G because it is dense in $[\omega]^\omega$. Similarly, if $f : \omega \to \omega$ is $V[G]$, then $f \in V$ and $\{Y \in [\omega]^\omega \mid f$ is one-to-one or constant on $Y\}$ is in V and dense in $[\omega]^\omega$ and therefore meets G. Thus, G is a selective ultrafilter on ω in $V[G]$. We shall refer to such ultrafilters as $[\omega]^\omega$-generic over V.

The notion of forcing $[\omega]^\omega$, or rather its separative quotient, can be identified with a dense subset of the collection of filters on ω that have countable bases, ordered so that larger filters count as extensions of smaller ones. A set $X \in [\omega]^\omega$ is identified with the filter $\{Y \subseteq \omega \mid X - Y$ is finite$\}$; that filters of this special form are dense among all countably generated filters is shown by the same argument as the countable closure of the separative quotient in the preceding paragraph. Thus, we may view $[\omega]^\omega$ forcing as constructing an ultrafilter by approximating it with countably generated filters. Other sorts of 'small' filters, particularly the F_σ filters and the analytic filters (with respect to the usual product topology on the power set of ω), can also be used to adjoin ultrafilters. Work of Teissier (née Daguenet) [9] shows, in topological rather than forcing terminology, that forcing with F_σ filters adjoins a P-point with no selective ultrafilter RK-below it and that forcing with analytic filters adjoins an ultrafilter with no P-point RK-below it.

6. The Lévy–Mahlo model

Throughout this section, we assume that κ is a Mahlo cardinal in V and that G is Lévy(κ)-generic over V. We study partition relations and ultrafilters in $V[G]$. The first result, a straightforward extension of a theorem of Mathias [19] (which is in turn based on work of Solovay [29]), establishes for HDVℝ sets in $V[G]$ some of the properties previously established for analytic sets in arbitrary models.

Theorem 11. *In $V[G]$, if $\mathcal{U} = (\mathcal{U}_s : s \in [\omega]^{<\omega})$ is a family of ultrafilters on ω, then every* HDVℝ *subset of $[\omega]^\omega$ belongs to the Boolean algebra $\mathbf{C}$ defined in Section* 3.

Proof. Let $\mathscr{X}$ be any HDVℝ subset of $[\omega]^\omega$ in $V[G]$. By properties of Lévy(κ) forcing proved in [19, 29] and recalled in Section 5 above, there exist a real $a \in V[G]$, a parameter $p \in V$, an ordinal $\kappa' < \kappa$, and a formula ψ in the language of set theory (with three free variables) such that:

(i) κ' is inaccessible in V.

(ii) $a \in V[G']$, where G' is G truncated at κ', called $G_{\kappa'}$ in Section 5.

(iii) The family $\mathscr{U}' = (\mathscr{U}'_s : s \in [\omega]^{<\omega})$ is in $V[G']$, where $\mathscr{U}'_s = \mathscr{U}_s \cap V[G']$.

(iv) For every $Y \in [\omega]^\omega$, $Y \in \mathscr{X} \Leftrightarrow V[G', Y] \models \psi(Y, a, p)$.

We shall show that $\mathscr{X}$ is in C by finding a Borel set $\mathscr{B}$ (which is in C by Theorem 3) such that the symmetric difference $\mathscr{X} \triangle \mathscr{B}$ is in I.

Let C' and I' be the Boolean σ-algebra and σ-ideal in $V[G']$ determined by $\mathscr{U}'$. Thus, a subset $\mathscr{Z}$ of $[\omega]^\omega$ in $V[G']$ belongs to I' if and only if there exists, for each $s \in [\omega]^{<\omega}$, a $\mathscr{U}'$-tree T_s (in $V[G']$) with stem s, such that no paths (in $V[G']$) through T_s are in $\mathscr{Z}$. Clearly, T_s is also a $\mathscr{U}$-tree (in $V[G]$) with stem s. If $\mathscr{Z}$ is a Borel set and $\bar{\mathscr{Z}}$ is its canonical extension to a Borel set in $V[G]$, then the fact that $\mathscr{Z}$ contains no paths through T_s is a Π^1_1 statement, hence absolute, so $\bar{\mathscr{Z}}$ contains no paths through T_s (see [29] for details of such absoluteness considerations). Thus, $\bar{\mathscr{Z}} \in I$. The number of Borel sets (of $V[G']$) $\mathscr{Z} \in I'$ is the power of the continuum in $V[G']$, which is smaller than κ, hence is countable in $V[G]$. Therefore, the union of all the corresponding $\bar{\mathscr{Z}}$'s is in I. Notice that this union is, by Theorem 8, precisely the set of $Y \in [\omega]^\omega$ (in $V[G]$) that are not $Q(\mathscr{U}')$-generic over $V[G']$. Thus, to complete the proof, it suffices to find a Borel set $\mathscr{B}$ such that $\mathscr{X} \triangle \mathscr{B}$ consists entirely of such Y's, i.e., such that, whenever Y is $Q(\mathscr{U}')$-generic over $V[G']$, then $Y \in \mathscr{X}$ if and only if $Y \in \mathscr{B}$.

Consider an arbitrary $Q(\mathscr{U}')$-generic subset of ω, say R_H where H is a $V[G']$-generic ultrafilter in C'/I'. Then by (iv),

$$R_H \in \mathscr{X} \Leftrightarrow V[G', R_H] \models \psi(R_H, a, p).$$

The model $V[G', R_H]$ is the $Q(\mathscr{U}')$-generic extension $V[G'][H]$ of $V[G']$. The parameters R_H, a, p occurring in ψ are the denotations, with respect to H, of names $\dot{R}, \check{a}$, and $\check{p}$ in the $Q(\mathscr{U}')$-forcing language over $V[G']$. (This uses (ii) to ensure that $\check{a}$ makes sense. $\dot{R}$ names the real associated to the canonical name for a generic set.) Thus, by elementary properties of generic extensions ("forcing equals truth"),

$$R_H \in \mathscr{X} \Leftrightarrow \|\psi(\dot{R}, \check{a}, \check{p})\| \in H.$$

The Boolean truth value $\|\psi(\dot{R}, \check{a}, \check{p})\|$ (calculated in $V[G']$) is independent of H and is the equivalence class modulo I' of some Borel set $\mathscr{B}'$ (by Theorem 3(iv)). The definition relating R_H to H shows that

$$\|\psi(\dot{R}, \check{a}, \check{p})\| = [\mathscr{B}'] \in H \Leftrightarrow R_H \in \mathscr{B}$$

where $\mathscr{B}$ is the canonical extension $\overline{\mathscr{B}'}$ of $\mathscr{B}'$ to a Borel set in $V[G]$. Summarizing, we have

$$R_H \in \mathscr{X} \Leftrightarrow R_H \in \mathscr{B},$$

as required. □

Corollary 11.1. *In* $V[G]$, *Theorems* 4, 5, 6, *and* 7 *continue to hold if the assumption* "$\mathscr{X}$ *is analytic*" *is weakeneed to* "$\mathscr{X} \in \text{HDVR}$".

Proof. In the proofs of these theorems, analyticity was used only to ensure that $\mathscr{X} \in C$. In $V[G]$, this follows from $\mathscr{X} \in \text{HDVR}$. □

The part of Corollary 11.1 that pertains to Theorem 4(d) was proved by Mathias [19, Section 5]. The following corollary is an immediate consequence of this result of Mathias. Recall from Section 5 that $[\omega]^\omega$, ordered by inclusion, is a (non-separative) notion of forcing that adjoins a selective ultrafilter.

Corollary 11.2. *In* $V[G]$, *every selective ultrafilter is* $[\omega]^\omega$*-generic over* HDVR.

Proof. We work in $V[G]$. Let $\mathscr{U}$ be a selective ultrafilter and let $\mathscr{D} \in \text{HDVR}$ be a dense subset of $[\omega]^\omega$; we must show that $\mathscr{U}$ meets $\mathscr{D}$. By Theorem 4(d), extended by Corollary 11.1, there is a set $H \in \mathscr{U}$ such that $\mathscr{D}$ contains all or none of the infinite subsets of H. As $\mathscr{D}$ is dense, 'none' is impossible, so we have 'all'. In particular, $H \in \mathscr{D}$. □

Corollary 11.2 implies that, in a certain sense, all selective ultrafilters in $V[G]$ look alike. More precisely, consider a statement σ about a selective ultrafilter in $V[G]$ such that

(i) if $\mathscr{U}$ satisfies σ, then so does every ultrafilter isomorphic to $\mathscr{U}$, and

(ii) σ can be expressed in the form $\text{HDVR}[\mathscr{U}] \models \varphi$, where the sentence φ can involve names for $\mathscr{U}$ and for members of HDVR.

Then σ holds either for all selective ultrafilters or for none. Indeed, if there is a selective $\mathscr{U}$ satisfying σ, then φ holds in the $[\omega]^\omega$-generic extension $\text{HDVR}[\mathscr{U}]$ of HDVR, hence must be forced over HDVR by some $X \in \mathscr{U}$. Then φ holds in $\text{HDVR}[\mathscr{U}']$, and therefore σ holds of $\mathscr{U}'$, for every selective $\mathscr{U}'$ containing X. But then (i) implies that σ holds for every selective $\mathscr{U}'$. (It is obvious that hypothesis (i) cannot be omitted, for then σ could say that $\mathscr{U}$ contains the set of even numbers. I conjecture that (ii) cannot be omitted either.)

Louveau [17] has shown that, in models of Martin's axiom and not CH, selective ultrafilters do not all look alike, for they can be distinguished by their degrees of stability, i.e., the smallest κ such that some κ sets in the ultrafilter do not all almost include a single set in the ultrafilter. This degree of stability is (for P-points) between $\aleph_1$ and $2^{\aleph_0}$, so in models of CH (like our $V[G]$) it cannot be used to distinguish between selective ultrafilters. Selective ultrafilters adjoined to a model of ZFC by forcing with $[\omega]^\omega$ always have degree of stability $2^{\aleph_0}$ (of the extension; this may be smaller than $2^{\aleph_0}$ of the ground model).

The next theorem strengthens Corollary 11.2 by showing that the selective ultrafilters in $V[G]$ are not only generic individually but mutually generic, except for isomorphic ones.

Theorem 12. *In $V[G]$, every sequence $(\mathcal{U}_n : n \in \omega)$ of pairwise non-isomorphic selective ultrafilters is generic over HDVℝ with respect to the product of countably many copies of $[\omega]^\omega$.*

Remark. Since forcing with $[\omega]^\omega$ does not adjoin new reals, the product forcing mentioned in the theorem can also be viewed as the ω-length iteration of $[\omega]^\omega$-forcing with countable support (i.e., inverse limit).

The theorem cannot be extended to uncountably many ultrafilters, since an $\aleph_1$-sequence of $\mathcal{U}_\alpha$'s could encode, via the choice whether $\mathcal{U}_\alpha$ contains the set of even numbers, an arbitrary subset $\aleph_1$; in particular (since CH holds in $V[G]$), it could encode another selective ultrafilter $\mathcal{V}$. Then $\mathcal{V}$ and the $\mathcal{U}_\alpha$'s cannot be mutually generic, since $\mathcal{V}$ is in the model generated over HDVℝ by the $\mathcal{U}_\alpha$'s.

Proof of Theorem 12. Let the sequence $(\mathcal{U}_n : n \in \omega)$ and a dense open subset $\mathcal{D} \in$ HDVℝ of the product notion of forcing be given. Fix an ω-sequence in V of natural numbers in which every natural number occurs infinitely often and the first occurrence of any n precedes the first occurrence of $n+1$. We write $\bar{i}$ for the ith term of this sequence. (For example, we could define $\bar{i}$ to be the largest n such that 2^n divides $i+1$.) We use the sequence to break every $X \in [\omega]^\omega$ into infinitely many disjoint infinite pieces $X(n)$ by putting the ith element of X into $X(\bar{i})$, for each i. Let

$$\mathcal{X} = \{X \in [\omega]^\omega \mid (X(n) : n \in \omega) \in \mathcal{D}\};$$

clearly, $\mathcal{X}$ is in HDVℝ.

We apply Theorem 7, as extended by Corollary 11.1, to $\mathcal{X}$ and the system $\mathcal{U}' = (\mathcal{U}'_s : s \in [\omega]^\omega)$ of selective ultrafilters defined by

$$\mathcal{U}'_s = \mathcal{U}_n \quad \text{where } n = \overline{|s|}\,.$$

We obtain, for each n, a set Z_n (called $Z(\mathcal{U}_n)$ in Theorem 7) such that $\mathcal{X}$ contains all or none of the infinite subsets $X = \{x_0 < x_1 < \cdots\}$ of ω that satisfy $x_i \in Z_{\bar{i}}$ for all i, i.e., that satisfy $X(n) \subseteq Z_n$ for all n. We check next that 'all' holds, rather than 'none'.

As $\mathcal{D}$ is dense in the product notion of forcing, it contains an extension $(Y_n : n \in \omega)$ of the condition $(Z_n : n \in \omega)$; so $Y_n \subseteq Z_n$ for all n. Define a set $X = \{x_0 < x_1 < \cdots\}$ by inductively choosing x_i to be a member of $Y_{\bar{i}}$ larger than all previously chosen x_j $(j < i)$. Then $X(n) \subseteq Y_n \subseteq Z_n$ for all n and, as $(X(n) : n \in \omega)$ is an extension of $(Y_n : n \in \omega)$ and $\mathcal{D}$ is open, $X \in \mathcal{X}$. This shows that the 'none' alternative in the homogeneity given by Theorem 7 cannot hold, so 'all' holds. That is, if $X \in [\omega]^\omega$ and $X(n) \subseteq Z_n$ for every n, then $X \in \mathscr{X}$. We intend to find such an X with $X(n) \in \mathscr{U}_n$ for every n. Then, by definition of $\mathscr{X}$, we shall have $(X(n): n \in \omega) \in \mathscr{D} \cap \prod_{n\in\omega} \mathscr{U}_n$, which will conclude the proof of genericity.

For each fixed $k \in \omega$, let $\lambda = \lambda(k) \in \omega$ be so large that every $n \leq k$ occurs as $\bar{i}$ for some i with $k < i \leq \lambda$. Then choose a sequence of natural numbers $n_0 < n_1 < \cdots < n_\lambda$ such that $k < n_0$ and each $n_i \in Z_{\bar{i}}$. (It is trivial to find such n_i's inductively, as each Z_n is infinite.) Now allow k to vary; n_λ, which of course depends on k, will be called $g(k)$.

By Corollary 7.2, there are sets $B_n \in \mathscr{U}_n$ such that

$$\forall x, y \in \bigcup_n B_n\, [x < y \rightarrow g(x) < y].$$

We may assume $B_n \subseteq Z_n$; just replace B_n with $B_n \cap Z_n$ if necessary. We may also assume that the sets B_n are pairwise disjoint; the argument for this is well known, but we include it for the sake of completeness. For each pair of distinct natural numbers $m < n$, since $\mathscr{U}_m$ and $\mathscr{U}_n$ are distinct, we can find $C(m, n) \in \mathscr{U}_m$ with $\omega - C(m, n) \in \mathscr{U}_n$. For each m, consider the countably many sets $C(m, n)$ for $n > m$ and $\omega - C(n, m)$ for $n < m$; they are in $\mathscr{U}_m$, and $\mathscr{U}_m$ is a P-point, so find $C_m \in \mathscr{U}_m$ almost included in each of these sets. Now, if $m < n$, then C_m and C_n are almost included in $C(m, n)$ and $\omega - C(m, n)$; so their intersection is finite. Thus, for each n,

$$C'_n = C_n - \bigcup_{m<n} C_m \in \mathscr{U}_n.$$

The sets C'_n are pairwise disjoint, and we replace B_n by $B_n \cap C'_n$ to achieve the desired disjointness of the B's.

We are now ready to construct the desired $X = \{x_0 < x_1 < \cdots\}$. It will consist of two sorts of elements, *special* elements chosen from appropriate B_n's, and *filler* elements chosen from appropriate Z_n's. The special elements will be chosen in increasing order; immediately after one has been chosen, some filler elements may be inserted between it and the next smaller special element. The first special element is the first element of B_0. No fillter elements are inserted below it, so it will be x_0. Notice that, by our choice of the sequence of $\bar{i}$'s, $\bar{0} = 0$, so we have $x_0 \in B_{\bar{0}}$.

Consider now a later stage of the induction, where the last special element chosen was k, and any filler elements to be inserted before it have also been chosen. Thus, this special element k is x_p where p is already known because all the predecessors of k in X have already been chosen. The next special element after k is defined to be the smallest $m > k$ such that $m \in \bigcup_{n\leq k} B_n$. Let n be the index such that $m \in B_n$. By the definition of $\lambda = \lambda(k)$, we can fix an i with $k < i \leq \lambda$ and $\bar{i} = n$. By our choice of the B's, $m > g(k)$, so we have $k < n_0 < n_1 < \cdots < n_\lambda < m$ with each $n_j \in Z_{\bar{j}}$. The filler elements to be inserted between k

and m are $n_{p+1} < \cdots < n_{i-1}$. (Since $p \leq k < i$, this makes sense; if $p = i - 1$, no filler elements are inserted.) Since k was x_p, we now have $x_j = n_j$ for $p + 1 \leq j < i$, so $x_j \in Z_j$ for such j. Also, $x_i = m \in B_n \subseteq Z_n = Z_i$. This completes the inductive definition of X and the verification that $X(n) \subseteq Z_n$ for all n. It remains to check that $X(n) \in \mathcal{U}_n$ for all n.

Fix an arbitrary $n \in \omega$. Once a special element $k \geq n$ has been chosen, every element $>k$ in B_n will be put into X as a special element and will be in $X(n)$. Thus, $X(n)$ contains all but finitely many elements of B_n and is therefore in $\mathcal{U}_n$. □

Corollary 12.1. *In* $V[G]$, *if* $\mathcal{U}$ *is a selective ultrafilter, then the only selective ultrafilters in* HDVℝ$[\mathcal{U}]$ *are those isomorphic to* $\mathcal{U}$.

Proof. If $\mathcal{V}$ is a selective ultrafilter not isomorphic to $\mathcal{U}$, then $\mathcal{U}$ and $\mathcal{V}$ are mutually $[\omega]^\omega$-generic over HDVℝ, by the theorem. So $\mathcal{V}$ is $[\omega]^\omega$-generic over HDVℝ$[\mathcal{U}]$, hence is certainly not in HDVℝ$[\mathcal{U}]$. □

In connection with Corollary 12.1, it should be mentioned that Shelah [26, (VI.5)] has shown that the existence of a unique isomorphism class of selective ultrafilters is consistent with ZFC (not merely ZF as here, and Shelah needs no large cardinals for his result).

Question. Can all ultrafilters in HDVℝ$[\mathcal{U}]$ be obtained from $\mathcal{U}$ by transfinite iteration of the two processes of (a) taking images under maps from ω to ω and (b) taking limits of previously obtained ultrafilters along previously obtained ultrafilters? An affirmative answer would imply by virtue of [6] that, in HDVℝ$[\mathcal{U}]$, the Rudin–Keisler ordering of ultrafilters is linear; this cannot happen in models of ZFC, by [14].

The same proof as for Corollary 12.1 also shows that, if any finitely or countably many of the selective ultrafilters in $V[G]$ are adjoined to HDVR, then these ultrafilters and their isomorphic images are the only selective ultrafilters in the resulting model.

By Corollary 11.1, indeed by the special case given in [19, Section 5], HDVℝ satisfies the partition relation $\omega \to (\omega)^\omega$. (In fact, as is shown in [19, (5.1)] this does not require that κ be Mahlo in V, only that it be inaccessible in V.) Thus, if $\mathcal{U}$ is any selective ultrafilter in $V[G]$, then the work of Henle, Mathias, and Woodin [13] on $[\omega]^\omega$-generic extensions of models of $\omega \to (\omega)^\omega$ is applicable to HDVℝ$[\mathcal{U}]$. For example, every set of ordinals in this model is already in HDVℝ; in particular, adjoining $\mathcal{U}$ does not adjoin a well-ordering of the continuum. (An alternate proof of this fact can be obtained by observing that, if ℝ could be well-ordered in HDVℝ$[\mathcal{U}]$, then its cardinality would have to be $\aleph_1 (= \kappa)$ because it cannot be mapped onto the next cardinal (of V or $V[G]$ or any intermediate model) κ^+ even in $V[G]$. So HDVR[$\mathcal{U}$] would satisfy CH and would therefore have many non-isomorphic selective ultrafilters, contrary to Corollary 12.1.) It also follows from [13] that, if $\{\mathcal{X}_\alpha \mid \alpha<\lambda\}$ is, in HDVR, a well-ordered family of subsets of $[\omega]^\omega$, each closed under finite alteration (i.e., if $A \in \mathcal{X}_\alpha$ and $A \Delta B$ is finite, then $B \in \mathcal{X}_\alpha$), and if $\mathcal{U}$ is a selective ultrafilter in $V[G]$, then $\mathcal{U}$ contains a set H homogeneous for all $\mathcal{X}_\alpha$ simultaneously (i.e., for every α, $\mathcal{X}_\alpha$ contains all or none of the infinite subsets of H). This can also be proved by finding a $\kappa'<\kappa$ such that the real parameter used to define $\{\mathcal{X}_\alpha \mid \alpha<\lambda\}$ is in $V[G']$ (where G' means $G_{\kappa'}$ as before), then using selectivity of $\mathcal{U}$ to find an $H \in \mathcal{U}$ that is $Q(\mathcal{U}')$-generic over $V[G']$ where $\mathcal{U}' = \mathcal{U} \cap V[G']$, then using the $Q(\mathcal{U}')$-genericity of all infinite subsets of H to show that $\mathcal{X}_\alpha$ contains all or none of those subsets that have a certain initial segment in common with H, and finally removing the requirement of a common initial segment because of the assumed invariance of $\mathcal{X}_\alpha$ under finite alterations.

7. Topological inhomogeneity

Assuming CH, W. Rudin [25] showed that any of the 2^c P-points can be mapped to any other by a self-homeomorphism of $\beta\omega - \omega$. Thus, the topological structure of $\beta\omega - \omega$ is inadequate, not merely for distinguishing between selective ultrafilters but even for distinguishing these from non-selective P-points. The latter defect can be remedied by the following considerations.

The two projection maps p_1 and p_2 from $\omega \times \omega$ to ω induce two maps $\hat{p}_i : \beta(\omega \times \omega) \to \beta\omega$, which we combine into a single map

$$\pi : \beta(\omega \times \omega) \to \beta\omega \times \beta\omega : \mathcal{U} \to (\hat{p}_1(\mathcal{U}), \hat{p}_2(\mathcal{U})).$$

Let Y be the subspace $\pi^{-1}((\beta\omega - \omega) \times (\beta\omega - \omega))$ of $\beta(\omega \times \omega)$; Y consists of the ultrafilters on $\omega \times \omega$ neither of whose projections are principal. It is shown in [5] that, for each $\mathcal{U} \in \beta\omega - \omega$, the cardinality of $\pi^{-1}(\mathcal{U}, \mathcal{U})$ is at least 3, with equality if and only if $\mathcal{U}$ is selective. Thus, the topological structure consisting of $\beta\omega - \omega$, Y, and the map $\pi : Y \to (\beta\omega - \omega)^2$ distinguishes selective from non-selective ultrafilters.

J. Baumgartner suggested to me in a conversation that the intuition "all selective ultrafilters look alike" could lead to the conjecture that this topological structure does not distinguish selective ultrafilters from each other. More precisely, the conjecture would assert that any selective ultrafilter can be mapped to any other by a self-homeomorphism ξ of $\beta\omega - \omega$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{\eta} & Y \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ (\beta\omega - \omega)^2 & \xrightarrow[\xi \times \xi]{} & (\beta\omega - \omega)^2 \end{array}$$

commutes for some self-homeomorphism η of Y. (In view of the definition of π, commutativity of this diagram means that $\xi \circ \hat{p}_i = \hat{p}_i \circ \eta$ for both values of i.) The results cited above show that every such ξ preserves selectivity. At the time Baumgartner formulated this conjecture, I felt that it was plausible and should perhaps be extended by requiring ξ to lift not only to Y but to similar subspaces of $\beta(\omega^k)$ for all finite k.

We shall see, however, that, even without this embellishment, the conjecture is false. The topological structure described above is sufficient to distinguish any two non-isomorphic ultrafilters (selective or not).

Theorem 13. *Assume* CH. *Let* ξ *and* η *be self-homeomorphisms of* $\beta\omega - \omega$ *and* Y*, respectively, such that the diagram above commutes. Then, for every ultrafilter* $\mathcal{U}$*,* $\xi(\mathcal{U}) \cong \mathcal{U}$.

Remark. The proof of this theorem uses far less than the full strength of CH. My first proof used the existence of infinitely many non-isomorphic selective ultrafilters. Immediately after seeing that proof, S. Glazer reduced the hypothesis to the existence of either two non-isomorphic selective ultrafilters or infinitely many P-points with no common RK-predecessor. These improvements led me to reduce the hypothesis further to the existence of two P-points with no common RK-predecessor; it is this hypothesis that is used in the following proof.

Proof of Theorem 13. We use without further comment the notation introduced in Section 1. Our first objective is to show that, for any ultrafilter $\mathcal{W} \in Y$, the following two assertions are equivalent: (a) p_1 is one-to-one on some set $A \in \mathcal{W}$, (b) $\hat{p}_1$ is one-to-one on some (basic) neighborhood $\hat{A} \cap Y$ of $\mathcal{W}$ in Y. The implication from (a) to (b) is easy, for if A is as in (a), then there exists $f : \omega \to A$ such that $f \circ p_1$ is the identity on A, from which it follows that $\hat{f} \circ \hat{p}_1$ is the identity on the closure $\hat{A}$, so (b) holds. For the converse, suppose (b) holds, so we have an $A \in \mathcal{W}$ with $\hat{p}_1$ one-to-one on $\hat{A} \cap Y$. We consider two cases. Suppose first that, for each $n \in \omega$, there are two points in A such that their first coordinates are equal, their second coordinates are distinct, and all these coordinates are larger than n. Then we can inductively define two sequences of points a_n and b_n in A such that $p_1(a_n) = p_1(b_n)$ but no other equalities hold between any coordinates of any of the chosen points. Let $\mathcal{U}$ be any ultrafilter on ω, and observe that its images under the two maps $\omega \to A$ given by $n \mapsto a_n$ and $n \mapsto b_n$ are two distinct elements of $\hat{A} \cap Y$ with the same image under $\hat{p}_1$. This contradicts (b), so this case is impossible. There remains the case that, by removing from A finitely many rows and columns, we obtain a set

$$A' = \{(x, y) \in A \mid x, y > n\}, \quad \text{for some } n,$$

that is the graph of a partial function. Since $A \in \mathcal{W}$ and neither projection of $\mathcal{W}$ is principal, we have $A' \in \mathcal{W}$, so A' is as required in (a). This completes the proof that (a) and (b) are equivalent.

The hypotheses on ξ and η, particularly the requirement that $\xi \circ \hat{p}_1 = \hat{p}_1 \circ \eta$, easily imply that property (b) is preserved by η. Therefore, so is (a).

Our next step is to show that ξ preserves the RK ordering. Suppose, therefore, that $\mathcal{U} \leq \mathcal{V}$, and let $f : \omega \to \omega$ be such that $f(\mathcal{V}) = \mathcal{U}$. Let $\mathcal{W}$ be the ultrafilter obtained by 'lifting $\mathcal{V}$ to the graph of f', i.e., $\mathcal{W}$ is the image of $\mathcal{V}$ under the function $g : \omega \to \omega^2 : n \mapsto (n, f(n))$. Thus, $\pi(\mathcal{W}) = (\mathcal{V}, \mathcal{U})$, and p_1 is one-to-one on the set $g(\omega) \in \mathcal{W}$. By the preceding paragraph, it follows that p_1 is one-to-one on some set in $\eta(\mathcal{W})$, so

$$\eta(\mathcal{W}) \cong \hat{p}_1 \eta(\mathcal{W}) = \xi \hat{p}_1(\mathcal{W}) = \xi(\mathcal{V}).$$

Similarly, $\hat{p}_2 \eta(\mathcal{W}) = \xi(\mathcal{U})$. Therefore, $\xi(\mathcal{U}) \leq \eta(\mathcal{W}) \cong \xi(\mathcal{V})$, so the RK ordering is preserved by ξ.

Now let $\mathcal{U}$ be any ultrafilter on ω, and let $\mathcal{V}$ and $\mathcal{V}'$ be two P-points with no common RK-predecessor. By definition of the Rudin–Frolík ordering, we have $\mathcal{U} <_{\mathrm{RF}} \mathcal{U} \otimes \mathcal{V}$. As ξ^{-1} is a homeomorphism, it follows that $\mathcal{U} <_{\mathrm{RF}} \xi^{-1}(\mathcal{U} \otimes \mathcal{V})$. (To see this, use the description of $<_{\mathrm{RF}}$ in terms of limits of discrete sequences.) A fortiori, $\mathcal{U} < \xi^{-1}(\mathcal{U} \otimes \mathcal{V})$ in the RK order, and, since ξ preserves this order, $\xi(\mathcal{U}) \leq \mathcal{U} \otimes \mathcal{V}$. (The inequality is actually strict, but we won't need this.) By the result cited at the end of Section 1, $\xi(\mathcal{U})$ is one of three sorts of ultrafilters: (a) an RK-predecessor of $\mathcal{U}$, (b) an RK-predecessor of $\mathcal{V}$, (c) an isomorph of $\mathcal{U}_0\text{-}\sum_n \mathcal{V}_n$ with $\mathcal{U}_0 \leq \mathcal{U}$ and each $\mathcal{V}_n \leq \mathcal{V}$.

Similarly, using $\mathcal{V}'$ in place of $\mathcal{V}$, we find that $\xi(\mathcal{U})$ is one of the following: (a) an RK-predecessor of $\mathcal{U}$ (as before), (b′) an RK-predecessor of $\mathcal{V}'$, (c′) an isomorph of $\mathcal{U}'_0\text{-}\sum_n \mathcal{V}'_n$ with $\mathcal{U}'_0 \leq \mathcal{U}$ and each $\mathcal{V}'_n \leq \mathcal{V}'$.

We consider various combinations of these alternatives and eliminate most of them, until we arrive at the conclusion that (a) holds. We cannot have both (b) and (b′) because $\mathcal{V}$ and $\mathcal{V}'$ have no common RK-predecessors. Nor can we have both (b) and (c′), for (b) implies that $\xi(\mathcal{U})$ is, like $\mathcal{V}$, a P-point, hence RF minimal. Symmetrically, we cannot have (c) and (b′). Finally, suppose we had (c) and (c′). By the result from [23] cited in the last paragraph of Section 1, the isomorphisms

$$\mathcal{U}_0\text{-}\textstyle\sum_n \mathcal{V}_n \cong \xi(\mathcal{U}) \cong \mathcal{U}'_0\text{-}\textstyle\sum_n \mathcal{V}'_n$$

imply that either some of the $\mathcal{V}_n$ are not RF-minimal, or some of the $\mathcal{V}'_n$ are not RF-minimal, or some of the $\mathcal{V}_n$ are isomorphic to some of the $\mathcal{V}'_n$. Since the $\mathcal{V}_n$ and $\mathcal{V}'_n$ are RK-predecessors of $\mathcal{V}$ and $\mathcal{V}'$, the first two of these alternatives contradict the assumption that $\mathcal{V}$ and $\mathcal{V}'$ are P-points, and the third alternative contradicts the assumption that $\mathcal{V}$ and $\mathcal{V}'$ have no common RK-predecessors. Thus, we cannot have both (c) and (c′). The only possibility that remains is (a).

We have shown that $\xi(\mathcal{U}) \leq \mathcal{U}$ for all $\mathcal{U}$. The same argument with ξ and η replaced by their inverses and $\mathcal{U}$ replaced by $\xi(\mathcal{U})$ shows that $\mathcal{U} = \xi^{-1}(\xi(\mathcal{U})) \leq \xi(\mathcal{U})$. So $\mathcal{U} \cong \xi(\mathcal{U})$. $\square$

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