Sensitivity Analyses of Parameters of a $M(t)/G/\infty$ Stochastic Service System

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ABSTRACT

Parameter sensitivity analyses were conducted on a $M(t)/G/\infty$ stochastic service system in which (1) the number of constants in an approximating nonhomogeneous Poisson process of inputs, (2) the mean of a Weibull c.d.f. of service time, and (3) the variance of the c.d.f. of service time were traded off in analyses of 24 cases for each of two fitting criteria: an L_1 metric implemented by a linear goal program, and an L_2 metric implemented by a multilinear least squares regression. The model goodness of fit and estimated total input to the system are both more sensitive to the mean service time than to its variance or to the number of constants in the approximating Poisson input. The fitting criteria give consistent results, but the L_2 criterion gives slightly higher estimates of total input to the system over a fixed period of time.

INTRODUCTION

An $M(t)/G/\infty$ service system was analyzed by Patterson [1], in which a method for approximating a nonhomogeneous Poisson arrival process with unknown intensity $\lambda(t)$ by the superposition of homogeneous Poisson processes was demonstrated. A linear regression approach to estimating the constant arrival intensities of the approximating Poisson process was demon-

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strated, and six cases were numerically analyzed using both an L_1 metric (linear goal program) and an L_2 metric (least squares regression). Additional sensitivity analyses were indicated, and the objective of this paper is to present the results of a systematic investigation of the sensitivity of that model to variation of input and service time parameters using the same data set as used in the earlier study [1].

A total of 24 distinct parameter combinations were analyzed, labeled cases 1–24. For each case the same two fitting criteria were used as in the previous study: (1) an L_1 metric implemented by the linear goal program IBM-PC/MP1-LPROG and (2) an L_2 metric implemented by the multilinear regression program MICHIGAN AMDAHL/MIDAS-REGRESS with both zero and nonzero intercepts. A total of 72 numerical fitting exercises were computed, 48 of which are presented. Linear regression models with nonzero intercepts gave inadequate fits to the time series, and those results are not shown.

DEFINITIONS OF CASES INVESTIGATED

Twenty-four cases were def d by distinct combinations of the number of coefficients defining the piecewise constant Poisson input intensity function $\lambda(t)$ (four cases), the mean of the c.d.f. of service (residence) time $B(z)$ (two cases), and the variance of $B(z)$ (three cases).

TABLE 1

EXPECTED NUMBER OF INDIVIDUALS IN A TYPICAL CELL AT TIME t

$$E[N^k(t)] = \begin{cases} 0 & (t \leqslant 0) \\ \lambda_1 \int_0^t [1 - B(x)]\,dx & (0 < t \leqslant t_1) \\ \lambda_1 \int_0^{t_1} [1 - B(t - x)]\,dx + \lambda_2 \int_0^{t - t_1} [1 - B(x)]\,dx & (t_1 < t \leqslant t_2) \\ \vdots & \vdots \\ \lambda_1 \int_0^{t_1} [1 - B(t - x)]\,dx + \lambda_2 \int_0^{t_2 - t_1} [1 - B(t - t_1 - x)]\,dx & \\ \quad + \cdots + \lambda_{k-1} \int_0^{t_{k-1} - t_{k-2}} [1 - B(t - t_{k-2} - x)]\,dx & \\ \quad + \lambda_k \int_0^{t - t_k} [1 - B(x)]\,dx & (t_{k-1} < t \leqslant t_k) \\ \lambda_1 \int_0^{t_1} [1 - B(t - x)]\,dx + \cdots + \lambda_k \int_0^{t_k - t_{k-1}} [1 - B(t - x)]\,dx & (t_k < t) \end{cases}$$

As developed by Patterson [1], a time dependent mean value function $E[N^k(t)]$ was fitted to each of 17 points in a time series showing concentrations of live larvae (*Dorosoma cepedanium*) in a typical 100 cubic meter cell of water in Western Lake Erie distributed over a 59 day sampling period April–June 1977. $E[N^k(t)]$ is the expected number of live individuals present in the larval life stage in a typical 100 cubic meter cell at a fixed point in time t within the period of abundance of the species. The random variable $N^k(t)$ (= number of individuals in a typical cell at time t) was shown by Patterson [1] to be Poisson distributed with expectation shown in Table 1.

Approximating the Unknown Input Intensity $\lambda(t)$

A piecewise constant input intensity approximating the actual but unknown intensity $\lambda(t)$,

$$\lambda(t) \doteq \begin{cases} 0 & (t<0), \\ \lambda_1 & (0<t \leqslant t_1), \\ \lambda_2 & (t_1<t \leqslant t_2), \\ \vdots & \\ \lambda_{k-1} & (t_{k-2}<t \leqslant t_{k-1}), \\ \lambda_k & (t_{k-1}<t \leqslant t_k) \end{cases} \tag{1}$$

was fitted to the time series, requiring that the subintervals defined in Equation (1) be specified in advance. The constants $\lambda_1,\ldots,\lambda_k$ were estimated by a linear regression model with zero intercept. The target species was assumed to be present on day 126 and was present in abundance on the last day ($t=189$) on which measurements were taken. Four cases of the index k defined in Equation (1) were selected: $k=1,2,3,6$ (see Table 2). When $k=1$, the Poisson arrival process is homogeneous; when $k>1$ it is nonhomogeneous.

Residence Time C.D.F. $B(z)$

A Weibull form for the c.d.f. $B(z)$ describing times of residence of individuals in the live larval state was assumed:

$$B(z)=\begin{cases} 0, & z<0, \\ 1-\exp[-(z/b)^c], & z \geqslant 0. \end{cases}$$

Two values of the mean of $B(z)$ were selected as well as three values of the

TABLE 2

ENDPOINTS FOR TIME SUBINTERVALS FOR PIECEWISE CONSTANT ARRIVAL INTENSITIES

No. of Constants in $\lambda(t)$	Endpoints of time subintervals for constant arrival intensities					
1	(126,189]					
2	(126,163]	(163,189]				
3	(126,141]	(141,163]	(163,189]			
6	(126,141]	(141,157]	(157,163]	(163,176]	(176,187]	(187,189]

variance (Table 3). The six cases in Table 3 when combined with the four cases of the index k in Table 2 give the twenty-four cases (Table 4) analyzed by both the L_1 and L_2 fitting criteria. Other cases were considered. A least squares regression model with nonzero intercept was employed to fit the same 24 cases to the time series, but the results were inferior on both biological and statistical grounds and are not reported here. In [1] a mean $B(z)$ of 50 days was used initially, but was reduced to 10 days. Considerations of the biology of the species indicated that residence times in the live larval state of 5 to 10 days are more realistic than 50 days.

Three cases for the variance of $B(z)$ were chosen in order to compare relative effects of errors in estimates of the mean versus the variance of $B(z)$ on estimated total larval production and on goodness of fit of the models. The models become conceptually more complex whenever $B(z)$ departs from the exponential case, i.e., whenever the coefficient c in the Weibull form is unequal to one. When $c \neq 1$ the stochastic process $[N^k(t)]$ denoting the

TABLE 3

PARAMETER COMBINATIONS OF $B(z)$ USED FOR MEAN AND VARIANCE OF RESIDENCE TIME OF INDIVIDUALS IN LIVE LARVAL STATE

Mean of $B(z)$, m (days)	Variance of $B(z)$, v (day^2)		
	$v = m$	$v = 2m$	$v = 8m$
5	$b = 5.000$ $c = 1.000$	$b = 4.311$ $c = 0.773$	$b = 2.712$ $c = 0.524$
10	$b = 10.000$ $c = 1.000$	$b = 8.622$ $c = 0.773$	$b = 5.424$ $c = 0.524$

TABLE 4

PARAMETER COMBINATIONS FOR TWENTY-FOUR CASES

Case	No. of time sub-intervals	Mean of $B(z)$, m	Variance of $B(z)$		
			$v = m$	$v = 2m$	$v = 8m$
1	6	10	10	—	—
2	6	5	5	—	—
3	3	10	10	—	—
4	3	5	5	—	—
5	2	10	10	—	—
6	2	5	5	—	—
7	1	10	10	—	—
8	1	5	5	—	—
9	6	10	—	20	—
10	6	5	—	10	—
11	3	10	—	20	—
12	3	5	—	10	—
13	2	10	—	20	—
14	2	5	—	10	—
15	1	10	—	20	—
16	1	5	—	10	—
17	6	10	—	—	80
18	6	5	—	—	40
19	3	10	-	—	80
20	3	5	—	—	40
21	2	10	—	—	80
22	2	5	—	—	40
23	1	10	—	—	80
24	1	5	—	—	40

number of individuals in the live larval state at time t is a one state semi-Markov process, whereas it is a Markov process when $c = 1$. Thus, an additional purpose of selecting two values of c unequal to unity was to investigate whether the more complex semi-Markov model of net abundance, plausible on biophysical grounds, obtained a more adequate fit to the time series than did the more simplistic Markov model. Thus a three way comparison of relative effects of (1) variations in the mean of $B(z)$, (2) variations in the variance of $B(z)$, and (3) variations in the permissible degree of nonhomogeneity of the fitted input intensity $\lambda(t)$ was conducted. Relative effects were examined in terms of (1) total seasonal production of larvae and (2) goodness of fit of the models to the time series using both L_1 and L_2 fitting criteria.

RESULTS AND DISCUSSION

Tables 2–4 show all assumptions of numerical values of parameters needed to fit the 24 cases. The resulting 48 fits are summarized in Table 5(a)–(c). Other outputs of the numerical analyses are not shown in separate tables, as they are accounted for in Table 5(a)–(c). Except for identification of the constants in Equation (1) which are significantly different from zero and those which are negative, all important comparisons, of which there are seven, can be made with data from Table 5(a)–(c).

The seven comparisons are:

(1) L_1 objective function values (col. 3) against the number of constants (col. 8) in the input intensity $\lambda(t)$;

(2) L_2 objective function values (col. 5) against the number of constants (col. 8) in the input intensity $\lambda(t)$;

(3) L_1 objective function values (col. 3) against coefficient of variation of $B(z)$ (col. 7);

(4) L_2 objective function values (col. 5) against coefficient of variation of $B(z)$ (col. 7);

(5) Estimated total production (col. 2) against coefficient of variation of $B(z)$ (col. 7);

(6) Estimated total production (col. 4) against coefficient of variation of $B(z)$ (col. 7);

(7) Estimated total production using L_2 criterion (col. 4) against estimated total production using L_1 criterion (col. 2).

Comparisons (1)–(2), (3)–(4), and (5)–(6) are matched, the purpose being to determine whether L_1 and L_2 fitting criteria give the same or at least consistent solutions. Comparison (7) shows whether the estimated total larval production is the same or different when L_1 and L_2 fitting criteria are used.

Comparisons (1) and (2) are consistent, and both show a rapid improvement in model fit from one to three subdivisions of the time axis. Only a slight improvement is shown when the number of subdivisions is increased from three to six, except for cases (2) and (4), in which a considerable improvement is demonstrated.

Comparisons (3) and (4) are consistent, and both show case (2) as providing the best-fitting model. The more complex semi-Markov process representation of the number of live larvae provides poorer model fits than the Markovian representation.

Comparisons (5) and (6) are consistent and are significant in terms of comparing relative effects of the mean and variance of $B(z)$ and the number of constants in the input intensity function. The important point from these comparisons is that the mean of $B(z)$ has (i) a much greater effect on the

TABLE 5

COMPARISONS OF EIGHT CASES

(1) Case	(2) Estimated total production, optimized L_1 fit	(3) Optimized objective function value L_1	(4) Estimated total production, optimized L_2 fit	(5) R^2, optimized L_2 fit	(6) No. of signif. input constants, L_2 fit	(7) Coeff. of Variation, $B(z)$	(8) No. of constants in $\lambda(t)$
			(a) $v = m$				
1	1886	1936	1548	0.764	2 out of 6	0.32	6
2	3512	1697	3119	0.782	3 out of 6	0.45	6
3	1671	1994	1638	0.693	1 out of 3	0.32	3
4	3144	2214	3615	0.584	2 out of 3	0.45	3
5	2177	2806	2385	0.536	1 out of 2	0.32	2
6	4635	3478	4634	0.345	1 out of 2	0.45	2
7	1487	4073	2179	0.166	1 out of 1	0.32	1
8	2955	4217	4106	0.183	1 out of 1	0.45	1
			(b) $v = 2m$				
9	1764	1944	1544	0.730	2 out of 6	0.45	6
10	3614	1740	3135	0.788	3 out of 6	0.63	6
11	1888	2090	1686	0.705	1 out of 3	0.45	3
12	3126	2046	3577	0.625	1 out of 3	0.63	3
13	2351	2843	2408	0.563	1 out of 2	0.45	2
14	4466	3202	4713	0.411	1 out of 2	0.63	2
15	1528	4084	2248	0.133	1 out of 1	0.45	1
16	2954	4173	4184	0.174	1 out of 1	0.63	1
			(c) $v = 3m$				
17	1723	2298	1790	0.780	2 out of 6	0.89	6
18	3962	1949	3298	0.788	2 out of 6	1.26	6
19	2081	2318	1919	0.717	1 out of 3	0.89	3
20	3545	2036	3747	0.669	1 out of 3	1.26	3
21	2858	3028	2644	0.590	1 out of 2	0.89	2
22	4730	3022	4988	0.494	1 out of 2	1.26	2
23	1753	4119	2566	0.086	1 out of 1	0.89	1
24	3121	4145	4510	0.128	1 out of 1	1.26	1

estimated total production than either of the other two factors and (ii) a substantial effect on the estimated total production.

Comparison (7) shows that when all 24 cases are considered, the estimated total production is slightly higher for the L_2 fitting criterion than for the L_1 criterion.

Column 6 shows the number of constants in the fitted input intensity which are significantly different from zero. In all cases where constants are not significantly different from zero, they represent time subintervals at the ends of the period of larval abundance. In some cases nonsignificant constants were positive, and in others they were negative.

Not shown in Table 5(a)–(c) are sensitivity analyses produced by the linear goal program. Since the time series measurements had coefficients of variation of up to 17 percent, it is important to assess the stability of the estimated parameters to variations in the means of the points in the time series. The sensitivity analysis produced by the linear program indicated that means can be varied, one at a time, up to plus or minus one standard error without altering the fitted constants in each case. As primal degeneracy was present in most cases, the results of the sensitivity analysis must be taken as tentative. Reruns of each case in which the means of points in the time series were varied would verify the sensitivity analysis or would give new estimates of constants of the input intensities. No dual degeneracy was present in the final tableaux of the linear programs, so that no question of multiple optimal estimates of input constants for each case arose. Thus, the estimates obtained from the L_1 criterion for every case were unique.

CONCLUSIONS

(1) The L_1 and L_2 fitting criteria, as implemented in this study, give consistent results.

(2) Linear goal programs and least squares multiple linear regressions complement each other in the information they provide for analyzing model fits. The linear program provides sensitivity analyses of variations in time series means, while the ANOVA table from least squares regression provides direction on revising subdivisions of the time axis for additional rounds of model fitting.

(3) The mean of the c.d.f. $B(z)$ exercises a greater effect on the estimated total larval production than does either its variance or the number of fitted constants in the input intensity $\lambda(t)$, provided that number is either three or six;

(4) A best fitted model was found from the cases analyzed, which was the conceptually simpler Markovian representation of case 2.

(5) A chi-square test of fit of any of the cases 1–24 to the time series is inconclusive, as many of the means in the series are not independent due to their proximity in time. Further model fitting should be preceded by reprocessing of the raw data into a revised time series in which data are grouped into time subintervals 130, 132; 153, 154; 161, 166; 172, 176, 186, 189. One point of the time series would represent all raw measurements collected within each time subinterval. The degree of dependence between points of the time series would then be minimized.

REFERENCE

1 R. L. Patterson, Regression estimates of inputs to $M(t)/G/\infty$ service systems, *J. Appl. Math. & Comp. Environ.*, 24:47–63 (1987).