ADIABATIC PHASES FROM AN EFFECTIVE ACTION

Dieter W. DÜSEDAU

Randall Laboratory of Physics, University of Michigan, Ann Arbor, MI 48109-1120, USA

Berry's phase is calculated as a term in the derivative expansion of the effective action of a spin-$\frac{1}{2}$ system in an external field.

A quantum mechanical system depending on an external parameter, taken slowly around a closed loop in parameter space exhibits in addition to the familiar dynamical phase variation a geometric phase factor [1]. In this short note we show that this phase factor occurs naturally in the derivative expansion of the effective action [2]: Berry's phase can be calculated by straightforward and direct evaluation of a functional determinant.

We shall concentrate on the ground state of a spin-$\frac{1}{2}$ system in an external magnetic field $\boldsymbol{B}$ with lagrangian [2,3]

$$L = \tfrac{1}{2}i(\xi^n \dot{\xi}^n + \epsilon_{mnj}\xi^m \xi^n B^j) , \tag{1}$$

where the ξ's are real anticommuting variables. To obtain the effective action $\Gamma(\boldsymbol{B})$ we have to evaluate the path integral

$$\exp[i\Gamma(\boldsymbol{B})] = \int [\mathrm{D}\xi^n]\exp\left(i \int \mathrm{d}t[\tfrac{1}{2}i(\xi^n \dot{\xi}^n + \epsilon_{mnj}\xi^m \xi^n B^j)] \right) , \tag{2}$$

which is easily done, since L is only quadratic in the ξ's. We learn that

$$\exp[i\Gamma(\boldsymbol{B})] = \det^{1/2}(\omega\delta_{mn} + i\epsilon_{mnj}B^j) , \tag{3}$$

or

$$\Gamma(\boldsymbol{B}) = -\tfrac{1}{2}i\mathrm{Tr}\ln(\omega\delta_{mn} + i\epsilon_{mnj}B^j) , \tag{4}$$

where ω is the energy operator and the trace Tr is understood in a functional sense. Gerbert [3] arrives at the same expression for $\Gamma(\boldsymbol{B})$, (3), only he goes on to evaluate Γ making strong use of symmetry principles. We shall instead evaluate (4) in a straightforward way. Since $\boldsymbol{B}(t)$ is time dependent, the operator $(\omega\delta_{mn} + i\epsilon_{mnj}B^j)$ is not diagonal in the energy representation, and (4) cannot be evaluated exactly. Instead we may expand the density of $\Gamma(\boldsymbol{B})$ in powers of $\dot{\boldsymbol{B}}$ [4,5]. Schematically, we may write

$$\Gamma(\boldsymbol{B}) = \int \mathrm{d}t[-V_{\mathrm{eff}}(\boldsymbol{B}) + A(\boldsymbol{B})\dot{\boldsymbol{B}} + \tfrac{1}{2}i\dot{B}^i M_{ij}\dot{B}^j + ...] . \tag{5}$$

Clearly $V_{\mathrm{eff}}(\boldsymbol{B})$ is just the effective potential and $A(\boldsymbol{B})$ Berry's connection. $V_{\mathrm{eff}}(\boldsymbol{B})$ can be evaluated by taking $\boldsymbol{B}(t) = \boldsymbol{B}_0$ to be a constant [5,6]. Then (4) reduces to an integral in momentum space:

$$\Gamma(\boldsymbol{B}_0) = -\tfrac{1}{2}i\,\mathrm{Tr}\ln(\omega\delta_{mn}+i\epsilon_{mnj}B_0^j) = -\tfrac{1}{2}i\int dt\int\frac{d\omega}{2\pi}\ln[\omega^3(1-B_0^2/\omega^2)]$$

$$=\tfrac{1}{2}\int dt\int\frac{d\omega_E}{2\pi}\ln(1+B_0^2/\omega_E^2) = \int dt\,\frac{B_0}{2}, \tag{6}$$

where we subtracted an infinite constant and made the substitution $\omega=i\omega_E$. Comparing (5) and (6) we learn that $V_{\mathrm{eff}}(\boldsymbol{B}) = -\tfrac{1}{2}B = -\tfrac{1}{2}\sqrt{\boldsymbol{B}^2}$, which is the standard ground state energy [#1].

To evaluate the higher order terms in (5) we employ a technique due to Fraser [5]; set $B^i(t)=B_0^i+\tilde{B}^i(t)$, where $\boldsymbol{B}_0$ is again a constant background, and expand (4) in powers of $\tilde{B}^i$ and $\dot{\tilde{B}}^i$. This expansion around $\boldsymbol{B}_0$ is then compared with the corresponding expansion of (5), and the coefficients A and M_{ij} are identified. Let us first expand (5) to second order in $\tilde{B}^i$:

$$\Delta=\Gamma(B_0^i+\tilde{B}^i)-\Gamma(B_0^i)$$

$$=\int dt\left(-\tilde{B}^i\frac{\partial V}{\partial B_0^i}-\tfrac{1}{2}\tilde{B}^i\tilde{B}^j\frac{\partial^2 V}{\partial B_0^i\partial B_0^j}+\dot{\tilde{B}}^iA_i(B_0)+\dot{\tilde{B}}^i\frac{\partial A_i}{\partial B_0^j}\tilde{B}^j+\tfrac{1}{2}\dot{\tilde{B}}^i\dot{\tilde{B}}^jM_{ij}(B_0)+...\right). \tag{7}$$

From (4) we also know that

$$\Delta=-\tfrac{1}{2}i\,\mathrm{Tr}\{\ln[\omega\delta_{mn}+i\epsilon_{mnj}(B_0^j+\tilde{B}^j)]-\ln(\omega\delta_{mn}+i\epsilon_{mnj}B_0^j)\}$$

$$=-\tfrac{1}{2}i\,\mathrm{Tr}\ln[\delta_{ml}+i\epsilon_{mnj}\tilde{B}^j(\omega\delta^{nl}-B_0^nB_0^l/\omega-i\epsilon^{nlk}B_0^k)/(\omega^2-B_0^2)]. \tag{8}$$

Expanding the logarithm in (8) generates a power series in $\tilde{B}^i$, which we later may compare with (7)

$$\Delta=-\tfrac{1}{2}i\,\mathrm{Tr}\left(\frac{\tilde{B}^j\epsilon_{mnj}\epsilon^{nmk}B_0^k}{\omega^2-B_0^2}+\tfrac{1}{2}\epsilon_{mnj}\tilde{B}^j\frac{\omega\delta^{nl}-B_0^nB_0^l/\omega-i\epsilon^{nlk}B_0^k}{\omega^2-B_0^2}\epsilon_{lpq}\tilde{B}^q\frac{\omega\delta^{pm}-B_0^pB_0^m/\omega-i\epsilon^{pmr}B_0^r}{\omega^2-B_0^2}\right). \tag{9}$$

Here, the term linear in $\tilde{B}^i$ is just the first derivative of the effective potential, and for the calculation of A irrelevant. It comes as no surprise to us that formula (9) misses the term $\dot{\tilde{B}}^iA_i(B_0)$, which is present in (7), since it is a total derivative. After some algebra and using the cyclicity of the trace the term quadratic in $\tilde{B}^i$ may be rewritten as

$$\Delta^{(2)}=-\tfrac{1}{2}i\,\mathrm{Tr}\left(-\tilde{B}^j\frac{\omega}{\omega^2-B_0^2}\tilde{B}^j\frac{\omega}{\omega^2-B_0^2}-\tilde{B}\cdot B_0\frac{\omega}{\omega^2-B_0^2}\tilde{B}\cdot B_0\frac{1/\omega}{\omega^2-B_0^2}+B_0^2\tilde{B}^j\frac{\omega}{\omega^2-B_0^2}\tilde{B}^j\frac{1/\omega}{\omega^2-B_0^2}\right.$$

$$\left.-i\epsilon_{jlk}B_0^k\tilde{B}^j\frac{\omega}{\omega^2-B_0^2}\tilde{B}^l\frac{1}{\omega^2-B_0^2}+i\epsilon_{lkj}B_0^2B_0^k\tilde{B}^j\frac{1/\omega}{\omega^2-B_0^2}\tilde{B}^l\frac{1}{\omega^2-B_0^2}-\tilde{B}\cdot B_0\frac{1}{\omega^2-B_0^2}\tilde{B}\cdot B_0\frac{1}{\omega^2-B_0^2}\right). \tag{10}$$

Next, we will have to evaluate the trace. We can use the identity [5]

$$\left[\tilde{B}^i,\frac{1}{\omega^2-m^2}\right]=\frac{1}{(\omega^2-m^2)^2}[\omega^2,\tilde{B}^i]+\frac{1}{(\omega^2-m^2)^3}[\omega^2,[\omega^2,\tilde{B}^i]]+... \tag{11}$$

to move all the operators ω to the right and all the $\tilde{B}^i$ to the left. Then the traces can be performed independently. The commutators needed can be evaluated to be

$$[\omega,\tilde{B}^i]=i\dot{\tilde{B}}^i,\quad[\omega^2,\tilde{B}^i]=\ddot{\tilde{B}}^i+2i\omega\dot{\tilde{B}}^i,\quad[\omega^2,[\omega,\tilde{B}^i]]=i\dddot{\tilde{B}}^i-2\omega\ddot{\tilde{B}}^i,\quad[\omega^2,[\omega^2,\tilde{B}^i]]=\ddddot{\tilde{B}}^i+4i\omega\dddot{\tilde{B}}^i-4\omega^2\ddot{\tilde{B}}^i, \tag{12}$$

To find Berry's connexion $A(\boldsymbol{B})$ we need only collect terms with one time derivative. Commuting all ω's to the right and collecting only the terms containing one time derivative we obtain

$$\Delta_1^{(2)} = -\tfrac{1}{2}i\mathrm{Tr}\left(-\frac{2i\omega}{(\omega^2-B_0^2)^2}(\mathring{B}\tilde{B}) + \frac{i/\omega}{\omega^2-B_0^2}(\mathring{B}\tilde{B}) - \frac{4i\omega}{(\omega^2-B_0^2)^3}(\mathring{B}B_0)(\tilde{B}B_0) \right.$$

$$\left. + \frac{1/\omega}{\omega^2-B_0^2}(\mathring{B}B_0)(\tilde{B}B_0) + \frac{2}{(\omega^2-B_0^2)^2}\epsilon_{jlk}\tilde{B}^j\mathring{B}^l B_0^k \right). \tag{13}$$

Note that the first four terms in (13) can be rewritten as a time derivative. Since the trace involves an integral over time we discard them. Then

$$\Delta_1^{(2)} = -\tfrac{1}{2}i\mathrm{Tr}\left(\frac{2}{(\omega^2-B_0^2)^2}\epsilon_{ijk}\tilde{B}^i\mathring{B}^j B_0^k \right) = -\tfrac{1}{2}i\int\frac{d\omega}{2\pi}\frac{2}{(\omega^2-B_0^2)^2}\int dt\,\epsilon_{ijk}\tilde{B}^i\mathring{B}^j B_0^k = \int dt\,\tfrac{1}{4}\epsilon_{ijk}\frac{B_0^k}{B_0^3}\mathring{B}^j\tilde{B}^i. \tag{14}$$

Our final expression (14) must now be compared with the corresponding term in (7). From there we obtain

$$\Delta_1^{(2)} = \int dt\,\mathring{B}^i\frac{\partial A_i}{\partial B^j}\tilde{B}^j = \tfrac{1}{2}\int dt\left(\frac{\partial A_i}{\partial B_j} - \frac{\partial A_j}{\partial B_i}\right)\mathring{B}^i\tilde{B}^j, \tag{15}$$

where the second line follows by partial integration.

Equating (14) and (15) we finally learn that

$$\tfrac{1}{2}\int dt\left(\frac{\partial A_j}{\partial B_i} - \frac{\partial A_i}{\partial B_j}\right)\mathring{B}^j\tilde{B}^i = \int dt\,\tfrac{1}{4}\epsilon_{ijk}\frac{B_0^k}{B_0^3}\mathring{B}^j\tilde{B}^i \tag{16}$$

or

$$\left(\frac{\partial A_j}{\partial B_i} - \frac{\partial A_i}{\partial B_j}\right) = \tfrac{1}{2}\epsilon_{ijk}\frac{B_0^k}{B_0^3}. \tag{17}$$

From (17) we see immediately that $\nabla_B\times A = \tfrac{1}{2}B/B^3$, where A is the potential of a magnetic monopole with strength $\tfrac{1}{2}$. The curvature $\nabla_B\times A$ is just the field of a magnetic monopole, as Berry's analysis [1] has shown. Hence, this method can indeed be used to evaluate the adiabatic phase directly.

The higher order terms in (5) can be obtained by the same methods, one only has to collect terms with more time derivatives. Since there are no matters of principle involved and they are calculated elsewhere [3], we omit the calculation here.

References

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