JOURNAL OF DIFFERENTIAL EQUATIONS 73, 45–77 (1988)

Asymptotic Stability of Rarefaction Waves for 2 * 2 Viscous Hyperbolic Conservation Laws

ZHOUPING XIN*

Department of Mathematics,
The University of Michigan, Ann Arbor, Michigan 48109

Received May 7, 1987; revised July 10, 1987

This paper concerns the asymptotic behavior toward rarefaction waves of the solution of a general 2×2 hyperbolic conservation laws with positive viscosity. We prove that if the initial data is close to a constant state and its values at $\pm\infty$ lie on the kth rarefaction curve for the corresponding hyperbolic conservation laws, then the solution tends as $t \to \infty$ to the rarefaction wave determined by these states.

1. INTRODUCTION

In this paper, we will study the asymptotic behavior of the solution for the initial value problem for 2×2 viscous conservation laws of the form

$$u_t + (f(u))_x = \frac{\partial}{\partial x}\left(B(u)\frac{\partial u}{\partial x}\right), \qquad -\infty < x < +\infty,\ t \geqslant 0, \tag{1.1}$$

with initial data

$$\begin{aligned} &u(x,0) = u_0(x), \qquad x \in R^1, \\ &\lim_{x \to \pm\infty} u_0(x) = u_\pm; \end{aligned} \tag{1.2}$$

here $u = (u_1, u_2) \in \Omega$, Ω is some region in R^2, $f(u) = (f_1(u), f_2(u)) \in R^2$ is a smooth vector-valued function of $u \in \Omega$, $B(u)$ is a smooth matrix, and u_+ and u_- are two constant states which can be connected by a rarefaction wave solution of the associated system of conservation laws without viscosity.

The hypothesis on f is that the 2×2 matrix $(\partial f/\partial u)$ has real and distinct

* Research supported in part by the NSF under Grant DMS-830123.

0022-0396/88 $3.00

eigenvalues; $\lambda_1(u) < \lambda_2(u)$, with corresponding right and left eigenvectors r_i, l_i satisfying

$$\frac{\partial f}{\partial u} r_i(u) = \lambda_i(u)\, r_i(u),$$
$$l_i(u) \frac{\partial f}{\partial u} = \lambda_i(u)\, l_i(u), \tag{1.3}$$
$$l_i(u)\, r_j(u) = \delta_{ij}, \qquad i, j = 1, 2.$$

Define the 2×2 matrices I, Λ, L, R, and A, respectively, by $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2)$, $L = (l_1^t, l_2^t)^t$, $R = (r_1, r_2)$, and $A = (\partial f/\partial u)$, then (1.3) says that $LAR = \Lambda$, $LR = I$.

We will also assume that we can choose L and R such that LBR is a positive definite matrix. Another main hypothesis is that (1.1) is strongly coupled in the sense that

$$\frac{\partial f_1}{\partial u_2} \cdot \frac{\partial f_2}{\partial u_1} \neq 0, \qquad \forall u \in \Omega. \tag{1.4}$$

Notice that the Smoller–Johnson systems (cf. [1]) satisfy the condition (1.4).

As suggested by the results for scalar equations (cf. [2]) and for the p-system [3], we expect that the asymptotic behavior of the solution for (1.1) and (1.2) is closely related to the Riemann problem for the corresponding conservation laws:

$$u_t + (f(u))_x = 0, \tag{1.5}$$
$$u(x, 0) = u_0^r(x), \tag{1.6}$$

where

$$u_0^r(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0. \end{cases}$$

If the k-characteristic field is genuinely nonlinear, i.e.,

$$\nabla \lambda_k(u) \cdot r_k(u) = 1, \qquad k = 1 \text{ or } 2, \tag{1.7}$$

then it is well known that if u_+ is very close to u_- and u_+ is on the k-rarefaction curve $R_k(u_-)$, then the Riemann problem (1.5) and (1.6) has a unique rarefaction wave solution denoted by $u_r(x, t)$. Our purpose is to show that when the initial data $u_0(x)$ and $u_0^r(x)$ are suitably close and the rarefaction wave is weak (i.e., $\delta = \|u_+ - u_-\|$ is small), the solution $u(x, t)$

of (1.1) and (1.2) will tend to $u^r(x, t)$ as $t \to +\infty$. More precisely, we have the following theorem.

THEOREM 1.1. *Suppose* (1.3) *and* (1.4) *hold, and the* kth *characteristic field is genuinely nonlinear* ($k=1$ *or* 2). *Then for each fixed* $u_- \in \Omega$ *there exists a positive constant* ε, *such that if* $u_+ \in R_k(u_-)$ *and* $\|u_0 - u_0^r\|_{L^2}^2 + \|u_{0x}\|_{H^1}^2 + \|u_+ - u_-\| \leqslant \varepsilon$, *then the problem* (1.1) *and* (1.2) *has a unique global solution* $u(x, t)$, $t \geqslant 0$, *satisfying*

$$\begin{aligned} u - u_0^r &\in C^0(0, +\infty; L^2), \\ u_x &\in C^0(0, +\infty; H^2), \end{aligned} \tag{1.8}$$

and

$$\lim_{t \to +\infty} \sup_{x \in R^1} |u^r(x, t) - u(x, t)| = 0. \tag{1.9}$$

In order to prove this theorem, as for the p-system (cf. [3]), we first approximate $u^r(x, t)$ by a smooth function $U(x, t)$; this will be done in Section 2. Then the proof of Theorem 1.1 consists mainly of energy estimates on the perturbation of U. The method we use here is strongly motivated by the interesting work of J. Goodman for nonlinear stability of viscous shock waves (cf. [4]). We first give a careful diagonalization of the hyperbolic part of the equations for the perturbation. With this diagonalization, the basic estimates on the primary wave will be given in Section 3, the nonprimary wave will be estimated in Section 4 (see Section 3 for the definition of the primary wave and nonprimary wave). Note that in the analysis of the stability of viscous shock wave (cf. [4, 6]), due to the fact that viscous shock waves are compressive, if one works on the antiderivative of the perturbation of the viscous shock wave, then the hyperbolic part would not cause much difficulty in the energy estimates. However, since rarefaction waves are expansive, the treatment of the hyperbolic part (especially the nonlinear terms) becomes the main difficulty in our analysis. Nevertheless, the assumption (1.4) that the system (1.1) is strongly coupled will enable us to overcome this difficulty (see the proof of Lemma 3.1, and also Section 4.). Finally, we apply the estimates in Section 4 to study the existence and large time behavior of the solutions of (1.1) and (1.2).

2. SMOOTH APPROXIMATE SOLUTION OF THE RIEMANN PROBLEM

In this section, we will define a smooth approximation U of $u^r(x, t)$ and give some estimates on U. We suppose that k is fixed and the k-characteristic field is genuinely nonlinear. It is well known (cf. [5]) that the

k-rarefaction curve $R_k(u_-)$ in a small neighborhood N of u_- can be defined as follows:

$$R_k(u_-) = \{u \in \Omega \cap N;\ w_i(u) = w_i(u_-),\ i \neq k;\ \lambda_k(u) \geqslant \lambda_k(u_-)\}, \tag{2.1}$$

where w_i is a k-Riemann invariant. Now suppose $u_+ \in R_k(u_-)$, and let $w^r(x, t)$ be the solution for the following problem

$$\begin{gathered} w_t + \left(\frac{w^2}{2}\right)_x = 0, \qquad x \in R^1,\ t \geqslant 0, \\ w(x, 0) = w_0^r(x), \end{gathered} \tag{2.2}$$

with

$$w_0^r(x) = \begin{cases} \lambda_k(u_-), & x < 0, \\ \lambda_k(u_+), & x > 0. \end{cases}$$

Then if we define $w^r(x, t)$ by

$$w^r(x, t) = \begin{cases} \lambda_k(u_-), & \dfrac{x}{t} \leqslant \lambda_k(u_-), \\[2ex] \dfrac{x}{t}, & \lambda_k(u_-) \leqslant \dfrac{x}{t} \leqslant \lambda_k(u_+), \\[2ex] \lambda_k(u_+), & \dfrac{x}{t} \geqslant \lambda_k(u_+), \end{cases} \tag{2.3}$$

it is easy to check that the unique solution $u^r(x, t)$ for Riemann problems (1.5) and (1.6) is given by

$$\begin{aligned} &\lambda_k(u^r(x, t)) = w^r(x, t), \\ &w_i(u^r(x, t)) = w_i(u_-), \qquad i \neq k, \end{aligned} \tag{2.4}$$

(cf. [5]). We define $w(x, t)$ to be the smooth solution of the following initial value problem

$$\begin{gathered} w_t + \left(\frac{w^2}{2}\right)_x = 0, \qquad x \in R^1,\ t \geqslant 0, \\ w(x, 0) = w_0(x), \end{gathered} \tag{2.5}$$

with

$$w_0(x) = \frac{\lambda_k(u_-) + \lambda_k(u_+)}{2} + \frac{\lambda_k(u_+) - \lambda_k(u_-)}{2} \tanh x. \tag{2.6}$$

By the characteristic method, it is easy to see that the solution is given by

$$w(x, t) = w_0(x_0(x, t)), \qquad x = x_0(x, t) + w_0(x_0(x, t))t. \tag{2.7}$$

Then we define $U(x, t)$ by

$$U(x, t) \in R_k(u_-), \qquad \lambda_k(U(x, t)) = w(x, t). \tag{2.8}$$

Since $\nabla\lambda_k$ and ∇w_i are linearly independent [5], it follows that (2.8) uniquely determines $U(x, t)$, and from (2.7) we have

$$U(x, t) = U(x_0(x, t)). \tag{2.9}$$

The following lemma shows that $U(x, t)$ is a desired smooth approximation of $u^r(x, t)$.

LEMMA 2.1. *$U(x, t)$ is an approximation to $u^r(x, t)$ in the following sense*

(1) $U_t + (f(U))_x = 0$,

(2) $\lim_{t\to\infty} \sup_{x\in R^1} |u^r(x, t) - U(x, t)| = 0$.

Proof. By (1.7) and (1.8), we have (cf. [6])

$$U_t = \frac{\partial w}{\partial t} r_k(U), \qquad U_x = \frac{\partial w}{\partial x} r_k(U);$$

therefore $U_t + (f(U))_x = (\partial w/\partial t + w(x, t)\, \partial w/\partial x)\, r_k(U(x, t)) = 0$, and this gives (1). (2) follows from (2.1), (2.4), (2.8) and the facts that $\nabla\lambda_k$ and ∇w_i are linearly independent and $\sup_{x\in R^1} |w^r(x, t) - w(x, t)|$ tends to zero as $t\to\infty$. ∎

Concerning $U(x, t)$, we list the following properties which we will need later.

LEMMA 2.2. *The smooth function $U(x, t)$ constructed above has the following properties*:

(1) $(\partial/\partial x)(\lambda_k(U(x, t))) > 0$, $\forall x \in R^1$, $t \geq 0$.

(2) $\exists$ *positive constants c_1 and c_2 such that*

$$c_1 \left|\frac{\partial}{\partial x}\lambda_k(U)\right| \geq \left|\frac{\partial U}{\partial x}\right| \geq c_2 \left|\frac{\partial}{\partial x}\lambda_k(U)\right|. \tag{2.10}$$

(3) $\forall p$ $(1 \leq p \leq +\infty)$, $\exists c_p > 0$ *such that* $\forall t \geq 0$ *and* $\forall\delta$ $(0 < \delta \leq \delta_0)$

$$\left\|\frac{\partial U}{\partial x}\right\|_{L^p} \leq c_p\, \delta^{1/p}(1+t)^{-1+1/p}, \qquad \|U_x\|_{L^\infty} \leq c_\infty \delta. \tag{2.11}$$

(4) *For* $l \geqslant 2$, $\forall p$ $(1 \leqslant p < +\infty)$, $\exists c_{p,l} > 0$ *such that* $\forall t \geqslant 0$,

$$\left\| \frac{\partial^l}{\partial x_l} U \right\|_{L^p} \leqslant c_{p,l} \min(\delta, (1+t)^{-1}), \qquad \forall \delta \ (0 < \delta \leqslant \delta_0). \qquad (2.12)_l$$

(5) $|U_t| \leqslant c\,|U_x|$, *here c is a positive constant.*

(6) $\|U_x(x, t)\|_{L^\infty} \leqslant c\, \|U_x(x, 0)\|_{L^\infty}$, $\forall t \geqslant 0$.

Proof. (1) follows from (2.8) and $\partial w/\partial x > 0$. From (2.8) we get $\nabla \lambda_k \cdot U_x = w_x$, and $\nabla w_i \cdot U_x = 0$, therefore (2) holds since the coefficient matrix of U_x is nonsingular. The other conclusions follow from (2.8) and the corresponding estimates on $w(x, t)$ (cf. [3]). ∎

3. First Energy Estimates

In what follows, we will use H^l $(l \geqslant 1)$ to denote the usual Sobolev space with the norm $\| \ \|_l$ and $\| \ \|$ will denote the usual L_2-norm.

We begin by displaying the equations (1.1) in convenient form. Suppose that u is a solution of (1.1) and (1.2), and U is the smooth function constructed in Section 2. Let $\phi = u - U$. Then we can rewrite the initial value problem (1.1) and (1.2) as

$$\phi_t + [f'(U)\phi]_x + Q(U, \phi)_x = \frac{\partial}{\partial x}\left(B(U+\phi) \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial x}\left(B(U+\phi) \frac{\partial U}{\partial x} \right), \quad (3.1)$$

$$\phi(x, 0) = \phi_0(x) \equiv u_0(x) - U(x, 0) \in H^2, \qquad (3.2)$$

where $Q(U, \phi) = f(U+\phi) - f(U) - f'(U)\phi$ satisfies $|Q(U, \phi)| \leqslant c\,|\phi|^2$ for all x if $|\phi|$ is small enough. Notice that the condition on initial data in Theorem 1.1 implies (3.2).

Let us define the solution space of (3.1) by

$$X(0, T) = \{\phi \in C^0(0, T; H^2);\ \phi_x \in L^2(0, T; H^2)\} \qquad (3.3)$$

with $0 < T \leqslant +\infty$. We will obtain a priori estimate on the solution of (3.1) and (3.2). Thus suppose $\phi \in X(0, T)$ (for some $T > 0$) is such a solution; put

$$N(t) = \sup_{0 \leqslant \tau \leqslant t} \|\phi(\tau)\|_2 \qquad \text{for} \quad t \in [0, T], \qquad (3.4)$$

and assume $N(T) \leqslant \varepsilon_0$, where ε_0 is some positive constant.

In order to estimate the solution, first we need a careful diagonalization of the hyperbolic part of (3.1). Notice that $L = L(U)$, $R = R(U)$, and $A = (f'(U))$ depend only on x_0 by (2.7), so following the construction of

J. Goodman in [4], we can easily check that there exists a positive constant δ_0 such that for any δ, $0<\delta\leqslant\delta_0$, we can find L and R satisfying

$$Lf'(U)R=\Lambda, \qquad LR=I. \tag{3.5}$$

Furthermore, if we define $M(x,t)=L_x(x,t)\,R(x,t)$ and $N(x,t)=L_t(x,t)\,R(x,t)$, then

$$M_{pp}=N_{pp}=0, \qquad \forall x\in R^1, \forall t\geqslant 0,\ p=1,2. \tag{3.6}$$

With this diagonalization, our requirement on B is that

$$\bar{B}(x,t)=L(x,t)\,B(x,t)\,R(x,t) \tag{3.7}$$

be positive definite for all x and $t\geqslant 0$. From now on, we will always use this diagonalization. Put $V(x,t)=L(x,t)\,\phi(x,t)$; we will call V_k and V_i $(i\neq k)$ the *primary wave and nonprimary wave*, respectively. Then

$$\begin{gathered}\phi(x,t)=R(x,t)\,V(x,t),\\ V_x=MV+L\phi_x, \qquad V_t=NV+L\phi_t,\\ V_{xx}=(MV)_x+MV_x-M^2V+L\phi_{xx}.\end{gathered} \tag{3.8}$$

Multiplying (3.1) on the left by $L=L(x,t)$ and using (3.8), we get

$$\begin{aligned}V_t&+[\Lambda V]_x-M\Lambda V-NV+L(Q(U,RV))_x\\ &=L\frac{\partial}{\partial x}(B(U)\,RV_x)-L\frac{\partial}{\partial x}(BRMV)+L\frac{\partial}{\partial x}(\Delta BRV_x)\\ &\quad-L\frac{\partial}{\partial x}(\Delta BRMV)+L\frac{\partial}{\partial x}\left(B(U+RV)\frac{\partial U}{\partial x}\right),\end{aligned} \tag{3.9}$$

where $\Delta B=B(U+\phi)-B(U)$.

Now we proceed to estimate V. Multiply (3.9) on the left by V^t and integrate over $R^1\times[0,t]$, this gives 9 terms which we denote in order by I_l, $1\leqslant l\leqslant 9$. We estimate each term as follows,

$$I_1=\int_0^t\int (V^tV_t)\,dx\,d\tau=\frac{1}{2}\|V(t)\|^2-\frac{1}{2}\|V(0)\|^2,$$

$$I_2=\int_0^t\int V(\Lambda V)_x\,dx\,d\tau=\int_0^t d\tau\int V^t\Lambda_x V\,dx+\int_0^t d\tau\int V^t\Lambda V_x\,dx;$$

for the last term on the right above, integration by parts gives

$$\int_0^t d\tau\int V^t\Lambda V_x\,dx=-\frac{1}{2}\int_0^t\int V^t\Lambda_x V\,dx\,d\tau.$$

Since $\lambda_{kx} > 0$, we get

$$I_2 = \frac{1}{2}\int_0^t \int \left|\frac{\partial}{\partial x}(\lambda_k)\right| V_k^2\, dx\, d\tau + \frac{1}{2}\int_0^t \int \frac{\partial}{\partial x}(\lambda_i)\, V_i^2\, dx\, d\tau.$$

Then, (2.10) implies that for $i \neq k$,

$$I_2 \geqslant \frac{1}{2}\int_0^t \int |\lambda_{kx}|\, V_k^2\, dx\, d\tau - c\int_0^t \int |\lambda_{kx}|\, V_i^2\, dx\, d\tau, \tag{3.10}$$

where c is a positive constant. From now on, we will use c to denote any positive constant which does not depend on t and δ. In the following estimates, the idea is to use term $\frac{1}{2}\int_0^t \int |\lambda_{kx}|\, V_k^2\, dx\, d\tau$ to control terms involving the primary wave V_k, and the terms involving the nonprimary wave V_i will be controlled by estimates on $\int_0^t \int |\lambda_{kx}|\, V_i^2\, dx\, d\tau$. Note that since $|M(x, t)| \leqslant c\,|U_x|$ and $|N(x, t)| \leqslant c\,|U_x|$, using Cauchy inequality, (2.7) and (3.6), we get

$$\begin{aligned} I_3 &= \int_0^t \int V^t M \Lambda V\, dx\, d\tau = \sum_{p,j=1}^{2} \int_0^t \int \lambda_p V_p V_j M_{pj}\, dx\, d\tau \\ &\geqslant -\beta \int_0^t \int |\lambda_{kx}|\, V_k^2\, dx\, d\tau - c\int_0^t \int |\lambda_{kx}|\, V_i^2\, dx\, d\tau, \\ I_4 &= \int_0^t \int V^t N V\, dx\, d\tau = \sum_{p,j=1}^{2} \int_0^t \int V_p N_{pj} V_j\, dx\, d\tau \\ &\geqslant -\beta \int_0^t \int |\lambda_{kx}|\, V_k^2\, dx\, d\tau - c\int_0^t \int |\lambda_{kx}|\, V_i^2\, dx\, d\tau, \end{aligned} \tag{3.11}$$

where β is a small positive constant to be chosen later.

Now we estimate the four terms on the right hand side of (3.9),

$$\begin{aligned} I_6 &= \int_0^t \int V^t L \frac{\partial}{\partial x}(B(U)\, R V_x)\, dx\, d\tau \\ &= \int_0^t \int (V^t L)_x B(U)\, R V_x\, dx\, d\tau \\ &= -\int_0^t \int V_x^t (LBR)\, V_x\, dx\, d\tau - \int_0^t \int V^t L_x B R V_x\, dx\, d\tau \\ &\leqslant -B_0 \int_0^t \int \|V_x(\tau)\|^2\, d\tau + c\int_0^t \int |V^t|\, |U_x|\, |V_x|\, dx\, d\tau; \end{aligned}$$

here we have used the condition that LBR is positive definite, so there exists a positive constant B_0 such that $V_x^t(LBR)V_x \geqslant B_0|V_x|^2$. The last term on the right of I_6 can be estimated as follows,

$$\begin{aligned}
c\int_0^t\int |V^t|\,|U_x|\,|V_x|\,dx\,d\tau
&\leqslant c\int_0^t\int |V^t(|U_x|^2+|V_x|^2)\,dx\,d\tau\\
&\leqslant c\int_0^t (\|V^t\|^{1/2}\,\|V_x^t\|^{1/2}\|U_x\|^2+N(t)\|V_x\|^2)\,d\tau\\
&\leqslant cN(t)\int_0^t \|V_x\|^2\,d\tau+cN(t)^{1/2}\left\{\int_0^t \|V_x\|^2\,d\tau+\int_0^t \|U_x\|^{8/3}\,d\tau\right\}\\
&\leqslant c(N(t)+N(t)^{1/2})\int_0^t \|V_x\|^2\,d\tau+c(N(t))^{1/2}\,\delta^{4/3},
\end{aligned}$$

where Sobolev's inequality, Young's inequality, and (2.11) have been used. Thus we get

$$I_6\leqslant -B_0\int_0^t \|V_x\|^2\,d\tau+c(N(t)+N(t)^{1/2})\int_0^t \|V_x\|^2\,d\tau+cN(t)^{1/2}\,\delta^{4/3}. \tag{3.12}$$

Using the fact that $|L_xBRM|\leqslant c_1\,|U_x|^2\leqslant c\delta\,|\lambda_{kx}|$, we arrive at

$$\begin{aligned}
I_7&=-\int_0^t\int V^tL\frac{\partial}{\partial x}(BRMV)\,dx\,d\tau\\
&=\int_0^t\int V_x^tLBRMV\,dx\,d\tau+\int_0^t\int V^tL_xBRMV\,dx\,d\tau\\
&\leqslant \beta\int_0^t\int |\lambda_{kx}|\,V_k^2\,dx\,d\tau+c\delta\int_0^t\int |V_x|^2\,dx\,d\tau\\
&\quad+\delta\int_0^t\int |\lambda_{kx}|\,V_k^2\,dx\,d\tau+c\int_0^t\int |\lambda_{kx}|\,V_i^2\,dx_{d\tau}.
\end{aligned} \tag{3.13}$$

Similarly, we have

$$\begin{aligned}
I_8&=\int_0^t\int V^tL\frac{\partial}{\partial x}[\Delta B(RV_x-RMV)]\,dx\,d\tau\\
&=-\int_0^t\int (V_x^tL\Delta BRV_x-V_xL\Delta BRMV)\,dx\,d\tau\\
&\quad-\int_0^t\int (V^tL_x\,\Delta BRV_x-V^tL_x\,\Delta BRMV)\,dx\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leqslant c\int_0^t\int |V_x^t|\,|\Delta B|\,|V_x|\,dx\,d\tau + c\int_0^t\int |\lambda_{kx}|\,|V_x|\,|V|\,dx\,d\tau \\
&\quad + c\int_0^t\int |\lambda_{kx}|^2\,|V|^2\,dx\,d\tau \\
&\leqslant (c\delta+\beta)\int_0^t\int |\lambda_{kx}|\,V_k^2\,dx\,d\tau \\
&\quad + c(N(t)+\delta)\int_0^t \|V_x\|^2\,d\tau + c\int_0^t\int |\lambda_{kx}|\,V_i^2\,dx\,d\tau. \qquad (3.14)
\end{aligned}
$$

The last term on the right hand side of (3.9) is

$$
\begin{aligned}
I_9 &= \int_0^t\int V^t L\frac{\partial}{\partial x}\left(B(U+RV)\frac{\partial U}{\partial x}\right)dx\,d\tau \\
&= \int_0^t\int V^t L\left(\frac{\partial U}{\partial x}+\frac{\partial}{\partial x}(RV)\right)^t \nabla B(U+RV)\frac{\partial U}{\partial x}\,dx\,d\tau \\
&\quad + \int_0^t\int V^t LB(U+RV)\frac{\partial^2 U}{\partial x^2}\,dx\,d\tau \\
&\leqslant c\int_0^t\int |V^t|\left|\frac{\partial U}{\partial x}\right|^2 dx\,d\tau + c\int_0^t\int |V|^2\,|U_x|^2\,dx\,d\tau \\
&\quad + c\int_0^t\int |U_x|\,|V|\,|V_x|\,dx\,d\tau + c\int_0^t\int |V^t|\,|U_{xx}|\,dx\,d\tau \\
&\leqslant c\delta\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \beta\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau \\
&\quad + c\delta\int_0^t\int |V_x|^2\,dx\,d\tau + c\int_0^t\int |V^t|\,(|U_x|^2+|U_{xx}|)\,dx\,d\tau, \qquad (3.15)
\end{aligned}
$$

where β is a small positive constant to be chosen later. Applying the Sobolev inequality and Lemma 2.2 to the last term on the right above leads to

$$
\begin{aligned}
&c\int_0^t\int |V^t|\,(|U_x|^2+|U_{xx}|)\,dx\,d\tau \\
&\qquad \leqslant c\int_0^t\int \|V\|^{1/2}\,\|V_x\|^{1/2}(|U_x|^2+|U_{xx}|)\,dx\,d\tau \\
&\qquad = c\int_0^t \|V\|^{1/2}\,\|V_x\|^{1/2}(\|U_x\|^2+\|U_{xx}\|_{L^1})
\end{aligned}
$$

$$\leqslant cN(t)^{1/2}\int_0^t \|V_x\|^2\,d\tau$$

$$+cN(t)^{1/2}\int_0^t \|V_x\|^{8/3}\,d\tau+cN(t)^{1/2}\int_0^t \|U_{xx}\|_{L^1}^{4/3}\,d\tau$$

$$\leqslant cN(t)^{1/2}\int_0^t \|V_x\|^2\,d\tau+cN(t)^{1/2}\,\delta^{4/3}+cN(t)^{1/2}\,\delta^{1/6};$$

thus

$$I_9\leqslant(c\delta+\beta)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau$$

$$+c(\delta+N(t)^{1/2})\int_0^t \|V_x\|^2\,d\tau+cN(t)^{1/2}\,\delta^{1/6}. \tag{3.16}$$

Concerning the nonlinear term I_5, we have the following lemma which will be proved later.

LEMMA 3.1. *There exist positive constants ε_1 and c which are independent of T and δ such that if $N(t)+\delta\leqslant\varepsilon_1$, then*

$$|I_5|=\left|\int_0^t\int V^tLQ(U,RV)_x\,dx\,d\tau\right|$$

$$\leqslant c\left\{N(t)\,\|V(t)\|^2+N(0)\,\|V(0)\|^2\right.$$

$$\left.+N(t)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+N(t)\int_0^t\|V_x\|^2\,d\tau+N(t)\,\delta^{1/2}\right\}. \tag{3.17}$$

Combining all these results gives

$$\frac{1}{2}\,\|V(t)\|^2+\frac{1}{2}\int_0^t\int|\lambda_{kx}|\,V_k^2\,ddx\,d\tau+B_0\int_0^t\|V_x\|^2\,d\tau$$

$$\leqslant\frac{1}{2}\,\|V(0)\|^2+cN(0)\,\|V(0)\|^2+cN(t)\,\|V(t)\|^2$$

$$+5\beta\int_0^t\int|\lambda_{kx}|\,V_k^2\,dx\,d\tau$$

$$+c(\delta+N(t)+N(t)^{1/2})\int_0^t\int|\lambda_{kx}|\,V_k^2\,dx\,d\tau$$

$$+c(\delta+N(t)+N(t)^{1/2})\int_0^t \|V_x\|^2\,d\tau$$

$$+c\int_0^t\int|\lambda_{kx}|\,V_i^2\,dx\,d\tau+cN(t)\,\delta^{1/6}; \tag{3.18}$$

thus if we choose $\beta=\frac{1}{20}$, we have proved the following lemma.

LEMMA 3.2. *There exist positive constants $\varepsilon_2(\leqslant\varepsilon_1)$ and c independent of T and δ such that if $N(t)+\delta\leqslant\varepsilon_2$, then*

$$\|V(t)\|^2+\int_0^t\int|\lambda_{kx}|\,V_k^2\,dx\,d\tau+\int_0^t\|V_x(\tau)\|^2\,d\tau$$

$$\leqslant c\{\|V(0)\|^2+\delta^{1/6}\}+c\int_0^t\int|\lambda_{kx}|\,V_i^2\,dx\,d\tau\qquad(i\neq k). \tag{3.19}$$

Proof. In the rest of this section, we will prove Lemma 3.1, and the nonprimary wave will be estimated in Section 4. Integrating by parts gives

$$I_5=\int_0^t\int V^tLQ(U,RV)_x\,dx=\int_0^t\int\phi^tL^tLQ(U,\phi)_x\,dx\,d\tau$$

$$=-\int_0^t\int\phi^t(L^tL)_xQ(U,\phi)\,dx\,d\tau-\int_0^t\int\phi_x^tL^tLQ(U,\phi)\,dx\,d\tau.$$

Note that since $|L_x|\leqslant c\,|U_x|\leqslant c\,|\lambda_{kx}|$, the first term on the right above can be estimated as

$$\left|\int_0^t\int\phi^t(L^tL)_xQ(U,\phi)\,dx\,d\tau\right|\leqslant c\int_0^t\int|\lambda_{kx}|\,|\phi|^3\,dx\,d\tau$$

$$\leqslant c\int_0^t\int|\lambda_{kx}|\,|V|^3\,dx\,d\tau\leqslant cN(t)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau; \tag{3.20}$$

thus the main point is to estimate the last term on the right of I_5. We first rewrite this integral as

$$\int_0^t\int\phi_x^tL^tLQ(U,\phi)\,dx\,d\tau$$

$$=\int_0^t\int\phi_x^tL^tL\phi^t\nabla^2f(U)\phi\,dx\,d\tau+\int_0^t\int\phi_x^tL^tLG(U,\phi)\,dx\,d\tau, \tag{3.21}$$

where G satisfies $|G(U, \phi) \leqslant c\, |\phi|^3$ if $|\phi|$ is samll. Using Sobolev's inequality, we get

$$\left|\int_0^t \int \phi_x^t L^t L G(U, \phi)\, dx\, d\tau\right| \leqslant c \int_0^t \int |\phi_x^t|\, |\phi|^3\, dx\, d\tau$$

$$\leqslant c \int_0^t \|\phi\|\, \|\phi_x\| \int |\phi|\, |\phi_x|\, dx\, d\tau \leqslant cN(t)^2 \int_0^t \|\phi_x\|^2\, d\tau$$

$$\leqslant cN(t)^2 \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau + cN(t)^2 \int_0^t \|V_x(\tau)\|^2\, d\tau. \tag{3.22}$$

The first term on the right hand side of (3.21) can be written as

$$\int_0^t \int \phi_x^t L^t L \phi^t \nabla^2 f(U) \phi\, dx\, d\tau$$

$$\begin{aligned} &= \int_0^t \int [g_{11}(U)\, \phi_1^2 \phi_{1x} + g_{12}(U)\, \phi_2 \phi_1 \phi_{1x} \\ &\quad + g_{13}(U)\, \phi_2^2 \phi_{1x} + g_{21}(U)\, \phi_1^2 \phi_{2x} \\ &\quad + g_{22}(U)\, \phi_1 \phi_2 \phi_{2x} + g_{23}(U)\, \phi_2^2 \phi_{2x}]\, dx\, d\tau, \end{aligned} \tag{3.23}$$

where $g_{ij}(U)$ are smooth functions. Now we write the right hand side above as a sum of 6 terms, I_p' $(1 \leqslant p \leqslant 6)$, each of which we will estimate separately. After integration by parts, we have

$$I_1' = \int_0^t \int g_{11}(U)\, \phi_1^2 \phi_{1x}\, dx\, d\tau = -\int_0^t \int \nabla g_{11} \cdot U_x \frac{\phi_1^3}{3}\, dx\, d\tau$$

$$\leqslant c \int_0^t \int |U_x|\, |\phi_1|^3\, dx\, d\tau \leqslant cN(t) \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau. \tag{3.24}$$

In order to estimate I_3', we rewrite Eq. (3.1) as

$$\begin{aligned} &\phi_{1t} + [f_1(U_1 + \phi_1, U_2 + \phi_2) - f_1(U_1, U_2)]_x \\ &\quad = \frac{\partial}{\partial x}\left(B_{11} \frac{\partial \phi_1}{\partial x} + B_{12} \frac{\partial \phi_2}{\partial x} + B_{11} \frac{\partial U_1}{\partial x} + B_{12} \frac{\partial U_2}{\partial x}\right), \end{aligned} \tag{3.25}$$

$$\begin{aligned} &\phi_{2t} + [f_2(U_1, U_2 + \phi_2) - f_2(U_1, U_2)]_x \\ &\quad = \frac{\partial}{\partial x}\left(B_{21} \frac{\partial \phi_1}{\partial x} + B_{22} \frac{\partial \phi_2}{\partial x} + B_{21} \frac{\partial U_1}{\partial x} + B_{22} \frac{\partial U_2}{\partial x}\right). \end{aligned}$$

For any smooth function $K(u)$, $u \in \Omega$, we will use ΔK to denote the difference: $K(U+\phi)-K(U)=K(U_1+\phi_1, U_2+\phi_2)-K(U_1, U_2)$. Now define

$$K_1(U,\phi)=(f_{2u_1}+\Delta f_{2u_1})^{-1}, \qquad K_2(U,\phi)=(f_{1u_2}+\Delta f_{1u_2})^{-1},$$

then by assumption (1.4), we see that $\exists\varepsilon_1'>0$, $(\varepsilon_1'\leqslant\varepsilon_0)$, such that if $N(T)\leqslant\varepsilon'_1$, then K_1 and K_2 are well defined and smooth. Thus we can solve ϕ_{1x} and ϕ_{2x} from (3.25) to obtain

$$\phi_{1x}=K_1(U,\phi)\Big\{-\phi_{2t}-f_{2u_2}\phi_{2x}-\Delta f_{2u_2}(\phi_{2x}+U_{2x})-\Delta f_{2u_1}U_{1x} \\ +\frac{\partial}{\partial x}(B_{21}\phi_{1x}+B_{22}\phi_{2x}+B_{21}U_{1x}+B_{22}U_{2x})\Big\}, \tag{3.26$_1$}$$

and

$$\phi_{2x}=K_2(U,\phi)\Big\{-\phi_{1t}-f_{1u_1}\phi_{1x}-\Delta f_{1u_1}(\phi_{1x}+U_{1x})-\Delta f_{1u_2}U_{2x} \\ +\frac{\partial}{\partial x}(B_{11}\phi_{1x}+B_{12}\phi_{2x}+B_{11}U_{1x}+B_{12}U_{2x})\Big\}. \tag{3.26$_2$}$$

Using $(3.26)_1$, we get

$$\begin{aligned}I_3'&=\int_0^t\int g_{13}(U)\,\phi_2^2\phi_{1x}\,dx\,d\tau\\ &=\int_0^t\int K_1g_{13}\phi_2^2\Big[-\phi_{2t}-f_{2u_2}\phi_{2x}-\Delta f_{2u_2}\phi_{2x}-\Delta f_{2u_2}U_{2x}-\Delta f_{2u_1}U_{1x}\\ &\quad+\frac{\partial}{\partial x}(B_{21}\phi_{1x}+B_{22}\phi_{2x}+B_{21}U_{1x}+B_{22}U_{2x})\Big]\,dx\,d\tau.\end{aligned}$$

Now we estimat each term on the right above; thus

$$\begin{aligned}&\int_0^t\int K_1g_{13}\phi_2^2(-\phi_{2t})\,dx\,d\tau\\ &\quad=-\int K_1g_{13}\frac{1}{3}\phi_2^3\Big|_{\tau=0}^{\tau=t}dx+\int_0^t\int\frac{\partial}{\partial t}(K_1g_{13})\frac{1}{3}\phi_2^3\,dx\,d\tau\\ &\quad\leqslant cN(t)\,\|V(t)\|^2+cN(0)\,\|V(0)\|^2+\int_0^t\int\frac{\partial}{\partial t}(K_1g_{13})\frac{1}{3}\phi_2^3\,dx\,d\tau,\end{aligned}$$

and it is easy to see that

$$\int_0^t \int \frac{\partial}{\partial t}(K_1 g_{13}) \frac{1}{3} \phi_2^3 \, dx \, d\tau$$

$$\leqslant c \int_0^t \int |U_t| \, |\phi_2|^3 \, dx \, d\tau + \left| \int_0^t \int g_1(U, \phi) \frac{1}{3} \phi_2^3 \phi_{1t} \, dx \, d\tau \right|$$

$$+ \left| \int_0^t \int g_2(U, \phi) \frac{1}{3} \phi_2^3 \phi_{2t} \, dx \, d\tau \right|, \tag{3.27}$$

where g_1 and g_2 are some smooth functions. Using Eqs. (3.25), we can estimate the last two terms in (3.27) as follows,

$$\left| \int_0^t \int g_1(U, \phi) \frac{1}{3} \phi_2^3 \phi_{1t} \, dx \, d\tau \right|$$

$$= \left| - \int_0^t \int g_1(U, \phi) \frac{1}{3} \phi_2^3 (\Delta f_1)_x \, dx \, d\tau \right.$$

$$+ \left| \int_0^t \int g_1(U, \phi) \frac{1}{3} \phi^3 \frac{\partial}{\partial x} (B_{11} \phi_{1x} + B_{12} \phi_{2x} \right.$$

$$\left. + B_{11} U_{1x} + B_{12} U_{2x}) \, dx \, d\tau \right|$$

$$\leqslant c \int_0^t \int |(\Delta f_1)_x| \, |\phi_2|^3 \, dx \, d\tau$$

$$+ \left| \int_0^t \int \frac{\partial}{\partial x} \left(g_1(U, \phi) \frac{1}{3} \phi_2^3 \right) (B_{11} \phi_{1x} + B_{12} \phi_{2x} \right.$$

$$\left. + B_{11} U_{1x} + B_{12} U_{2x}) \, dx \, d\tau \right|,$$

$$c \int_0^t \int |(\Delta f_1)_x| \, |\phi_2|^3 \, dx \, d\tau$$

$$\leqslant c \int_0^t \int |\lambda_{kx}| \, |\phi_2|^3 \, dx \, d\tau + c \int_0^t \int |\phi_x| \, |\phi_2|^3 \, dx \, d\tau$$

$$\leqslant cN(t) \int_0^t \int |\lambda_{kx}| \, |V|^2 \, dx \, d\tau + cN(t)^2 \int_0^t \| V_x(\tau) \|^2 \, d\tau,$$

$$\left|\int_0^t\int \frac{\partial}{\partial x}\left(g_1(U,\phi)\frac{1}{3}\phi_2^3\right) B_{11}\phi_{1x}\,dx\,d\tau\right|$$

$$\leqslant c\int_0^t\int |U_x|\,|\phi_2|^3\,|\phi_{1x}|\,dx\,d\tau$$

$$+\,c\int_0^t\int |\phi_x|\,|\phi_2|^3\,|\phi_{1x}|\,dx\,d\tau + c\int_0^t\int |\phi_2|^2\,|\phi_{2x}|\,|\phi_{1x}|\,dx\,d\tau$$

$$\leqslant cN(t)^2\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + cN(t)^2\int_0^t \|V_x(\tau)\|^2\,d\tau,$$

where Sobolev and Cauchy inqualities have been used. Similarly

$$\left|\int_0^t\int \frac{\partial}{\partial x}\left(g_1(U,\phi)\frac{1}{3}\phi_2^3\right) B_{12}\phi_{2x}\,dx\,d\tau\right|$$

$$\leqslant cN(t)^2\left(\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \int_0^t \|V_x(\tau)\|^2\,d\tau\right),$$

$$\left|\int_0^t\int \frac{\partial}{\partial x}\left(g_1(U,\phi)\frac{1}{3}\phi_2^3\right) B_{11}U_{1x}\,dx\,d\tau\right|$$

$$\leqslant c\int_0^t\int |U_x|\,|\phi_2|^3\,|U_{1x}|\,dx\,d\tau$$

$$+\,c\int_0^t\int |\phi_x|\,|\phi_2|^3\,|U_{1x}|\,dx\,d\tau + c\int_0^t\int |\phi_2|^2\,|\phi_{2x}|\,|U_{1x}|\,dx\,d\tau$$

$$\leqslant cN(t)\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + c\,\delta N(t)\int_0^t \|V_x(\tau)\|^2\,d\tau \equiv \alpha_0.$$

Finally, we can get in the same way as above

$$\left|\int_0^t\int \frac{\partial}{\partial x}\left(g_1(U,\phi)\frac{1}{3}\phi_2^3\right) B_{12}U_{2x}\,dx\,d\tau\right| \leqslant \alpha_0,$$

so that

$$\left|\int_0^t\int g_1(U,\phi)\frac{1}{3}\phi_2^3\phi_{1t}\,dx\,d\tau\right| \leqslant cN(t)\left\{\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \int_0^t \|V_x(\tau)\|^2\,d\tau\right\}.$$

Similarly, one can show

$$\left|\int_0^t\int g_2(U,\phi)\frac{1}{3}\phi_2^3\phi_{2t}\,dx\,d\tau\right| \leqslant cN(t)\left\{\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \int_0^t \|V_x(\tau)\|^2\,d\tau\right\}.$$

These and (3.27) give

$$\left|\int_0^t\int\frac{\partial}{\partial t}(K_1 g_{13})\frac{1}{3}\phi_2^3\,dx\,d\tau\right|\leqslant cN(t)\left\{\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+\int_0^t\|V_x(\tau)\|^2\,d\tau\right\},$$

so that

$$\begin{aligned}\int_0^t\int K_1 g_{13}\phi_2^2(-\phi_{2t})\,dx\,d\tau&\\ \leqslant cN(0)\,\|V(0)\|^2+cN(t)\,\|V(t)\|^2&\\ +cN(t)\left\{\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+\int_0^t\|V_x(\tau)\|^2\,d\tau\right\}&\equiv\alpha_1. \qquad (3.28)\end{aligned}$$

Now for the other terms on the right of I_3', we have

$$\begin{aligned}&\int_0^t\int K_1 g_{13}\phi_2^2(-f_{2u_2}-\Delta f_{2u_2})\,\phi_{2x}\,dx\,d\tau\\ &=\frac{1}{3}\int_0^t\int\frac{\partial}{\partial x}(K_1 g_{13}(f_{2u_2}+\Delta f_{2u_2}))\,\phi_2^3\,dx\,d\tau\\ &\leqslant cN(t)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+cN(t)^2\int_0^t\|V_x(\tau)\|^2\,d\tau, \qquad (3.29)\end{aligned}$$

$$\begin{aligned}&\int_0^t\int K_1 g_{13}\phi_2^2(-\Delta f_{2u_2}U_{2x}-\Delta f_{2u_1}U_{1x})\,dx\,d\tau\\ &\leqslant cN(t)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau, \qquad (3.30)\end{aligned}$$

where we used the fact that $|\Delta f_{2u_2}|$, $|\Delta f_{2u_1}|\leqslant c\,|\phi|$ if $N(T)$ is small.

$$\begin{aligned}&\int_0^t\int K_1 g_{13}\phi^2\frac{\partial}{\partial x}(B_{21}\phi_{1x})\,dx\,d\tau\\ &=-\int_0^t\int\frac{\partial}{\partial x}(K_1 g_{13}\phi_2^2)\,B_{21}\phi_{1x}\,dx\,d\tau\\ &\leqslant c\int_0^t\int|U_x|\,\phi_2^2\,|\phi_{1x}|\,dx\,d\tau+c\int_0^t\int\phi_2^2\,|\phi_{1x}|\,dx\,d\tau\\ &\quad+c\int_0^t\int|\phi_2|\,|\phi_{2x}|\,|\phi_{1x}|\,dx\,d\tau\\ &\leqslant cN(t)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+(c\delta N(t)+cN(t))\int_0^t\|\phi_x(\tau)\|^2\,d\tau\\ &\leqslant cN(t)\left\{\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+\int_0^t\|V_x(\tau)\|^2\,d\tau\right\}. \qquad (3.31)\end{aligned}$$

Similarly, we have

$$\int_0^t \int K_1 g_{13}\phi_2^2 \frac{\partial}{\partial x}(B_{22}\phi_{2x})\,dx\,d\tau$$

$$\leqslant cN(t)\left\{\int_0^t \int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \int_0^t \|V_x(\tau)\|^2\,d\tau\right\}.$$

Now, we estimate the last term on the right hand side of I_3':

$$\begin{aligned}
&\int_0^t \int K_1 g_{13}\phi_2^2 \frac{\partial}{\partial x}(B_{21}U_{1x} + B_{22}U_{2x})\,dx\,d\tau \\
&\quad = \int_0^t \int K_1 g_{13}\phi_2^2(\nabla B_{21}(U_x+\phi_x)U_{1x} + \nabla B_{22}(U_x+\phi_x)U_{2x})\,dx\,d\tau \\
&\qquad + \int_0^t \int K_1 g_{13}\phi_2^2(B_{21}U_{1xx} + B_{22}U_{2xx})\,dx\,d\tau \\
&\quad \leqslant (cN(t)+c\delta)\int_0^t \int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \delta N(t)\int_0^t \|V_x(\tau)\|^2\,d\tau \\
&\qquad + cN(t)\int_0^t \|\phi_x(\tau)\|^2\,d\tau + cN(t)\int_0^t \|U_{xx}\|_{L^1}^2\,d\tau \\
&\quad \leqslant c(\delta+N(t))\int_0^t \int |\lambda_{kx}|\,|V|^2\,dx\,d\tau \\
&\qquad + cN(t)\int_0^t \|V_x(\tau)\|^2\,d\tau + cN(t)\,\delta^{1/2}.
\end{aligned} \tag{3.32}$$

Combining these results, we obtain

$$\begin{aligned}
I_3' \leqslant c\Big\{&N(0)\,\|V(0)\|^2 + N(t)\,\|V(t)\|^2 \\
&+ (\delta+N(t))\int_0^t \int |\lambda_{kx}|\,|V|^2\,dx\,d\tau \\
&+ N(t)\int_0^t \|V_x(\tau)\|^2\,d\tau + N(t)\,\delta^{1/2}\Big\} \equiv \alpha_2.
\end{aligned} \tag{3.33}$$

Applying integration by parts to I_2', we find

$$I_2' = \int_0^t \int g_{12}(U)\, \phi_2 \phi_1 \phi_{1x}\, dx\, d\tau$$

$$= -\frac{1}{2} \int_0^t \int \nabla g_{12}(U)\, U_x \phi_2 \phi_1^2\, dx\, d\tau - \frac{1}{2} \int_0^t \int g_{12} \phi_{2x} \phi_1^2\, dx\, d\tau$$

$$\leqslant cN(t) \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau + \left| \int_0^t \int \left(\frac{1}{2}\, g_{12}(U)\right) \phi_1^2 \phi_{2x}\, dx\, d\tau \right|,$$

and let last term on the right above can be treated in the same way as for I_3'; thus

$$I_2' \leqslant \alpha_2. \tag{3.34}$$

Similarly, by using $(3.26)_2$, we can show that

$$I_4' \leqslant \alpha_2, \qquad I_5' \leqslant \alpha_2, \qquad \text{and} \qquad I_6' \leqslant \alpha_2; \tag{3.35}$$

thus we have shown that

$$\int_0^t \int \phi_x^t L^t L \phi^t \nabla^2 f(U) \phi\, dx\, d\tau \leqslant c \Big\{ \dots \|V(0)\|^2 + N(t)\, \|V(t)\|^2$$

$$+ (\delta + N(t)) \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau$$

$$+ N(t) \int_0^t \|V_x(\tau)\|^2\, d\tau + N(t) \delta^{1/2} \Big\}, \tag{3.36}$$

provided that $N(t) + \delta$ is suitably small. Now the conclusion of Lemma 3.1 follows from (3.20), (3.22), and (3.36). The proof is complete. ∎

The next main point is to estimate the nonprimary wave V_i. By construction, we know that $U(x, t)$ is a function of x_0 and $x_0 = x - w_0(x_0)t$ (cf. (2.7)). Therefore, by a change of variable, we see that

$$\int_0^t \int \frac{\partial}{\partial x} (\lambda_k)\, V_i^2\, dx\, d\tau = \int_0^t \int \frac{\partial}{\partial x_0} (\lambda_k(x_0))\, \frac{\partial x_0}{\partial x}\, V_i^2(x_0 + w_0(x_0)\tau, \tau)\, \frac{\partial x}{\partial x_0}\, dx_0\, d\tau$$

$$= \int_0^t \int \frac{\partial}{\partial x_0} (\lambda_k(x_0))\, V_i^2(x_0 + w_0(x_0)\tau, \tau)\, dx_0\, d\tau$$

$$= \int_{-\infty}^{+\infty} \lambda_{kx_0}\, dx_0 \int_0^t V_i^2(x_0 + w_0(x_0)\tau, \tau)\, d\tau.$$

Thus we must estimate the integral $\int_0^t V_i^2(x_0 + w_0(x_0)\tau, \tau)\, d\tau$; this will be done in next section.

4. Second Energy Estimate

In this section, first we will obtain a "vertical" estimate on the nonprimary wave V_i, i.e., we will give an estimate on time integral $\int_0^t V_i^2(x_0+w_0(x_0)\tau, \tau)\,d\tau$. The idea is to relate this time integral to the space-time integral which can be estimated by the energy method. Then we can estimate $\int_0^t \int |\lambda_{kx}|\, V_i^2\,dx\,d\tau$ by combining the "vertical" estimate with Lemma 3.2 and the fact that for the weak rarefaction wave, $\int_{-\infty}^{+\infty} |\lambda_{kx_0}(x_0)|\,dx_0$ is small.

First, we remark that we can assume that the k-rarefaction wave propagates very slowly, i.e., $|w_0|$ is small. To see this, we use the transformation $(x, t)\to(\xi, t)$, where $\xi=x-\lambda_k(u_-)t$; then in the new coordinates (ξ, t), Eqs. (1.1) take the form

$$u_t+(f(u)-\lambda_k(u_-))_\xi=\frac{\partial}{\partial\xi}(B(u)u_\xi).$$

The eigenvalues of the matrix $(f'(u)-\lambda_k(u_-))$ are $\lambda_k(u)-\lambda_k(u_-)$ and $\lambda_i(u)-\lambda_k(u_-)$; thus by (2.6), we see that the corresponding $w_0(\xi)$ satisfies

$$|w_0|\leqslant c\,\|u_+-u_-\|_{R^2}=c\delta. \tag{4.1}$$

Therefore, without lose of generality, we can assume that (4.1) holds.

Next since we are only interested in the nonprimary wave V_i, we may assume that

$$\lambda_i(U(x,t))\geqslant\alpha>0, \qquad \text{for all} \quad x\in R^1,\ t\geqslant 0; \tag{4.2}$$

the case $\lambda_i(U(x,t))\leqslant-\alpha_0<0$ can be treated similarly. Now we define

$$E_i(x_0, t)=\int_0^t V_i^2(x_0+w_0(x_0)\tau, \tau)\,d\tau$$

and rewrite $E_i(x_0, t)$ as

$$\begin{aligned} E_0(x_0, t)&=\int_{-\infty}^{x_0}\frac{\partial}{\partial x_0}E_i(x', t)\,dx' \\ &=\int_{-\infty}^{x_0}\int_0^t 2V_i(x'+w_0(x')\tau, \tau)\,V_{ix}(x'+w_0(x')\tau, \tau) \\ &\quad\times(1+w_0(x')\tau)\,dx'\,d\tau. \end{aligned}$$

Changing variables gives

$$E_i(x_0, t) = \int_0^t \int_{-\infty}^{x_0 + w_0(x_0)\tau} 2V_i(x, \tau)\, V_{ix}(x, \tau)\, dx\, d\tau. \tag{4.3}$$

Now we have the following lemma.

LEMMA 4.1. *There exist positive constants* ε_3 ($\leqslant \varepsilon_2$) *and c independent of T and* δ *such that if* $N(T) + \delta \leqslant \varepsilon_3$, *then for any fixed* $x_0 \in R^1$, $0 \leqslant t \leqslant T$,

$$\begin{aligned} E_i(x_0, t) \leqslant c \Big\{ & \|V(0)\|^2 + N(t)\, \|V(t)\|^2 \\ & + (N(t) + \delta) \int_0^t |V^2(x_0 + w_0(x_0)\tau, \tau)|^2\, d\tau \\ & + \int_0^t |V_x(x_0 + w_0(x_0)\tau, \tau)|^2\, d\tau + \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau \\ & + \int_0^t \|V_x(\tau)\|^2\, d\tau + \delta^{1/6} \Big\}. \end{aligned} \tag{4.4}$$

Proof. Fix $x_0 \in R^1$ and assume that $w_0(x_0) > 0$ (the case $w_0(x_0) \leqslant 0$ is similar). For ease of notation, we use $\bar{x}(\tau)$ to denote $x_0 + w_0(x_0)\tau$ whenever this term appears as an upper limit on a spatial integral; we also employ summation convention. Using (3.9), we can rewrite $E_i(x_0, t)$ as

$$\begin{aligned} E_i(x_0, t) = \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} \Big[& -V_{it} - \lambda_{ix} V_i - \lambda_i M_{ij} V_j - N_{ij} V_j - L_{ij}(Q_j)_x \\ & + L_{ij} \frac{\partial}{\partial x} ((BR)_{jn} V_{nx} - (BRM)_{jn} V_n + (\Delta BR)_{jn} V_{nx} \\ & - (\Delta BRM)_{jn} V_n + \bar{B}_{jm} U_{mx}) \Big]\, dx\, d\tau, \end{aligned} \tag{4.5}$$

where $\bar{B}_{jm} = B_{jm}(U + \phi)$. Now we denote each term on the right above in order by L_p ($1 \leqslant p \leqslant 10$); each term will be studied separately. By changing the order of integration we get (denoting $(w_0(x_0))^{-1}(x - x_0)$ by $\bar{t}$)

$$\begin{aligned} L_1 = \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} (-V_{it})\, dx\, d\tau = & -\int_{-\infty}^{x_0} \int_0^t \frac{1}{\lambda_i} \frac{\partial}{\partial t} (V_i^2(x, \tau))\, d\tau\, dx \\ & - \int_{x_0}^{\bar{x}(t)} \int_{\bar{t}}^t \frac{1}{\lambda_i} \frac{\partial}{\partial t} (V_i^2(x, \tau))\, d\tau\, dx, \end{aligned}$$

$$-\int_{-\infty}^{x_0}\int_0^t \frac{1}{\lambda_i}\frac{\partial}{\partial t}(V_i^2(x,\tau))\,d\tau\,dx$$

$$=\int_{-\infty}^{x_0}\frac{1}{\lambda_i}V_i^2(x,\tau)|_{\tau=0}^{\tau=t}\,dx-\int_{-\infty}^{x_0}\int_0^t(\lambda_i)^{-2}\lambda_{ix}V_i^2(x,\tau)\,d\tau\,dx$$

$$\leqslant\frac{1}{\alpha}\|V_i(0)\|^2+c\int_{-\infty}^{x_0}\int_0^t|\lambda_{kx}|\,V_i^2(x,\tau)\,dx\,d\tau.$$

Similarly,

$$-\int_{x_0}^{\bar{x}(t)}\int_{\bar{t}}^t\frac{1}{\lambda_i}\frac{\partial}{\partial t}(V_i^2(x,\tau))\,dx\,d\tau$$

$$\leqslant c\int_{x_0}^{\bar{x}(t)}V_i^2(x,(w_0(x_0))^{-1}(x-x_0))\,dx+c\int_{x_0}^{\bar{x}(t)}\int_{\bar{t}}^t|\lambda_{kx}|\,V_i^2(x,\tau)\,dx\,d\tau,$$

so that

$$L_1\leqslant c\left\{\|V_i(0)\|^2+\int_0^t\int|\lambda_{kx}|\,V_i^2(x,\tau)\,dx\,d\tau\right.$$
$$\left.+\int_{x_0}^{\bar{x}(t)}V_i^2(x,(w_0(x_0))^{-1}(x-x_0))\,dx\right\}.$$

We change variables in the last term on the right above to obtain

$$\int_{x_0}^{\bar{x}(t)}V_i^2(x,(w_0(x_0))^{-1}(x-x_0))\,dx$$

$$=\int_0^t V_i^2(x_0+w_0(x_0)\tau,\tau)\,w_0(x_0)\,d\tau\leqslant c\delta E_i(x_0,t),$$

where inequality (4.1) has been used. Thus we get

$$L_1\leqslant c\left\{\delta E_i(x_0,t)+\|V_i(0)\|^2+\int_0^t\int|\lambda_{kx}|\,V_i^2(x,\tau)\,dx\,d\tau\right\}. \tag{4.6}$$

Using (4.2) and the fact that $|M_{ij}|\leqslant c\,|\lambda_{kx}|$ and $|N_{ij}|\leqslant c\,|\lambda_{kx}|$ gives

$$L_2=\int_0^t\int_{-\infty}^{\bar{x}(\tau)}\frac{1}{\lambda_i}(-2V_i^2(x,t)\lambda_{ix})\,dx\,d\tau\leqslant c\int_0^t\int|\lambda_{kx}|\,V_i^2(x,\tau)\,dx\,d\tau,$$

$$L_3+L_4=\int_0^t\int_{-\infty}^{\bar{x}(\tau)}2V_i\frac{1}{\lambda_i}(\lambda_iM_{ij}V_j-N_{ij}V_j)\,dx\,d\tau$$

$$\leqslant c\int_0^t\int|\lambda_{kx}|\,|V(x,\tau)|^2\,dx\,d\tau.$$

Applying integration by parts and Cauchy inequality leads to

$$L_6 = \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij} \frac{\partial}{\partial x} (B_{jm} R_{mn} V_{nx})\, dx\, d\tau$$

$$= \int_0^t \frac{2}{\lambda_i} L_{ij} B_{jm} R_{mn} V_i V_{nx}(x_0 + w_0(x_0)\tau, \tau)\, d\tau$$

$$- \int_0^t \int_{-\infty}^{\bar{x}(\tau)} \frac{\partial}{\partial x} (\lambda_i^{-1} L_{ij} V_i)\, B_{jm} R_{mn} V_{nx}\, dx\, d\tau$$

$$\leqslant \frac{1}{10} E_i(x_0, t) + c \int_0^t |V_x(x_0 + w_0(x_0)\tau, \tau)|^2\, d\tau$$

$$+ c \int_0^t \|V_x(\tau)\|^2\, d\tau + c \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau,$$

$$L_7 = -\int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij} \frac{\partial}{\partial x} (B_{jm} (RM)_{mn} V_n)\, dx\, d\tau$$

$$= -\int_0^t \left(\frac{2}{\lambda_i} L_{ij} B_{jm} (RM)_{mn} V_i V_n \right) (x_0 + w_0(x_0)\tau, \tau)\, d\tau$$

$$+ \int_0^t \int_{-\infty}^{\bar{x}(\tau)} \frac{\partial}{\partial x} (\lambda_i^{-1} L_{ij} V_i)\, B_{jm} (RM)_{mn} V_n\, dx\, d\tau$$

$$\leqslant c\delta \int_0^t |V_i(x_0 + w_0(x_0)\tau, \tau)|^2\, d\tau$$

$$+ c \int_0^t \|V_x(\tau)\|^2\, d\tau + c \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau,$$

$$L_8 = \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij} \frac{\partial}{\partial x} (\Delta B_{jm} R_{mn} V_{nx})\, dx\, d\tau$$

$$\leqslant \frac{1}{10} E_i(x_0, t) + c \int_0^t |V_x(x_0 + w_0(x_0)\tau, \tau)|^2\, d\tau$$

$$+ c \int_0^t \|V_x(\tau)\|^2\, d\tau + c \int_0^t \int |_{kx}|\, |V|^2\, dx\, d\tau,$$

$$L_9 = -\int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij} \frac{\partial}{\partial x} (\Delta B_{jm} (RM)_{mn} V_n)\, dx\, d\tau$$

$$\leqslant c\delta \int_0^t |V_i(x_0 + w_0(x_0)\tau, \tau)|^2\, d\tau$$

$$+ c \int_0^t \|V_x(\tau)\|^2\, d\tau + c \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau.$$

The last term on the right of (4.5) can be estimated as follows:

$$L_{10} = \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij} \frac{\partial}{\partial x} (\bar{B}_{jm} U_{mx})\, dx\, d\tau$$

$$= \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij} (\nabla \bar{B}_{jm}(U_x + \phi_x)\, U_{mx}\, dx\, d\tau + \bar{B}_{jm} U_{mxx})\, dx\, d\tau$$

$$\leqslant c \int_0^t \int |V_i|\, (|U_x|^2 + |U_x|^2\, |V| + |U_x|\, |V_x|)\, dx\, d\tau$$

$$+ c \int_0^t \int |V_i|\, |U_{xx}|\, dx\, d\tau$$

$$\leqslant c\delta + c \int_0^t \|V_x(\tau)\|^2\, d\tau + c \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau + c \int_0^t \int |V_i|\, |U_{xx}|\, dx\, d\tau.$$

Using Sobolev's inequality and $(2.12)_2$, we see that

$$\int_0^t \int |V_i|\, |U_{xx}|\, dx\, d\tau$$

$$\leqslant c \int_0^t \|V_i(\tau)\|^{1/2}\, \|V_{ix}(\tau)\|^{1/2}\, \|U_{xx}(\tau)\|_{L^1}^2\, dx\, d\tau$$

$$\leqslant c(N(t))^{1/2} \int_0^t \int \|V_{ix}(\tau)\|^2\, d\tau + c(N(t))^{1/2} \int_0^t \int \|U_{xx}(\tau)\|_{L^1}^{4/3}\, d\tau$$

$$\leqslant c(N(t))^{1/2} \int_0^t \int \|V_{ix}(\tau)\|^2\, d\tau + c(N(t))^{1/2}\, \delta^{1/6},$$

from which we obtain

$$L_{10} = c \left\{ \delta^{1/6} + \int_0^t \|V_x(\tau)\|^2\, d\tau + \int_0^t \int |\lambda_{kx}|\, |V|^2\, dx\, d\tau \right\}.$$

To estimate L_5, we will use a similar argument as I_5 in Section 3. However, since the upper limit on the spatial integral is a variable depending on t, we need to deal with some time integrals introduced by integration by parts with respect to the space variable. Integration by parts and using Sobolev's inequality gives

$$
\begin{aligned}
L_5 &= \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2V_i \frac{1}{\lambda_i} L_{ij}(Q_j)_x \, dx \, d\tau \\
&= \int_0^t \left(\frac{2}{\lambda_i} L_{ij} V_i Q_j \right) (x_0 + w_0(x_0)\tau, \tau) \, d\tau \\
&\quad - 2 \int_0^t \int_{-\infty}^{\bar{x}(\tau)} \frac{\partial}{\partial x} (\lambda_i^{-1} L_{ij}) \, V_i Q_j \, dx \, d\tau - \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} Q_j V_{ix} \, dx \, d\tau \\
&\leqslant cN(t) \left\{ \int_0^t |V(x_0 + w_0(x_0)\tau, \tau)|^2 \, d\tau + \int_0^t \int |\lambda_{kx}| \, |V|^2 \, dx \, d\tau \right\} \\
&\quad + \left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} Q_j V_{ix} \, dx \, d\tau \right|.
\end{aligned}
$$

As in Section 3, to estimate the last term on the right above, it is convenient to use ϕ instead of V. Since $V_{ix} = M_{il} V_l + L_{il} \phi_{lx}$, we have

$$
\begin{aligned}
&\left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} Q_j V_{ix} \, dx \, d\tau \right| \\
&\quad \leqslant \left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} M_{il} V_l Q_j \, dx \, d\tau \right| \\
&\quad\quad + \left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} L_{il} \phi_{lx} Q_j \, dx \, d\tau \right| \\
&\quad \leqslant cN(t) \int_0^t \int |\lambda_{kx}| \, |V|^2 \, dx \, d\tau \\
&\quad\quad + cN(t) \int_0^t \| V_x(\tau) \|^2 \, d\tau + \left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} L_{il} \phi_{lx} \phi' \nabla^2 f \phi \, dx \, d\tau \right|.
\end{aligned}
$$

Now the last term on the right above can be written as

$$
\begin{aligned}
&\left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} 2\lambda_i^{-1} L_{ij} L_{i1} \phi_{lx} \phi' \nabla^2 f \phi \, dx \, d\tau \right| \\
&\quad = \left| \int_0^t \int_{-\infty}^{\bar{x}(\tau)} (h_{11} \phi_1^2 \phi_{1x} + h_{12} \phi_1 \phi_2 \phi_{1x} \right. \\
&\quad\quad \left. + h_{13} \phi_2^2 \phi_{1x} + h_{21} \phi_1^2 \phi_{2x} + h_{22} \phi_2^2 \phi_{2x} + h_{23} \phi_1 \phi_2 \phi_{2x}) \, dx \, d\tau \right|,
\end{aligned}
$$

where $h_{ij} = h_{ij}(U)$ are smooth functions. Integrating by parts gives

$$\left|\int_0^t \int_{-\infty}^{\bar{x}(\tau)} (h_{11}\phi_1^2\phi_{1x} + h_{22}\phi_2^2\phi_{2x})\,dx\,d\tau\right|$$
$$\leqslant cN(t)\left\{\int_0^t |V(x_0 + w_0(x_0)\tau, \tau)|^2\,d\tau\right.$$
$$\left. + \int_0^t \int |\lambda_{kx}|\,|V|^2\,dx\,d\tau + \int_0^t \|V_x(\tau)\|^2\,d\tau\right\}. \tag{4.7}$$

Making use of (3.26), we have

$$\int_0^t \int_{-\infty}^{\bar{x}(\tau)} h_{13}\phi_2^2\phi_{1x}\,dx\,d\tau$$
$$= \int_0^t \int_{-\infty}^{\bar{x}(\tau)} K_1 h_{13}\phi_2^2 \Big[-\phi_{2t} - f_{2u_2}\phi_{2x} - \Delta f_{2u_2}\phi_{2x} - \Delta f_{2u_2} U_{2x} - \Delta f_{2u_1} U_{1x}$$
$$+ \frac{\partial}{\partial x}(B_{21}\phi_{1x} + B_{22}\phi_{2x} + B_{21}U_{1x} + B_{22}U_{2x})\Big]\,dx\,d\tau.$$

Integrating by parts and using Cauchy–Schwartz inequality gives

$$\left|\int_0^t \int_{-\infty}^{\bar{x}(\tau)} K_1 h_{13}\phi_2^2 \Big[-f_{2u_2}\phi_{2x} - \Delta f_{2u_2}\phi_{2x} - \Delta f_{2u_2} U_{2x} - \Delta f_{2u_1} U_{1x}\right.$$
$$\left. + \frac{\partial}{\partial x}(B_{21}\phi_{1x} + B_{22}\phi_{2x} + B_{21}U_{1x} + B_{22}U_{2x})\Big]\,dx\,d\tau\right|$$
$$\leqslant c\int_0^t \int |\lambda_{kx}|\,|V|^2\,dx\,d\tau$$
$$+ \int_0^t \|V_x(\tau)\|^2\,d\tau + c(N(t)+\delta)\int_0^t |V(x_0 + w_0(x_0)\tau, \tau)|^2\,d\tau$$
$$+ cN(t)\int_0^t |V_x(x_0 + w_0(x_0)\tau, \tau)|^2\,d\tau \equiv \alpha_3. \tag{4.8}$$

Changing the order of integration as for L_1, then integrationg by parts gives

$$\left|\int_0^t \int_{-\infty}^{\bar{x}(\tau)} K_1 h_{13}\phi_2^2\phi_{2t}\,dx\,d\tau\right|$$
$$\leqslant cN(0)\,\|V(0)\|^2 + cN(t)\,\|V(t)\|^2$$
$$+ c\delta N(t)\int_0^t |V_x(x_0 + w_0(x_0)\tau, \tau)|^2\,d\tau$$

$$+cN(t)\int_0^t\int |\lambda_{kx}|\,|V|^2\,dx\,d\tau$$

$$+\left|\int_0^t\int_{-\infty}^{\bar{x}(\tau)}(h_1\phi_{1t}+h_2\phi_{2t})\frac{1}{3}\phi_2^3\,dx\,d\tau\right|, \tag{4.9}$$

where h_1 and h_2 are two smooth functions of U and ϕ. Applying a similar argument as in Section 3 after (3.27) to the last term on the right above, we get

$$\left|\int_0^t\int_{-\infty}^{\bar{x}(\tau)}(h_1\phi_{1t}+h_2\phi_{2t})\frac{1}{3}\phi_2^3\,dx\,d\tau\right|\leqslant\alpha_3.$$

Thus,

$$\left|\int_0^t\int_{-\infty}^{\bar{x}(\tau)}K_1h_{13}\phi_2^2\phi_{2t}\,dx\,d\tau\right|\leqslant cN(0)\,\|V(0)\|^2+cN(t)\,\|V(t)\|^2+\alpha_3\equiv\alpha_4. \tag{4.10}$$

Combining (4.8)–(4.10) shows that

$$\left|\int_0^t\int_{-\infty}^{\bar{x}(\tau)}h_{13}\phi_2^2\phi_{1x}\,dx\,d\tau\right|\leqslant\alpha_4. \tag{4.11}$$

Similarly, we can show that

$$\left|\int_0^t\int_{-\infty}^{\bar{x}(\tau)}(h_{12}\phi_1\phi_2\phi_{1x}+h_{21}\phi_1^2\phi_{2x}+h_{23}\phi_1\phi_2\phi_{2x})\,dx\,d\tau\right|\leqslant\alpha_4.$$

Putting all these results together shows that

$$L_5\leqslant\alpha_4. \tag{4.12}$$

The conclusion in Lemma 4.1 now follows from the estimates on L_p, $(1\leqslant p\leqslant 10)$; thus the proof is complete. ∎

We can now estimate the nonprimary wave V_i. If $N(T)+\delta\leqslant\varepsilon_3$, then Lemma 4.1 and (3.37) give

$$\int_0^t\int\frac{\partial}{\partial x}(\lambda_k)\,V_i^2\,dx\,d\tau$$

$$=\int_{-\infty}^{+\infty}\lambda_{kx_0}E_i(x_0,t)\,dx_0$$

$$\leqslant c\left\{\delta\,\|V(0)\|^2+\delta(N(t)+\delta)\,\|V(t)\|^2\right.$$

$$+N(t)\int_{-\infty}^{+\infty}\lambda_{kx_0}(x_0)\int_0^t|V(x_0+w_0(x_0)\tau,\tau)|^2\,d\tau\,dx_0$$

$$+\delta\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau$$

$$+\int_{-\infty}^{+\infty}\lambda_{kx_0}(x_0)\int_0^t|V_x(x_0+w_0(x_0)\tau,\tau)|^2\,d\tau\,dx_0$$

$$\left.+\delta\int_0^t\|V_x(\tau)\|^2\,d\tau+\delta^{7/6}\right\}$$

$$=c\left\{\delta\,\|V(0)\|^2+\delta(N(t)+\delta)\,\|V(t)\|^2+(N(t)+\delta)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau\right.$$

$$\left.+\int_{-\infty}^{+\infty}\int_0^t|\lambda_{kx}|\,|V_x(x,t)|^2\,d\tau\,dx+\delta\int_0^t\|V_x(\tau)\|^2\,d\tau+\delta^{7/6}\right\}.$$

Since $|\lambda_{kx}|\leqslant c\delta$, it follows that

$$\int_0^t\int|\lambda_{kx}|\,V_i^2(x,t)\,dx\,d\tau$$

$$\leqslant c\left\{\delta\,\|V(0)\|^2+\delta(N(t)+\delta)\,\|V(t)\|^2\right.$$

$$\left.+(N(t)+\delta)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+\delta\int_0^t\|V_x(\tau)\|^2\,d\tau+\delta^{7/6}\right\}. \quad (4.13)$$

Combining Lemma 3.2 with (4.13) shows that

$$\|V(t)\|^2+\int_0^t\int|\lambda_{kx}|\,|V(x,t)|^2\,dx\,d\tau+\int_0^t\|V_x(\tau)\|^2\,d\tau$$

$$\leqslant c\left\{\|V(0)\|^2+\delta^{1/6}+\delta(N(t)+\delta)\,\|V(t)\|^2\right.$$

$$\left.+(N(t)+\delta)\int_0^t\int|\lambda_{kx}|\,|V|^2\,dx\,d\tau+\delta\int_0^t\|V_x(\tau)\|^2\,d\tau\right\}. \quad (4.14)$$

Thus (4.14) and Lemma 3.2 yield the following lemma.

LEMMA 4.2. *There exist positive constants ε_4 ($\leqslant\varepsilon_3$) and c independent of T and δ such that if $N(T)+\delta\leqslant\varepsilon_4$, then for any t, $0\leqslant t\leqslant T$,*

$$\|V(t)\|^2+\int_0^t\int|\lambda_{kx}|\,|V(x,t)|^2\,dx\,d\tau+\int_0^t\|V_x(\tau)\|^2\,d\tau$$

$$\leqslant c\{\|V(0)\|^2+\delta^{1/6}\}. \quad (4.15)$$

From Lemma 4.2 and (3.10), we obtain

PROPOSITION 4.3. *If $N(T)+\delta \leqslant \varepsilon_4$, then $\forall t$, $0 \leqslant t \leqslant T$,*

$$\|\phi(t)\|^2 + \int_0^t \int |\lambda_{kx}|\, |\phi(x,\tau)|^2\, dx\, d\tau + \int_0^t \|\phi_x(\tau)\|^2\, d\tau \leqslant c\{\|\phi(0)\|^2 + \delta^{1/6}\}, \tag{4.16}$$

where c is independent of δ and T.

5. GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

Having the stability estimate (4.16), the proof of the Theorem 1.1 becomes fairly straightforward. First we need to estimate the higher order derivatives of the solution. Differentiate (3.9) with respect to x, then multiply by V_x^t on the left, and integrate over $[0, t] \times R^1$ to get (using integration by parts)

$$\begin{aligned}
&\frac{1}{2}\|V_x(t)\|^2 - \frac{1}{2}\|V_x(0)\|^2 + \int_0^t \int V_x^t \frac{\partial}{\partial x}(L(f(U+RV)-f(U))_x)\, dx\, d\tau \\
&= \int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(B(U)RV_x)\right) dx\, d\tau \\
&\quad - \int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(B(U)\, RMV)\right) dx\, d\tau \\
&\quad + \int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(\Delta B(RV)_x)\right) dx\, d\tau \\
&\quad + \int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(\bar{B}U_x)\right) dx\, d\tau.
\end{aligned} \tag{5.1}$$

We estimate each term in (5.1) as follows

$$\begin{aligned}
&\left| \int_0^t \int V_x^t \frac{\partial}{\partial x}(L(f(U+RV)-f(U))_x)\, dx\, d\tau \right| \\
&= \left| \int_0^t \int V_{xx}^t L(f(U+RV)-f(U))_x\, dx\, d\tau \right| \\
&\leqslant \frac{B_0}{4} \int_0^t \|V_{xx}(\tau)\|^2\, d\tau + c \int_0^t \int |\lambda_{kx}|\, |V(x,t)|^2\, dx\, d\tau \\
&\quad + c \int_0^t \|V_x(\tau)\|^2\, d\tau + c(\delta + N(t)) \int_0^t \|V_{xx}(\tau)\|^2\, d\tau,
\end{aligned} \tag{5.2}$$

$$\int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(B(U)\, RV_x)\right) dx\, d\tau$$

$$= -\int_0^t \int V_{xx}^t L \frac{\partial}{\partial x}(B(U)\, RV_x)\, dx\, d\tau$$

$$= -\int_0^t \int V_{xx}^t LB(U)\, RV_{xx}\, dx\, d\tau - \int_0^t \int V_{xx}^t\, L \frac{\partial}{\partial x}(B(U)R)\, V_x\, dx\, d\tau$$

$$\leqslant -B_0 \int_0^t \|V_{xx}(\tau)\|^2\, d\tau + c\delta \int_0^t \|V_{xx}(\tau)\|^2\, d\tau + c \int_0^t \|V_x(\tau)\|^2\, d\tau, \tag{5.3}$$

$$-\int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(B(U) RMV)\right) dx\, d\tau$$

$$= \int_0^t \int V_{xx}^t\, L \frac{\partial}{\partial x}(B(U) RMV)\, dx\, d\tau$$

$$\leqslant c \int_0^t \int |\lambda_{kx}|\, |V(x, t)|^2\, dx\, d\tau + \int_0^t \|V_x(\tau)\|^2\, d\tau$$

$$+ c\delta \int_0^t \|V_{xx}(\tau)\|^2\, d\tau, \tag{5.4}$$

$$\int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(\Delta B(RV)_x)\right) dx\, d\tau$$

$$= -\int_0^t \int V_{xx}^t\, L \frac{\partial}{\partial x}(\Delta B(RV)_x)\, dx\, d\tau$$

$$\leqslant c \int_0^t \int \int |\lambda_{kx}|\, |V(x, t)|^2\, dx\, d\tau + \int_0^t \|V_x(\tau)\|^2\, d\tau$$

$$+ c(\delta + N(t)) \int_0^t \|V_{xx}(\tau)\|^2\, d\tau, \tag{5.5}$$

where the inequality $|V_x| \leqslant cN(t)$ and Sobolev's inequality have been used. Using Cauchy inequality, (2.11) and $(2.12)_2$ lead to

$$\int_0^t \int V_x^t \frac{\partial}{\partial x}\left(L \frac{\partial}{\partial x}(\bar{B}U_x)\right) dx\, d\tau$$

$$= -\int_0^t \int V_{xx}^t L \frac{\partial}{\partial x}(\bar{B}U_x)\, dx\, d\tau$$

$$\leqslant \frac{B_0}{4} \int_0^t \|V_{xx}(\tau)\|^2\, d\tau + c \int_0^t \int |\lambda_{kx}|\, |V(x, t)|^2\, dx\, d\tau + c \int_0^t \|V_x(\tau)\|^2\, d\tau$$

$$+c(\delta+N(t))\int_0^t \|V_{xx}(\tau)\|^2\,d\tau+c\int_0^t \|U_{xx}(\tau)\|_{L^2}^2\,d\tau$$
$$+c\int_0^t\int |U_x(x,\tau)|^4\,dx\,d\tau$$
$$\leqslant \frac{B_0}{4}\int_0^t \|V_{xx}(\tau)\|^2\,d\tau+c\int_0^t\int |\lambda_{kx}|\,V^2(x,t)\,dx\,d\tau$$
$$+c\int_0^t \|V_x(\tau)\|^2\,d\tau+c(\delta+N(t))\int_0^t \|V_{xx}(\tau)\|^2\,d\tau+c\delta^{1/2}. \tag{5.6}$$

The inequalities (5.1)–(5.6) show that

$$\|V_x(t)\|^2+\int_0^t \|V_{xx}(\tau)\|^2\,d\tau$$
$$\leqslant c\left\{\|V_x(0)\|^2+\delta^{1/2}+\int_0^t \|V_x(\tau)\|^2\,d\tau\right.$$
$$+\int_0^t\int |\lambda_{kx}|\,V^2(x,t)\,dx\,d\tau+(\delta+N(t))\int_0^t \|V_x(\tau)\|^2\,d\tau$$
$$\left.+\int_0^t\int |\lambda_{kx}|\,V^2(x,t)\,dx\,d\tau+(\delta+N(t))\int_0^t \|V_{xx}(\tau)\|^2\,d\tau\right\}. \tag{5.7}$$

Lemma 4.2 and (5.7) imply that there exist positive constants ε_5 ($\varepsilon_5\leqslant\varepsilon_4$) and c independent of T and δ such that for any t, $0\leqslant t\leqslant T$,

$$\|V_x(t)\|^2+\int_0^t \|V_{xx}(\tau)\|^2\,d\tau\leqslant c\{\|V(0)\|_1^2+\delta^{1/6}\}. \tag{5.8}$$

Putting (3.8), (4.16), and (5.8) together shows that the following lemma holds

LEMMA 5.1. *There exist positive constants ε_5 ($\leqslant\varepsilon_4$) and c independent of T and δ such that if $N(T)+\delta\leqslant\varepsilon_5$, then for any t, $0\leqslant t\leqslant T$,*

$$\|\phi_x(t)\|^2+\int_0^t \|\phi_{xx}(\tau)\|^2\,d\tau\leqslant c\{\|\phi(0)\|_1^2+\delta^{1/6}\}. \tag{5.9}$$

Similarly we have

LEMMA 5.2. *There exist positive constants* ε_6 $(\leqslant \varepsilon_5)$ *and* c *independent of* T *and* δ *such that if* $N(T)+\delta \leqslant \varepsilon_6$, *then for any* t, $0 \leqslant t \leqslant T$,

$$\|\phi_{xx}(t)\|^2 + \int_0^t \left\| \frac{\partial^3}{\partial x^3} \phi(\tau) \right\|^2 d\tau \leqslant c\{\|\phi(0)\|_2^2 + \delta^{1/6}\}. \tag{5.10}$$

Proposition 4.3, Lemma 5.1, and Lemma 5.2 give the following result.

PROPOSITION 5.3 (A priori estimate). *Suppose* $\phi \in X(0, T)$ *is a solution of* (3.1) *and* (3.2) *for some* $T(>0)$ *and* $\delta(0<\delta \leqslant \delta_0)$. *Then there exist positive constants* ε *and* c, *independent of* T *and* δ, *such that if* $N(T)+\delta \leqslant \varepsilon$, *then we have for any* t, $0 \leqslant t \leqslant T$,

$$\|\phi(t)\|_2^2 + \int_0^t \int |\lambda_{kx}|\, |\phi(x,\tau)|^2\, dx\, d\tau + \int_0^t \|\phi_x(\tau)\|_2^2\, d\tau \leqslant c\{\|\phi(0)\|_2^2 + \delta^{1/6}\}. \tag{5.11}$$

Since the local (in time) existence and uniqueness of the solution for initial value problem (3.1) and (3.2) can be obtained by the standard iteration method (cf. 4, 6]), it follows from Proposition 5.3 and the standard continuity augument (cf. [7]) that the following proposition holds.

PROPOSITION 5.4. *Fo each fixed* $u_- \in \Omega$, *there exist positive constants* ε_0 *and* c_0 *such that if* $u_+ \in R_k(u_-)$ *and* $N(0)+\delta \leqslant \varepsilon_0$, *then the problem* (3.1) *and* (3.2) *has a unique global solution* $\phi \in X(0, +\infty)$ *satisfying, for any* $t \geqslant 0$,

$$\|\phi(t)\|_2^2 + \int_0^t \int |\lambda_{kx}|\, |\phi(x,\tau)|^2\, dx\, d\tau + \int_0^t \|\phi_x(\tau)\|_2^2\, d\tau \leqslant c_0\{N(0) + \delta^{1/6}\}. \tag{5.12}$$

By (5.12) and the proof of Lemma 5.1, it is easy to check that there exists a constant c such that

$$\int_0^t \left| \frac{d}{dt} \int |\phi_x(x,\tau)|^2\, dx \right| d\tau \leqslant c\{N(0)+\delta^{1/6}\}, \qquad \forall t \geqslant 0;$$

thus

$$\int_0^{+\infty} \left(\|\phi_x(\tau)\|^2 + \left| \frac{d}{dt} \|\phi_x(\tau)\|^2 \right| \right) d\tau < +\infty.$$

It follows that

$$\lim_{t \to \infty} \|\phi_x(t)\|^2 = 0, \tag{5.13}$$

so that Sobolev's inequality implies that

$$\lim_{t \to \infty} \sup_{x \in R^1} |\phi(x, t)| = 0. \tag{5.14}$$

By Lemma 2.2, it is easy to see that

$$N(0) \leqslant \|u_0 - u_0^r\| + \|u_{0x}\|_1^2 + c\delta^2. \tag{5.15}$$

Therefore all statements in Theorem 1.1 follow fom Proposition 5.4, Lemma 2.1, (5.14), and (5.15). This completes the proof of Theorem 1.1.

Acknowledgments

This work comprises part of the author's Ph. D dissertation, written with the guidance and encouragement of Professor Joel Smoller at the University of Michigan. I am sincerely grateful to Professor J. Smoller for his help.

References

1. J. A. Smoller and J. L. Johnson, Global solution for an extended class of hyperbolic systems, *Arch. Rational Mech. Anal.* **32** (1969), 169–189.
2. A. M. Ilin and O. A. Oleinik, Asymptotic behavior of the solutions of the Cauchy problem for certain quasilinear equations for large time, *Mat. Sb.* **51** (1960), 191–216. [Russian]
3. A. Matsumura and K. Nishihara, Asymptotics toward the rarefaction waves of the solutions of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **3** (1986), 1–13.
4. J. Goodman, Nonlinear asymptotic stability of viscous shock profile for conservation laws, *Arch. Rational Mech. Anal.* **95** (4), 325–344.
5. J. A. Smoller, "Shock Waves and Reaction–Diffusion Equations," Springer-Verlag, New York/Berlin, 1983.
6. T. P. Liu, Nonlinear stability of shock waves for viscous conservation laws, *Mem. Amer. Math. Soc.* **328** (1985), 1–108.
7. S. Kawashima and A. Matsumura, Asymptotic stability of travelling wave solutions of system for one-dimensional gas motion, *Comm. Math. Phys.* **101** (1985), 97–127.