ON THE $K$-THEORY OF CURVES OVER FINITE FIELDS

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Let $X$ be a smooth projective curve over a finite field. The main result is that the odd-dimensional $K$-theory of the extension of $X$ to the algebraic closure is the sum of two copies of the $K$-theory of the field. Two plausible conjectures are advanced which would suffice to compute the $K$-theory of $X$ itself. These provisional computations are then related to the $L$-functions of $X$.

Introduction

Let $X$ be a smooth projective curve over a finite field $F$. Let $J$ denote the Jacobian variety of $X$. Write $F$ for the algebraic closure of $F$ and $X = X \times_F \bar{F}$. The main result of this paper is

**Theorem 1.** If $n \geq 0$, then $K_{2n+1}(X) = K_{2n+1}(\bar{F}) \oplus K_{2n+1}(\bar{F})$.

The theorem is proved in the first section. The techniques used are a shameless exploitation of the work of Quillen, Soulé, and Suslin. Quillen’s results on higher $K$-theory [9] and finite fields [8] form a solid and indispensible foundation. Soulé’s study [11] of the $K$-theory of varieties over finite fields and Suslin’s study [12] of the torsion in higher $K$-theory raise on this foundation an imposing edifice, from which vantage point the way to Theorem 1 can be clearly seen.

The remainder of the paper is built on a more conjectural foundation. The computation of the $K$-theory of $X$ is reduced to certain properties of function fields.

**Conjecture A.** Let $Y$ be a geometrically integral variety over a field $k$. Let $G = \text{Gal}(\bar{k}/k)$. Then

$$
(K_i(\bar{k}(Y))/K_i(\bar{k}))^G \approx K_i(k(Y))/K_i(k).
$$

**Conjecture B.** The $K$-theoretic product yields an isomorphism

$$
\tilde{F}(X)^* \otimes K_{2n-1}(\bar{F}) \to K_{2n}(\bar{F}(X)).
$$
Conjecture A was known ‘classically’ for \( i = 0, 1 \) and has been proven for \( i = 2 \) by Colliot–Thélène [2] and Suslin [13].

Conjecture B was proven for \( n = 1 \) by Tate [14] and generalized by Suslin [13]. Furthermore, it will be seen that Conjecture B is equivalent to an isomorphism

\[
K_{2n}(\bar{X}) \cong \text{Tor}(J(\bar{F}), K_{2n-1}(\bar{F})).
\]

The main consequence of these conjectures is

**Theorem 2.** If both Conjectures A and B hold, then

(i) \( K_{2n+1}(X) \cong K_{2n+1}(F) \oplus K_{2n+1}(F) \);

(ii) \( K_{2n}(X) \cong \text{Tor}(J(\bar{F}), K_{2n-1}(\bar{F}))^G \).

Theorem 1 and Theorem 2(i) will be proven in the first section of the paper. Theorem 2(ii) will be proven in the second section. The second section concludes with a study of the relationship between the \( K \)-theory of \( X \) and its \( L \)-function. The groups computed by Theorem 2 have the orders predicted by the Quillen–Lichtenbaum conjectures [6, 10].

**Notation.** Let \( A \) be an abelian group, \( n \) a positive integer, \( l \) a prime number. Write

- \( \frac{1}{n}A = n \)-torsion subgroup,
- \( A_{\text{tor}} \) = subgroup of all torsion elements,
- \( A(l) \) = \( l \)-primary torsion subgroup.

1. **Odd-dimensional \( K \)-groups**

Let \( X \) be a smooth projective geometrically integral curve over a field \( F \). Quillen [9] has constructed an exact localization sequence

\[
\cdots \rightarrow K_n(X) \rightarrow K_n(F(X)) \rightarrow \bigsqcup_{x \in X} K_{n-1}(F(x)) \rightarrow K_{n-1}(X) \rightarrow \cdots
\]

where \( F(X) \) is the function field of \( X \) and, for each closed point \( x \), \( F(x) \) is its residue field.

Let \( \mathcal{H}_n \) denote the Zariski sheaf on \( X \) associated to the presheaf \( U \rightarrow K_n(U) \). Then Quillen [9] has also constructed an acyclic resolution

\[
0 \rightarrow \mathcal{H}_n \rightarrow \eta_* K_n(F(X)) \rightarrow \bigsqcup_x i_* K_{n-1}(F(x)) \rightarrow 0.
\]

Consequently, the localization sequence can be decomposed into two different flavors of shorter exact sequences:
\[(A_n) \quad 0 \rightarrow \Gamma(X, \mathcal{H}_n) \rightarrow K_n(F(X)) \rightarrow \prod K_{n-1}(F(x)) \rightarrow H^1(X, \mathcal{H}_n) \rightarrow 0 ,\]
\[(B_n) \quad 0 \rightarrow H^1(X, \mathcal{H}_{n+1}) \rightarrow K_n(X) \rightarrow \Gamma(X, \mathcal{H}_n) \rightarrow 0 .\]

For the remainder of this section, let \( F \) be a finite field with algebraic closure \( \overline{F} \). Write \( \overline{X} \) for the curve \( X \times_{\text{Spec } F} \text{Spec } \overline{F} \) obtained by base extension.

As a consequence of the work of Quillen and Soulé, sequences \( (A_n) \) and \( (B_n) \) simplify considerably. There is, however, a difference depending on whether \( y \) is odd or even. Let \( n \geq 1 \). Quillen’s proof \({\cite{8}}\) that \( K^n(F) = 0 \) implies
\[(A_{2n+1}) \quad \Gamma(X, \mathcal{H}_{2n+1}) = K_{2n+1}(F(X)) .\]

Now Soulé \({\cite{11, Proposition 3}}\) has shown that \( H^i(X, \mathcal{H}_{n+1}) \approx K_n(F) \). Combined with \( (A_{2n+1}) \), this yields
\[(B_{2n+1}) \quad 0 \rightarrow K_{2n+1}(F) \rightarrow K_{2n+1}(X) \rightarrow K_{2n+1}(F(X)) \rightarrow 0 .\]

Using Soulé’s result in the case of the even-dimensional \( K \)-groups, one has
\[(B_{2n}) \quad K_{2n}(X) = \Gamma(X, \mathcal{H}_{2n}) .\]

Therefore
\[(A_{2n}) \quad 0 \rightarrow K_{2n}(X) \rightarrow K_{2n}(F(X)) \rightarrow \prod K_{2n-1}(F(x)) \rightarrow K_{2n-1}(F) \rightarrow 0 .\]

By passing to the direct limit over finite extensions of \( F \), one obtains the same sequences for \( \overline{X} \) over \( \overline{F} \). For the sake of completeness, note also that
\[
\begin{align*}
K_0(X) &= \mathbb{Z} \oplus \text{Pic}(X) , \\
K_1(X) &= F^* \oplus F^* .
\end{align*}
\]

**Lemma 1.1.** Let \( X \) be a curve over either a finite field or its algebraic closure. Let \( E \) be the function field of \( X \). If \( n \geq 2 \), then \( K_n(X) \) and \( K_n(E) \) are torsion groups.

**Proof.** Harder \({\cite{5}}\) showed that \( K_n(X) \) is finite for \( X \) defined over a finite field and \( n \geq 1 \). The result follows for \( X \) over the algebraic closure by passage to the direct limit. Finally, the result follows for function fields from sequences \( (A_{2n}) \) and \( (B_{2n+1}) \). \( \Box \)

**Proposition 1.2.** Let \( X \) be a smooth projective curve over an algebraically closed field \( L \). Let \( E = L(X) \) be the algebraic closure of its function field. Then there is an injection
\[
\Gamma(X_L, \mathcal{H}_n)_{\text{tor}} \hookrightarrow \Gamma(X_{\overline{F}}, \mathcal{H}_n)_{\text{tor}} .
\]

**Proof.** Write \( E = \lim \rightarrow A \) where \( A \supset L(X) \) is a finite algebraic extension field. Since the exact sequences \( (A_n) \) and \( (B_n) \) are stable under base change and \( K \)-theory
commutes with direct limits, it suffices to show that

\[ \Gamma(X_L, \mathcal{H}_n)_{\text{tor}} \leftrightarrow \Gamma(X_A, \mathcal{H}_n)_{\text{tor}}. \]

For $K$-theory with any finite coefficients, Suslin [12] has constructed specialization maps

\[ K_n(X_A; \mathbb{Z}/r) \rightarrow K_n(X_L; \mathbb{Z}/r) \]

which split off the $K$-theory of $X_L$. In particular,

\[ K_n(X_L)_{\text{tor}} \leftrightarrow K_n(X_A)_{\text{tor}}. \]

Since the sequences $(B_n)$ split compatibly with base change [11], the result follows. \( \square \)

**Theorem 1.3.** Let $X$ be a smooth projective curve over a finite field $F$. Then

(i) $K_{2n+1}(\bar{X}) = K_{2n+1}(\bar{F}) \oplus K_{2n+1}(\bar{F})$;

(ii) $K_{2n+1}(\bar{F}(X)) = K_{2n+1}(\bar{F})$.

**Proof.** The two parts are equivalent by sequence $(B_{2n+1})$. Write $E$ for the algebraic closure of the function field $\bar{F}(X)$. There is a commutative diagram

\[
\begin{array}{ccc}
K_{2n+1}(\bar{F}) & \rightarrow & K_{2n+1}(\bar{F}(X)) \\
\downarrow \alpha & & \downarrow \gamma \\
K_{2n+1}(E) & \rightarrow & \Gamma(X_F, \mathcal{H}_{2n+1}) \\
\end{array}
\]

where the isomorphism comes from $(A_{2n+1})$.

By Suslin [12], the composite $\beta \alpha$ is an isomorphism on torsion. Since $K_{2n+1}(\bar{F})$ is all torsion, $\alpha$ is an injection. By Proposition 1.2, the composite $\gamma \beta$ is also an injection on torsion. When $n \geq 1$, Lemma 1.1 shows that all of $K_{2n+1}(E)$ is torsion. So, $\beta$ is an injection. Finally, since $\beta \alpha$ is also surjective on torsion, $\alpha$ must be surjective on torsion and hence surjective. Since the case $n = 0$ has been noted earlier, the theorem is proved. \( \square \)

**Corollary 1.4.** Let $\Lambda$ be a smooth affine curve over a finite field $F$. If $n \geq 1$, then

\[ K_{2n+1}(\bar{\Lambda}) = K_{2n+1}(\bar{F}). \]

**Proof.** Let $X$ be a smooth completion. The corollary follows from the localization sequence
Corollary 1.5. Assume Conjecture A holds when \( i = 2n + 1 \), \( Y = X \) is a curve, and \( k = F \) is a finite field. Then

\[
K_{2n+1}(X) \cong K_{2n+1}(F) \oplus K_{2n+1}(F)
\]

Proof. Using exact sequence \((B_{2n+1})\), one sees that the corollary is equivalent to showing that there is an isomorphism \( K_{2n+1}(F(X)) \cong K_{2n+1}(F) \). By Theorem 1.3(ii), this isomorphism holds over \( E \). The conjecture then implies that it also holds over \( F \).

Remark. (i) Conjecture A is trivially true when \( i = 0 \), since both sides are zero. If \( i = 1 \), the conjecture follows from \( K_1(F) = F^* \) and Hilbert's Theorem 90. The conjecture has been proven for \( i = 2 \) by Colliot-Thélène [2] under mild hypotheses and in general by Suslin [13]. Their proofs rely on the Merkurjev and Suslin [7] version of Hilbert's Theorem 90 for \( K_2 \). Conjecture A should, therefore, be related to a generalization of Hilbert's Theorem 90 to higher K-theory.

(ii) The conjecture must be made in the context of a quotient of K-groups. In general, the map \( K_n(F) \to K_n(E)^G \) is not an isomorphism. For example, take \( n = 2 \), \( F = \mathbb{Q} \), \( E = \mathbb{Q}(i) \). Then \( \{-1, -1\} \) is a nontrivial element of the kernel.

(iii) Corollary 1.5 would also follow if the conjecture were only known for the torsion subgroup of \( K_i \). It is likely that it would suffice equally well to prove the conjecture for \( K \)-theory with finite coefficients.

2. Even dimensional K-groups

In this section, we study the even-dimensional \( K \)-groups of a curve \( X \) over a finite field. First, Conjecture B will be used to compute the \( K \)-theory of \( \bar{X} \). Then Conjecture A will be used to descend to \( X \). Finally, the orders of these \( K \)-groups will be compared with special values of the \( L \)-functions of \( X \). Throughout the section, write \( Y = \bar{X} \) and \( k = \bar{F} \).

Proposition 2.1. Assume Conjecture B holds. Then

\[
K_{2n}(Y) \cong \text{Tor}(J(k), K_{2n-1}(k))
\]
Proof. To use the multiplicative structure of \(K\)-theory, it is necessary to study the effect on

\[
(A_1) \quad 0 \to k^* \to k(Y)^* \to \prod \mathbb{Z} \to \text{Pic}(Y) \to 0
\]

of tensoring with \(K_{2n-1}(k)\). There is also an exact sequence

\[(*) \quad 0 \to J(k) \to \text{Pic}(Y) \to \mathbb{Z} \to 0\]

where \(J\) is the Jacobian variety of \(Y\). All the groups \(k^*, J(k), \text{and } K_{2n-1}(k)\) are divisible torsion groups. Therefore tensoring (*) with \(K_{2n-1}(k)\) yields the pair of isomorphisms:

\[
\text{Pic}(Y) \otimes K_{2n-1}(k) \cong K_{2n-1}(k),
\]

\[
\text{Tor}(J(k), K_{2n-1}(k)) \cong \text{Tor}(\text{Pic}(Y), K_{2n-1}(k)).
\]

Next, introduce the group \(D(Y) = k(Y)^*/k^*\) of principal divisors on \(Y\). Then

\[
k(Y)^* \otimes K_{2n-1}(k) \cong D(Y) \otimes K_{2n-1}(k).
\]

Finally, tensoring the short exact sequence

\[
0 \to D(Y) \to \prod \mathbb{Z} \to \text{Pic}(Y) \to 0
\]

with \(K_{2n-1}(k)\) and using the above isomorphisms, one obtains a four-term exact sequence which can be compared with \((A_{2n})\):

\[
0 \to \text{Tor}(J(k), K_{2n-1}(k)) \to k(Y)^* \otimes K_{2n-1}(k) \to \prod K_{2n-1}(k) \to K_{2n-1}(k) \to 0
\]

\[
0 \to K_{2n}(Y) \to K_{2n}(k(Y)) \to \prod K_{2n-1}(k(y)) \to K_{2n-1}(k) \to 0
\]

The proposition follows. \(\square\)

Remarks. (i) When \(n = 1\), Tate [14] has proven Conjecture B. Many of the consequences to be drawn from this conjecture are based on his arguments in [14].

(ii) Since Quillen [8] has shown that \(K_{2n-1}(k)(l) \cong \mathbb{Q}_l/\mathbb{Z}_l(n)\), one might look for a version of this conjecture computing torsion over any algebraically closed field \(k\). Suslin [13] has proven such a generalization of Tate’s result when \(n = 1\).

Theorem 2.2. If both Conjectures A and B hold, then

\[
K_{2n}(X) \cong \text{Tor}(J(\overline{F}), K_{2n-1}(\overline{F}))^G.
\]
Proof. Using Proposition 2.1, it suffices to show that $K_{2n}(X) = K_{2n}(\bar{X})^G$. Since $K_{2n}F = K_{2n}\bar{F} = 0$, it follows from Conjecture A that $K_{2n}(F(X)) = H^0(G, K_{2n}(\bar{F}(X)))$.

Write $C = K_{2n}(\bar{F}(X))/K_{2n}(\bar{X})$. There are exact sequences

$$0 \to H^0(G, K_{2n}(\bar{X})) \to K_{2n}(F(X)) \to H^0(G, C) \to H^1(G, K_{2n}(\bar{X})) \to \cdots$$

and

$$0 \to H^0(G, C) \to \prod_x K_{2n-1}F(x) \to K_{2n-1}F \to 0$$

obtained by taking the Galois cohomology of $(A_{2n})$ over $\bar{F}$. Then the result follows by comparing these sequences with $(A_{2n})$ over $F$. □

In order to get a more complete description of $K_{2n}(X)$, it is useful to keep track of the action of $G = \text{Gal}(\bar{F}/F)$. Recall [3, 6, 11, 14] the definitions of the standard $l$-adic Galois modules

$$T_l = \mathbb{Q}_l/F(l),$$

$$V_l = T_l \otimes \mathbb{Q}_l/\mathbb{Z}_l = \lim_{\to} \mu_l(\bar{F}).$$

$$Z_l(1) = \lim_{\to} \mu_{p^n}(\bar{F}),$$

$$Z_l(n) = Z_l(1) \otimes \cdots \otimes Z_l(1) \ (n \text{ copies}).$$

For any $l$-adic Galois module $M$, let $M(n) = M \otimes Z_l(n)$. Also, write $W_l = \mathbb{Q}_l/\mathbb{Z}_l$ so that

$$W_l(1) = \mathbb{Q}_l/\mathbb{Z}_l(1) = \bar{F}^*(l).$$

Lemma 2.3. Tor($J(\bar{F}), K_{2n-1}(\bar{F}) \{l\} = V_l(n)$.

Proof. Quillen's computation [8] of the $K$-theory of finite fields says that $K_{2n-1}(\bar{F})\{l\} = W_l(n)$. Since Tor($J(\bar{F}), W_l) = V_l$, the result follows. □

Proposition 2.4. Assume both Conjectures A and B hold. Let $X$ be a smooth projective curve over a finite field $F$ with $q = p^r$ elements. Let $l \neq p$ be prime. Let $f$ denote the Frobenius endomorphism of $T_l$. Then

$$K_{2n}(X)\{l\} = T_l/(1 - fq^n) T_l.$$

Proof. There is a commutative diagram

$$
\begin{array}{ccc}
0 & \to & T_l \to T_l \otimes \mathbb{Q}_l \to V_l \to 0 \\
\downarrow{1-fq^n} & & \downarrow{1-fq^n} \\
0 & \to & T_l \otimes \mathbb{Q}_l \to V_l \to 0
\end{array}
$$
By Deligne's proof [3] of the Weil conjectures, the middle vertical arrow is an isomorphism. So

\[ T_i/(1 - fq^n)T_i = \text{Ker}(1 - fq^n : V_i \rightarrow V_i) \]
\[ = \text{Ker}(1 - \text{Frob} : V_i(n) \rightarrow V_i(n)) \]
\[ = H^0(G, V_i(n)) = H^0(G, \text{Tor}(J, K_{2n-1})) \]
\[ = H^0(G, K_{2n}(\tilde{X}')(l)) \]
\[ = K_{2n}(\tilde{X})(l). \]

Let \( \phi \) be the action of the geometric Frobenius [3] on \( H^i(\tilde{X}, \mathbb{Q}_l) \). The \( L \)-function of \( X \) is defined as

\[ L(X, s) = P(X, q^{-s}) \]

where

\[ P(X, t) = \det(1 - \phi t). \]

For any pair of rational numbers \( a, b \), write \( a \sim b \) to mean \( a/b \) is a power of \( p \).

**Corollary 2.5.** Assume both Conjectures A and B hold. Then \( L(X, n + 1) \sim \#K_{2n}(X) \).

**Proof.** By the functional equation, \( L(X, n + 1) \sim L(X, -n) \). It is a standard fact that

\[ P(X, t) = \det(1 - ft|T_i). \]

So, \( L(X, -n) = \det(1 -fq^n|T_i) \). The \( l \)-part of the \( L \)-function is therefore given by

\[ \#T_i/(1 - q^n) = \#K_{2n}(X)(l). \]

**Remark.** This is precisely the relation between \( K \)-theory and \( L \)-functions predicted by the Quillen–Lichtenbaum conjectures [6, 10].

**References**


