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On the Structure of Contraction Operators. I*

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1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex, Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. In the last eight years considerable progress has been made in the structure theory of operators in $\mathscr{L}(\mathscr{H})$. The starting point for this development was the pioneering paper [7] which solved the invariant subspace problem for subnormal operators and, more importantly, introduced into operator theory the fruitful concept of a dual algebra of operators.

The study of dual algebras was then taken up by many authors, and the subsequent results have contributed substantially to our knowledge of invariant subspaces, dilation theory, and reflexivity of operators. (For an in-depth development of the theory of dual algebras and a comprehensive bibiliography as of 1984, see [5].) In particular, attention has been focused in the last three years on sufficient conditions that a contraction T in $\mathscr{L}(\mathscr{H})$ belong to the class $\mathbb{A}_1$ appearing in the theory of dual algebras (definition reviewed below), and substantial contributions to this circle of ideas were made in [2–6, 11–13, 17, 18, 20]. In this paper we continue the study of sufficient conditions for membership in the class $\mathbb{A}_1$ (or, more precisely, one of the classes $\mathbb{A}_1(r)$), using improvements of techniques introduced in [2, 16, 17]. We first establish (Theorem 4.4) an abstract

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geometric sufficient condition for membership in some $\mathbb{A}_1(r)$ that is an analog of the sufficient condition [5, Proposition 6.1] for membership in $\mathbb{A}_{\aleph_0}$ (definition reviewed below). This gives us a powerful tool which we employ in Section 5 to considerably improve the known spectral sufficient conditions (from [13, 18]) for membership in some $\mathbb{A}_1(r)$. Moreover, Theorem 4.4 is sufficiently strong that we are able to utilize it, in conjunction with the new techniques of [8], to deduce, in the sequel [10] to this paper, the fact that every contraction T in $\mathscr{L}(\mathscr{H})$ whose spectrum $\sigma(T)$ contains the unit circle has nontrivial invariant subspaces. The main results of this paper were presented at the conference "Functional Analysis and its Applications," Nice, France, August 25–29, 1986.

2. Preliminaries on Dual Algebras

The notation and terminology herein agree with that in [5]. Nevertheless, for the reader's convenience, we begin by reviewing a few pertinent definitions. It is well known that $\mathscr{L}(\mathscr{H})$ is the dual space of the Banach space (and ideal) $\mathscr{C}_1(\mathscr{H})$ of trace-class operators on $\mathscr{H}$ equipped with the trace norm $\| \ \|_1$. This duality is implemented by the bilinear functional

$$\langle T, L\rangle = \operatorname{tr}(TL), \qquad T \in \mathscr{L}(\mathscr{H}), \quad L \in \mathscr{C}_1(\mathscr{H}).$$

A subalgebra $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ that contains $1_{\mathscr{H}}$ and is closed in the weak* topology on $\mathscr{L}(\mathscr{H})$ is called a dual algebra. It follows from general principles (cf. [9]) that if $\mathscr{A}$ is a dual algebra, or, more generally, any weak* closed subspace of $\mathscr{L}(\mathscr{H})$, then $\mathscr{A}$ can be identified with the dual space of $Q_{\mathscr{A}} = \mathscr{C}_1(\mathscr{H})/{}^{\perp}\mathscr{A}$, where ${}^{\perp}\mathscr{A}$ is the preannihilator of $\mathscr{A}$ in $\mathscr{C}_1(\mathscr{H})$, under the pairing

$$\langle T, [L]\rangle = \operatorname{tr}(TL), \qquad T \in \mathscr{A}, \quad [L] \in Q_{\mathscr{A}}.$$

(Here and throughout the paper we write $[L]$, or $[L]_{\mathscr{A}}$ when there is a possibility of confusion, for the coset in $Q_{\mathscr{A}}$ containing the operator L in $\mathscr{C}_1(\mathscr{H})$.) If x and y are vectors in $\mathscr{H}$, then the associated rank-one operator $x \otimes y$ belongs to $\mathscr{C}_1(\mathscr{H})$ and satisfies $\operatorname{tr}(x \otimes y) = (x, y)$. Thus if $\mathscr{A}$ is any weak* closed subspace of $\mathscr{L}(\mathscr{H})$, then $[x \otimes y] \in Q_{\mathscr{A}}$. As is well known, every operator L in $\mathscr{C}_1(\mathscr{H})$ can be written as $L = \sum_{i=1}^{\infty} x_i \otimes y_i$ for certain square-summable sequences $\{x_i\}$ and $\{y_i\}$ (with convergence in the norm $\| \ \|_1$), and it follows trivially that every element of $Q_{\mathscr{A}}$ has the form $[L] = \sum_{i=1}^{\infty} [x_i \otimes y_i]$. A weak* closed subspace $\mathscr{A}$ of $\mathscr{L}(\mathscr{H})$ is said to have property $(\mathbb{A}_1)$ if every element $[L]$ of $Q_{\mathscr{A}}$ is the coset

$$[L] = [x \otimes y] \tag{1}$$

of some rank-one operator, and to have property $(\mathbb{A}_1(r))$ (for some $r \geqslant 1$) if $\mathscr{A}$ has property $(\mathbb{A}_1)$ and if, in addition, for every $[L] \in Q_{\mathscr{A}}$ and $\varepsilon > 0$, vectors x and y satisfying (1) can be found which also satisfy $\|x\| \, \|y\| \leqslant (r+\varepsilon) \, \|[L]\|$. Although it is the properties $\mathbb{A}_1(r)$ that play the central role in this paper, the corresponding properties $\mathbb{A}_n(r)$ for n a cardinal number satisfying $2 \leqslant n \leqslant \aleph_0$ are worth mentioning. If n is such a cardinal number, and if for every doubly indexed family $\{[L_{ij}]\}_{0 \leqslant i,j < n}$ of elements of $Q_{\mathscr{A}}$ there exist sequences $\{x_i\}_{0 \leqslant i < n}$ and $\{y_i\}_{0 \leqslant i < n}$ of vectors from $\mathscr{H}$ such that

$$[L_{ij}] = [x_i \otimes y_j], \qquad 0 \leqslant i, j < n, \tag{2}$$

then $\mathscr{A}$ is said to have property $(\mathbb{A}_n)$. The properties $(\mathbb{A}_n(r))$ can be defined in a way analogous to the properties $(\mathbb{A}_1(r))$, and express the existence of some control on the norms of the vectors x_i and y_j appearing in (2) (cf. [5, p. 8]).

Let $\mathbb{N}$ be the set of positive integers, and let $\mathbb{D}$ be the open unit disc in $\mathbb{C}$. A set $\Lambda \subset \mathbb{D}$ is said to be dominating for $\mathbb{T} = \partial\mathbb{D}$ if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence of points from Λ. The spaces $L^p = L^p(\mathbb{T})$ and $H^p = H^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, are the usual Lebesgue and Hardy function spaces relative to normalized Lebesgue measure m on $\mathbb{T}$. Furthermore, $H_0^1 = H_0^1(\mathbb{T})$ denotes the subspace of H^1 consisting of those functions f whose analytic extension $\hat{f}$ to $\mathbb{D}$ satisfies $\hat{f}(0) = 0$. Moreover, if Σ is an arbitrary Borel subset of $\mathbb{T}$, we will need the (closed) subspace $L^p(\Sigma)$ of $L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, defined to be the set of (equivalence classes of) functions f in $L^p(\mathbb{T})$ such that $f = 0$ almost everywhere on $\mathbb{T} \setminus \Sigma$.

If T is a contraction in $\mathscr{L}(\mathscr{H})$, we denote by $\mathscr{A}_T$ the dual algebra generated by T, by $\mathscr{W}_T$ the closure of $\mathscr{A}_T$ in the weak operator topology (WOT), and by Q_T the predual $Q_{\mathscr{A}_T}$. If T is also absolutely continuous (i.e., if the maximal unitary direct summand of T is either absolutely continuous or acts on the space (0)), then one knows (cf. [5, Theorem 4.1]) that the Sz.-Nagy–Foias functional calculus Φ_T is a weak* continuous, norm-decreasing, algebra homomorphism of H^∞ onto a weak* dense subalgebra of $\mathscr{A}_T$, and in [4] we defined the class $\mathbb{A} = \mathbb{A}(\mathscr{H})$ to be the set of all absolutely continuous contractions T in $\mathscr{L}(\mathscr{H})$ for which Φ_T is an isometry of H^∞ onto $\mathscr{A}_T$. (In this case Φ_T is a weak* homeomorphism; see [9].) Furthermore, for each cardinal number n satisfying $1 \leqslant n \leqslant \aleph_0$, we defined the class $\mathbb{A}_n = \mathbb{A}_n(\mathscr{H})$ to be the set of all T in $\mathbb{A}$ for which $\mathscr{A}_T$ has property $(\mathbb{A}_n)$, and the class $\mathbb{A}_n(r)$ to be the set of all T in $\mathbb{A}$ for which $\mathscr{A}_T$ has property $(\mathbb{A}_n(r))$.

If $T \in \mathbb{A}$, then it follows easily from general principles that there exists an isometry φ_T from Q_T onto L^1/H_0^1 (the predual of H^∞) such that $\varphi_T^* = \Phi_T$. If $\lambda \in \mathbb{D}$ and we let P_λ denote the Poisson kernel function

$$P_\lambda(e^{it}) = (1 - |\lambda|^2) \, |1 - \bar{\lambda} e^{it}|^{-2}, \qquad e^{it} \in \mathbb{T},$$

in L^∞, then we write $[C_\lambda] = \varphi_T^{-1}([P_\lambda])$, and observe that

$$\langle f(T), [C_\lambda] \rangle = \langle \Phi_T(f), [C_\lambda] \rangle = \langle f, \varphi_T([C_\lambda]) \rangle$$
$$= \langle f, P_\lambda \rangle = f(\lambda), \qquad f \in H^\infty. \tag{3}$$

If $T \in \mathscr{L}(\mathscr{H})$, we write, as usual, $\sigma_e(T)$ for the essential spectrum of T. Furthermore, we write $(\mathscr{F})$ and $(\mathscr{S}\mathscr{F})$ for the classes of Fredholm and semi-Fredholm operators in $\mathscr{L}(\mathscr{H})$, and also $i(T)$ for the Fredholm index of an operator T in $(\mathscr{S}\mathscr{F})$. Recall that $C_{0\cdot} = C_{0\cdot}(\mathscr{H})$ is the class of all contractions T in $\mathscr{L}(\mathscr{H})$ such that the sequence $\{\|T^n x\|\}$ converges to zero for every x in $\mathscr{H}$, and that $C_{\cdot 0}$ and C_{00} are defined by $C_{\cdot 0} = (C_{0\cdot})^*$, $C_{00} = C_{0\cdot} \cap C_{\cdot 0}$.

3. The Minimal Coisometric Extension

To establish our results, it will be convenient to use the well-known fact that every contraction T in $\mathscr{L}(\mathscr{H})$ has a minimal coisometric extension $B = B_T$ that is unique up to unitary equivalence. This theorem is due to Sz.-Nagy and Foias (cf. [19]); a useful alternate geometric construction was given by Douglas [15] using ideas of de Branges and Rovnyak. In this section we set up some notation connecting T to B_T, and establish some lemmas that will be needed in the remainder of the paper.

Thus, let T be an arbitrary contraction in $\mathscr{L}(\mathscr{H})$. Without loss of generality (cf. [19, 15]), we may suppose that there exists a Hilbert space $\mathscr{K}$ and a coisometry $B = B_T$ in $\mathscr{L}(\mathscr{K})$ satisfying

$$\mathscr{K} \supset \mathscr{H}, \tag{4}$$

$$B\mathscr{H} \subset \mathscr{H}, \tag{5}$$

and

$$B \mid \mathscr{H} = T. \tag{6}$$

Furthermore, we may suppose B to be minimal, which means that for subspaces $\mathscr{M}$ of $\mathscr{K}$,

$$\{(\mathscr{H} \subset \mathscr{M} \subset \mathscr{K}) \wedge (B\mathscr{M} \subset \mathscr{M}) \wedge (B^*\mathscr{M} \subset \mathscr{M})\} \Rightarrow \mathscr{M} = \mathscr{K}. \tag{7}$$

In other words, the smallest reducing subspace $\mathscr{M}$ for B containing $\mathscr{H}$ is $\mathscr{K}$ itself. It is easy to see that (7) implies the following additional relation on subspaces $\mathscr{M}$:

$$\{(\mathscr{H} \subset \mathscr{M} \subset \mathscr{K}) \wedge (B\mathscr{M} \subset \mathscr{M}) \wedge (B \mid \mathscr{M} \text{ is a coisometry})\} \Rightarrow \mathscr{M} = \mathscr{K}. \tag{8}$$

This is because the left-hand side of (8) implies that $(B \mid \mathcal{M})^*$ is an isometry, and since B^* is also an isometry, it follows immediately that $\mathcal{M}$ is invariant for B^* and hence reducing for B, so (7) applies. Therefore, in the remainder of the paper, we employ

Convention 3.1. When a contraction T in $\mathcal{L}(\mathcal{H})$ and its minimal coisometric extension $B = B_T$ in $\mathcal{L}(\mathcal{K})$ are being discussed, we always assume that (4), (5), (6), (7), and (8) are valid.

Given such T and B, one knows that there exists a canonical decomposition of the isometry B^* as

$$B^* = S \oplus R^*, \tag{9}$$

corresponding to a decomposition of the space

$$\mathcal{K} = \mathcal{S} \oplus \mathcal{R}, \tag{10}$$

where, if $\mathcal{S} \neq (0)$, S is a unilateral shift operator (of some multiplicity) in $\mathcal{L}(\mathcal{S})$, and, if $\mathcal{R} \neq (0)$, R is a unitary operator in $\mathcal{L}(\mathcal{R})$. (Of course, either $\mathcal{S}$ or $\mathcal{R}$ may be (0)). This is the von Neumann decomposition of an isometry into its unitary and pure isometric parts. Concerning the relation between T and R, we will need the following easy lemma, which can be deduced from [19, p. 84] or as in [18].

LEMMA 3.2. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ with minimal coisometric extension B in $\mathcal{L}(\mathcal{K})$, and the subspace $\mathcal{R}$ of $\mathcal{K}$ in* (10) *is nonzero* (*i.e., B^* is not a pure isometry*), *then the unitary operator R in* (9) *is absolutely continuous.*

Notational Convention 3.3. In the remainder of the paper, given a contraction T in $\mathcal{L}(\mathcal{H})$ and its minimal coisometric extension B in $\mathcal{L}(\mathcal{K})$, the projection of $\mathcal{K}$ onto $\mathcal{S}$ will be denoted by Q and the projection of $\mathcal{K}$ onto $\mathcal{R}$ will be denoted by A, so $Q = 1_{\mathcal{K}} - A$ and every vector x in $\mathcal{K}$ may be written uniquely as

$$x = Qx + Ax = Qx \oplus Ax. \tag{11}$$

(Projections are not usually denoted by A, but there is a good reason for this choice of notation; see [15].) Moreover the projection of $\mathcal{K}$ onto the subspace $\mathcal{H}$ will be denoted by P.

The following lemma is a trivial consequence of (5), (6), (9), (11), and the fact that the polynomials are sequentially weak* dense in $H^\infty(\mathbb{T})$.

LEMMA 3.4. *If T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and B*

is its minimal coisometric extension in $\mathscr{L}(\mathscr{K})$, *then for every vector* x *in* $\mathscr{H}$, *and every function* h *in* $H^\infty(\mathbb{T})$,

$$h(T)\,x = h(B)\,x = h(S^*)(Qx) \oplus h(R)(Ax) = Q(h(T)\,x) \oplus A(h(T)\,x), \quad (12)$$

so

$$h(S^*)(Qx) = Q(h(T)\,x), \qquad h(R)(Ax) = A(h(T)\,x). \quad (13)$$

The next lemma is a tool for passing back and forth between the preduals Q_T and Q_B when $T \in \mathbb{A}$.

LEMMA 3.5. *Suppose* $T \in \mathbb{A}(\mathscr{H})$ *and has minimal coisometric extension* B *in* $\mathscr{L}(\mathscr{K})$. *Then* $B \in \mathbb{A}(\mathscr{K})$, $\Phi_T \circ \Phi_B^{-1}$ *is an isometry and weak* homeomorphism from* $\mathscr{A}_B$ *onto* $\mathscr{A}_T$, *and* $j = \varphi_B^{-1} \circ \varphi_T$ *is a linear isometry of* Q_T *onto* Q_B. *Moreover,*

$$j([C_\lambda]_T) = [C_\lambda]_B, \qquad \lambda \in \mathbb{D}, \quad (14)$$

and

$$j([x \otimes y]_T) = [x \otimes y]_B, \qquad x, y \in \mathscr{H}. \quad (15)$$

Proof. That $B \in \mathbb{A}(\mathscr{K})$ follows immediately from the fact that T is a part of B, and that $\Phi_T \circ \Phi_B^{-1}$ and j have the stated properties follows from results mentioned in Section 2. To establish (14), we compute, for any $\lambda \in \mathbb{D}$ and $h \in H^\infty$,

$$\begin{aligned}\langle h(B), j([C_\lambda]_T)\rangle &= \langle j^*(h(B)), [C_\lambda]_T\rangle = \langle (\Phi_T \circ \Phi_B^{-1})\, \Phi_B(h), [C_\lambda]_T\rangle \\ &= \langle h(T), [C_\lambda]_T\rangle = h(\lambda) = \langle h(B), [C_\lambda]_B\rangle.\end{aligned}$$

The verification of (15) is just as easy and is omitted.

We shall also need the following rules for operating in the preduals Q_T and Q_B.

LEMMA 3.6. *If* T *belongs to* $\mathbb{A}(\mathscr{H})$ *and has minimal coisometric extension* B *in* $\mathscr{L}(\mathscr{K})$, $x, y \in \mathscr{H}$, *and* $w, z \in \mathscr{K}$, *then*

$$\|[x \otimes y]_T\| = \|[x \otimes y]_B\|, \quad (16)$$

$$[x \otimes z]_B = [x \otimes Pz]_B, \quad (17)$$

and

$$[w \otimes z]_B = [Qw \otimes Qz]_B + [Aw \otimes Az]_B. \quad (18)$$

Proof. Relation (16) follows immediately from Lemma 3.5. Relation (17) follows from the computation

$$\langle h(B), [x \otimes z]_B \rangle = (h(B)\,x, z) = (h(B)\,x, Pz)$$
$$= \langle h(B), [x \otimes Pz]_B \rangle,$$

valid for every h in H^∞. One can establish (18) by a similar computation using (9).

The following easy "vanishing lemma" will be central to what follows.

LEMMA 3.7. *If T belongs to $\mathbb{A}(\mathscr{H})$, with minimal coisometric extension B in $\mathscr{L}(\mathscr{K})$, and $\{x_n\}_{n=1}^\infty$ is a sequence from $\mathscr{H}$ such that*

$$\|[x_n \otimes y]_T\| \to 0, \qquad \forall y \in \mathscr{H}, \tag{19}$$

then

$$\|[x_n \otimes z]_B\| \to 0, \qquad \forall z \in \mathscr{K}, \tag{20}$$

$$\|[Qx_n \otimes z]_B\| \to 0, \qquad \forall z \in \mathscr{K}, \tag{21}$$

and

$$\|[Ax_n \otimes z]_B\| \to 0, \qquad \forall z \in \mathscr{K}. \tag{22}$$

Proof. Relation (20) follows trivially from (19), (16), and (17). To establish (21), we compute

$$\|[Qx_n \otimes z]_B\| = \sup_{\substack{h \in H^\infty \\ \|h\| = 1}} |\langle (h(B), [Qx_n \otimes z]_B \rangle|$$
$$= \sup_{\substack{h \in H^\infty \\ \|h\| = 1}} |(h(B)\,x_n, Qz)| = \|[x_n \otimes Qz]_B\|,$$

which tends to zero by (20); (22) follows in the same way.

The following "vanishing lemma" is not so trivial.

LEMMA 3.8. *Suppose $T \in \mathbb{A}(\mathscr{H})$ and has B in $\mathscr{L}(\mathscr{K})$ for its minimal coisometric extension. If $\{z_n\}$ is any sequence in $\mathscr{K}$ that converges weakly to zero, then*

$$\|[w \otimes z_n]_B\| \to 0, \qquad \forall w \in \mathscr{S}. \tag{23}$$

Proof. If $\mathscr{S}=(0)$, then $w=0$ and the result is trivial. If $\mathscr{S}\neq(0)$, then, for every w in $\mathscr{S}$,

$$\begin{aligned}\|[w\otimes z_n]_B\| &= \sup_{\substack{h\in H^\infty\\ \|h\|=1}} |(h(B)\,w, z_n)| \\ &= \sup_{\substack{h\in H^\infty\\ \|h\|=1}} |(h(B)\,w, Qz_n)| = \sup_{\substack{h\in H^\infty\\ \|h\|=1}} |(h(S^*)\,w, Qz_n)| \\ &= \|[w\otimes Qz_n]_{S^*}\|,\end{aligned}$$

and this last quantity tends to zero as n gets large by [12, Proposition 2.7], since $S^*\in\mathbb{A}(\mathscr{S})\cap C_{0\cdot}$.

In the proof of Theorem 4.4 we will need a mechanism for solving certain equations in $L^1(\Sigma)$, where Σ is a Borel subset of $\mathbb{T}$. The procedure we use is a simplified version of similar procedures in [17] and [18]. In preparation for the use of this mechanism, we make some remarks about absolutely continuous unitary operators.

Suppose U is an absolutely continuous unitary operator in $\mathscr{L}(\mathscr{N})$ with spectral measure E_U, and let μ be a scalar spectral measure for U. (One may suppose that E_U and μ are defined on all Borel subsets of $\mathbb{T}$.) Then one knows, via the absolute continuity, that there exists a Borel set $\Sigma\subset\mathbb{T}$ such that μ is equivalent to Lebesgue measure $m|\Sigma$ (where this measure is defined to be zero on Borel subsets of $\mathbb{T}\setminus\Sigma$). For any vectors x and y in $\mathscr{N}$, let us denote by $\mu_{x,y}$ the complex measure on $\mathbb{T}$ defined by

$$\mu_{x,y}(\mathscr{B})=(E_U(\mathscr{B})\,x, y) \tag{24}$$

for every Borel subset $\mathscr{B}$ of $\mathbb{T}$. Obviously all of these complex measures $\mu_{x,y}$ are absolutely continuous with respect to the measure $m\mid\Sigma$. Therefore, for each pair $x, y\in\mathscr{N}$, there is a function in $L^1(\Sigma)$, which we denote by $x\cdot y$ or $x\overset{U}{\cdot}y$, that is the Radon–Nikodym derivative of $\mu_{x,y}$ with respect to $m\mid\Sigma$. We thus have, of course,

$$(l(U)\,x, y)=\int_{\mathbb{T}} l\,d\mu_{x,y}=\int_\Sigma l\{x\cdot y\}\,dm, \qquad l\in L^\infty(\Sigma). \tag{25}$$

It is obvious that the function $\langle x, y\rangle\to x\cdot y$ is sesquilinear; moreover the inequality $\|x\cdot y\|_1\leqslant\|x\|\,\|y\|$ follows from (25) by taking $l=\overline{x\cdot y}/|x\cdot y|$. Furthermore, if U can be written as a direct sum $U=U_1\oplus U_2$ relative to a spatial decomposition $\mathscr{N}=\mathscr{N}_1\oplus\mathscr{N}_2$, and x, y are any vectors in $\mathscr{N}$ with corresponding decompositions $x=x_1\oplus x_2$, $y=y_1\oplus y_2$, then $x_i\overset{U_i}{\cdot}y_i=x_i\overset{U}{\cdot}y_i$, $x_1\cdot y_2=x_2\cdot y_1=0$, and $x\overset{U}{\cdot}y=x_1\overset{U_1}{\cdot}y_1+x_2\overset{U_2}{\cdot}y_2$; these relations are all established by computations similar to this:

$$\int_\Sigma l(x \overset{U}{\cdot} y)\, dm = (l(U)\, x, y)$$

$$= (l(U_1)\, x_1, y_1) + (l(U_2)\, x_2, y_2)$$

$$= \int_\Sigma l\{(x_1 \overset{U_1}{\cdot} y_1) + (x_2 \overset{U_2}{\cdot} y_2)\}\, dm, \qquad l \in L^\infty(\Sigma).$$

Since, by definition, $L^1(\Sigma)$ is a subspace of $L^1(\mathbb{T})$, we may write $[l]$ for the equivalence class of l in the quotient space $(L^1/H_0^1)(\mathbb{T})$. A connection between this present discussion and the earlier part of this section is given by

LEMMA 3.9. *Suppose $T \in \mathbb{A}(\mathscr{H})$ and has $B = S^* \oplus R$ as its minimal coisometric extension, with $\mathscr{R} \neq (0)$. Then, for every pair of vectors $w, z \in \mathscr{R}$, we have*

$$[w \overset{R}{\cdot} z] = \varphi_B([w \otimes z]_B). \tag{26}$$

Proof. For an arbitrary h in $H^\infty(\mathbb{T})$, we have

$$\langle h, \varphi_B([w \otimes z]_B)\rangle = \langle \Phi_B(h), [w \otimes z]_B\rangle = (h(B)\, w, z)$$

$$= (h(R)\, w, z) = \int_{\mathbb{T}} h(w \overset{R}{\cdot} z)\, dm = \langle h, [w \overset{R}{\cdot} z]\rangle, \tag{27}$$

and the result follows from the fact that H^∞ is the dual space of L^1/H_0^1.

The following proposition is another step in the production of our equation solving technique in $L^1(\Sigma)$. If Σ is a Borel subset of $\mathbb{T}$, we denote by $H^2(\Sigma)$ the closure in $L^2(\Sigma)$ of the linear manifold of those functions that agree with some polynomial on Σ. Of course, if $m(\mathbb{T} \setminus \Sigma) \neq 0$, we have $H^2(\Sigma) = L^2(\Sigma)$.

PROPOSITION 3.10. *Suppose T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$, and $B = S^* \oplus R$ is its minimal coisometric extension in $\mathscr{L}(\mathscr{K})$, with $\mathscr{R} \neq (0)$. Then there exists a Borel set $\Sigma \subset \mathbb{T}$ such that $m \mid \Sigma$ is a scalar spectral measure for R, and $\mathscr{R}$ contains a reducing subspace $\mathscr{R}_0$ for R such that*

(a) *$R_0 = R \mid \mathscr{R}_0$ is unitarily equivalent to the operator $M_{e^{it}}$ of multiplication by the position function on $L^2(\Sigma)$, and*

(b) *if we denote by $\mathscr{R}_0^+$ the subspace of $\mathscr{R}_0$ corresponding to $H^2(\Sigma)$ under the unitary equivalence in* (a), *then $\mathscr{R}_0^+ \subset (A\mathscr{H})^-$.*

Proof. Since B is minimal and $\mathscr{R} \neq (0)$, we know that $(A\mathscr{H})^- \neq (0)$. Suppose first that $(A\mathscr{H})^- = \mathscr{R}$. Then, since R is absolutely continuous (Lemma 3.2), it follows easily from the theory of spectral multiplicity that there exists a reducing subspace $\mathscr{R}_0$ for R such that (a) is satisfied, and (b) is automatic. Suppose now that $(A\mathscr{H})^- \neq \mathscr{R}$. It follows from (13) that $(A\mathscr{H})^-$ is an invariant subspace for R, and, of course, $R \mid (A\mathscr{H})^-$ is an isometry. But if $R \mid (A\mathscr{H})^-$ were unitary, then $\mathscr{S} \oplus (A\mathscr{H})^-$ would be reducing for B, which is impossible by (8), since B is minimal and $(A\mathscr{H})^- \neq \mathscr{R}$. Thus $R \mid (A\mathscr{H})^-$ has an invariant subspace $\mathscr{R}_0^+ \subset (A\mathscr{H})^-$ on which $R \mid (A\mathscr{H})^-$ acts as a unilateral shift of multiplicity one. Hence $\mathscr{R}_0^+ \in \operatorname{Lat}(R)$, $\sigma(R) = \mathbb{T}$, and $R \mid \mathscr{R}_0^+$ is a unilateral shift of multiplicity one. Thus there must be a reducing subspace $\mathscr{R}_0 \supset \mathscr{R}_0^+$ for R such that $R \mid \mathscr{R}_0$ is a bilateral shift of multiplicity one, and (a) and (b) follow at once.

The following result is our "equation solving procedure" in $L^1(\Sigma)$, which is patterned after [17, Lemma 6] and [18, Lemma 3.8].

THEOREM 3.11. *Suppose T belongs to $\mathbb{A}(\mathscr{H})$ and has minimal coisometric extension $B = S^* \oplus R$ in $\mathscr{L}(\mathscr{K})$ with $\mathscr{R} \neq (0)$. Let $\Sigma \subset \mathbb{T}$ and $\mathscr{R}_0 \subset \mathscr{R}$ be as in Proposition* 3.10, *and denote the projection of $\mathscr{K}$ onto $\mathscr{R}_0$ by A_0. Let also ε and ρ be arbitrary real numbers such that $\varepsilon > 0$ and $0 < \rho < 1$. If $a_1 \in \mathscr{H}$, $b \in \mathscr{R}$, and $h \in L^1(\Sigma)$ are given, and we write $h_1 = (Aa_1 \overset{R}{\cdot} b) + h$, then there exist $u \in \mathscr{H}$ and $c \in \mathscr{R}$ such that*

$$\|h_1 - A(a_1 + u) \overset{R}{\cdot} c\|_1 < \varepsilon, \tag{28}$$

$$\|Qu\| < \varepsilon, \tag{29}$$

$$\|(A - A_0)\, u\| < \varepsilon, \tag{30}$$

$$\|u\| \leqslant 2\, \|h\|_1^{1/2}, \tag{31}$$

$$\|c\| \leqslant \frac{1}{\rho}\, \{\|b\| + \|h\|_1^{1/2}\}, \tag{32}$$

and

$$c - b \in \mathscr{R}_0, \tag{33}$$

where the notation $\| \ \|_1$ indicates the norm on $L^1(\Sigma)$.

Proof. For brevity we set $A_1 = A - A_0$, the projection of $\mathscr{K}$ onto $\mathscr{R} \ominus \mathscr{R}_0$. If $h = 0$ in $L^1(\Sigma)$, we take $u = 0$, $c = b$, and the theorem is proved. Hence we may suppose $h \neq 0$. Let $z \to \{z\}$ denote the Hilbert space isomorphism from $\mathscr{R}_0$ onto $L^2(\Sigma)$, given by Proposition 3.10, that implements a unitary equivalence between $R_0 = R \mid \mathscr{R}_0$ and $M_{e^{it}}$ on $L^2(\Sigma)$. Thus objects of the form $\{z\}$ will be square integrable functions on $\mathbb{T}$ that vanish off Σ. Moreover, a quantifying expression such as "$e^{it} \in \Sigma$" frequently means "almost everywhere on σ." The proof of the theorem really involves two cases: $\Sigma = \mathbb{T}$ and $m(\mathbb{T} \backslash \Sigma) \neq 0$. But when $m(\mathbb{T} \backslash \Sigma) \neq 0$, the H^2-functions to be constructed can be constructed trivially, since

$H^2(\Sigma)=L^2(\Sigma)$, so we concentrate on the case $\Sigma=\mathbb{T}$. Choose $\delta>0$ so small that

$$2\delta<\varepsilon, \qquad 4\delta(\|A_1 b\|+\|h\|_1^{1/2})<\varepsilon, \tag{34}$$

and

$$(1+\rho)(\|h\|_1^{1/2}(1+\delta)+2\delta)<2(\|h\|_1^{1/2}). \tag{35}$$

Since $|\int_{\mathbb{T}} \ln(|h|+\eta)\, dm|<+\infty$ for arbitrarily small $\eta>0$, one knows (cf. [16, p. 53]) that there are functions $\{y_1\}$ in $H^2(\mathbb{T})$ and $\{z_1\}$ in $L^2(\mathbb{T})$ such that

$$\{y_1\}\overline{\{z_1\}}=h, \qquad \|y_1\|\leqslant\|h\|_1^{1/2}(1+\delta), \qquad \|z_1\|\leqslant\|h\|_1^{1/2}, \tag{36}$$

and since $y_1\in\mathscr{R}_0^+\subset(A\mathscr{H})^-$, there exists $x_1\in\mathscr{H}$ such that

$$\|Ax_1-y_1\|<\delta.$$

Moreover, as is easily seen from (36) and the identity

$$x \overset{R}{\cdot} y=\{x\}\overline{\{y\}}, \qquad \forall x, y\in\mathscr{R}_0, \tag{37}$$

we have

$$(R^n y_1) \overset{R}{\cdot} (R^n z_1)=h, \qquad \forall n\in\mathbb{N},$$

and also, from (13),

$$\|R^n y_1-A(T^n x_1)\|=\|R^n(y_1-Ax_1)\|=\|y_1-Ax_1\|<\delta.$$

Since $\|Q(T^n x_1)\|=\|S^{*n}(Qx_1)\|\to 0$ as n tends to infinity, we may choose n_1 sufficiently large that

$$\|Q(T^{n_1}x_1)\|<\delta,$$

and define

$$x_2=T^{n_1}x_1, \qquad y_2=R^{n_1}y_1, \qquad z_2=R^{n_1}z_1.$$

Then

$$\begin{aligned} &y_2\in\mathscr{R}_0^+, \quad z_2\in\mathscr{R}_0, \quad \|Qx_2\|<\delta,\\ &y_2 \overset{R}{\cdot} z_2=h, \qquad \|y_2-Ax_2\|<\delta, \end{aligned} \tag{38}$$

$$\|y_2\|=\|y_1\|\leqslant\|h\|_1^{1/2}(1+\delta), \qquad \|z_2\|=\|z_1\|\leqslant\|h\|_1^{1/2}, \tag{39}$$

and

$$\|A_1 x_2\| = \|A_1(y_2 - Ax_2)\| < \delta, \tag{40}$$

since A_1 is the projection onto $\mathscr{R} \ominus \mathscr{R}_0$.

Let us write h_0 and h_2 for the L^1-functions

$$\begin{aligned} h_0 &= A_0 a_1 \overset{R}{\cdot} A_0 b + A_0 x_2 \overset{R}{\cdot} z_2, \\ h_2 &= A a_1 \overset{R}{\cdot} b + A_0 x_2 \overset{R}{\cdot} z_2 = h_0 + A_1 a_1 \overset{R}{\cdot} A_1 b. \end{aligned} \tag{41}$$

Using (38), (39), (40), and (34), we see that

$$\begin{aligned} \|h_1 - h_2\|_1 &= \|(A_0 x_2 - y_2) \overset{R}{\cdot} z_2\| \leqslant \|(A_0 x_2 - y_2)\| \; \|z_2\| \\ &\leqslant \{\|(A_0 - A)\, x_2\| + \|Ax_2 - y_2\|\} \; \|h\|_1^{1/2} \\ &\leqslant 2\delta \; \|h\|_1^{1/2} < \varepsilon/2. \end{aligned} \tag{42}$$

Recall that we are looking for $u \in \mathscr{H}$ and $c \in \mathscr{R}$ such that $\|h_1 - A(a_1 + u) \overset{R}{\cdot} c\|_1$ is small; (42) shows that it suffices to make $\|h_2 - A(a_1 + u) \overset{R}{\cdot} c\|$ small. We now decompose $\mathbb{T}$; define

$$E = \{e^{it} \in \mathbb{T} \colon |\{A_0 x_2\}(e^{it})| \geqslant |\{A_0 a_1\}(e^{it})|\}. \tag{43}$$

Of course E is only determined up to a set of measure zero, but this will cause no problems. One knows (cf., e.g., [16, p. 53]) that there is a function ψ_1 in $H^\infty(\mathbb{T})$ such that

$$|\psi_1(e^{it})| = \begin{cases} 1 + \rho, & e^{it} \in E, \\ 1 - \rho, & e^{it} \in \mathbb{T} \setminus E. \end{cases} \tag{44}$$

Choose a sufficiently large positive integer n_2 that

$$\|QT^{n_2} \psi_1(T)\, x_2\| = \|S^{*n_2} \psi_1(S^*)\, Qx_2\| < \delta,$$

and define $\psi_2(e^{it}) = e^{in_2 t} \psi_1(e^{it})$, $e^{it} \in \mathbb{T}$, and

$$u = \psi_2(T)\, x_2.$$

Then $\|Qu\| < \delta < \varepsilon$, so (29) is satisfied, and $|\psi_2|$ also satisfies (44). Thus,

$$\|u\| \leqslant \|\psi_2\|_\infty \; \|x_2\| \leqslant (1 + \rho)\; \|x_2\|, \tag{45}$$

and, using Proposition 3.10, we get

$$\{A_0 u\} = \{\psi_2(R_0)\, A_0 x_2\} = \psi_2 \{A_0 x_2\}. \tag{46}$$

Similarly,

$$\|A_1 u\| = \|\psi_2(R)\, A_1\, x_2\| < 2\delta < \varepsilon$$

from (44) and (40), so (30) is satisfied.

Now let us obtain some inequalities on the size of the function $\{A_0(a_1+u)\}$. For almost all e^{it} in $\mathbb{T}\setminus E$ we have, from (46), (44), and (43),

$$\begin{aligned} |\{A_0(a_1+u)\}(e^{it})| &\geqslant |\{A_0 a_1\}(e^{it})| - |\{A_0 u\}(e^{it})| \\ &\geqslant |\{A_0 a_1\}(e^{it})| - (1-\rho)|\,\{A_0 x_2\}(e^{it})| \qquad (47) \\ &\geqslant \rho\, |\{A_0 a_1\}(e^{it})| \geqslant \rho\, |\{A_0 x_2\}(e^{it})|, \end{aligned}$$

while a similar computation shows that for almost all e^{it} in E,

$$|\{A_0(a_1+u)\}(e^{it})| \geqslant \rho\, |\{A_0 x_2\}(e^{it})| \geqslant \rho\, |\{A_0 a_1\}(e^{it})|. \qquad (48)$$

We next define the complex-valued function f by

$$\overline{f(e^{it})} = h_0(e^{it})/\{A_0(a_1+u)\}(e^{it}) \qquad (49)$$

whenever $\{A_0(a_1+u)\}(e^{it}) \neq 0$ and $f(e^{it})=0$ whenever $\{A_0(a_1+u)\}(e^{it})=0$. It follows easily from (41), (46), (47), (48), and (49) that

$$|f(e^{it})| \leqslant \frac{1}{\rho}\, |\{A_0 b\}(e^{it})| + \frac{1}{\rho}\, |\{z_2\}(e^{it})|, \qquad e^{it} \in \mathbb{T}. \qquad (50)$$

Thus $f \in L^2(\Sigma)$, so we may write $f = \{c_1\}$ for some $c_1 \in \mathscr{R}_0$, and it is easy to see that c_1 satisfies

$$\|c_1\| \leqslant \frac{1}{\rho}\,(\|A_0 b\| + \|z_2\|). \qquad (51)$$

Moreover, it is clear from (49) and (37) that

$$A_0(a_1+u) \overset{R}{\cdot} c_1 = h_0.$$

We are finally in a position to define

$$c = c_1 + A_1 b = A_0 c_1 + A_1 b,$$

and, employing (41), we see that

$$\begin{aligned} \|h_2 - A(a_1+u) \overset{R}{\cdot} c\|_1 &= \|h_2 - A_0(a_1+u) \overset{R}{\cdot} c_1 - A_1(a_1+u) \overset{R}{\cdot} A_1 b\|_1 \\ &= \|A_1 u \overset{R}{\cdot} A_1 b\|_1 \leqslant 2\delta\, \|A_1 b\|. \qquad (52) \end{aligned}$$

Combining (52) with (42), and using (34), we obtain

$$\begin{aligned}\|h_1 - A(a_1+u) \overset{R}{\cdot} c\|_1 &\leqslant \|h_1 - h_2\|_1 + \|h_2 - A_0(a_1+u) \overset{R}{\cdot} c_1 \\ &\quad - A_1(a_1+u) \overset{R}{\cdot} A_1 b\|_1 \\ &\leqslant 2\delta \, \|h\|_1^{1/2} + 2\delta \, \|A_1 b\| < \varepsilon.\end{aligned}$$

Thus (28) is established. To verify (33), we simply note that $c - b = c_1 + A_1 b - Ab = c_1 - A_0 b \in \mathscr{R}_0$, and to check that (31) is valid, we compute, using (45), (38), (39), and (35),

$$\begin{aligned}\|u\| \leqslant (1+\rho) \, \|x_2\| &= (1+\rho) \, \|y_2 - (y_2 - Ax_2 - Qx_2)\| \\ &\leqslant (1+\rho)(\|y_2\| + \|y_2 - Ax_2\| + \| - Qx_2\|) \\ &< (1+\rho)(\|h\|_1^{1/2}(1+\delta) + \delta + \delta) < 2 \, \|h\|_1^{1/2}.\end{aligned}$$

Finally, we estimate $\|c\|^2$, using (51) and the fact that c_1 and $A_1 b$ are orthogonal:

$$\begin{aligned}\|c\|^2 &= \|c_1\|^2 + \|A_1 b\|^2 \\ &\leqslant \frac{1}{\rho^2} (\|A_0 b\|^2 + 2 \, \|A_0 b\| \, \|z_2\| + \|z_2\|^2) + \frac{1}{\rho^2} \|A_1 b\|^2 \\ &\leqslant \frac{1}{\rho^2} (\|b\|^2 + 2 \, \|b\| \, \|h\|_1^{1/2} + \|h\|_1) \leqslant \frac{1}{\rho^2} (\|b\| + \|h\|_1^{1/2})^2,\end{aligned}$$

so (32) is valid, and the theorem is proved.

4. A Geometric Criterion for Membership in $\mathbb{A}_1(r)$

In this section we establish an abstract geometric sufficient condition for membership in various clases $\mathbb{A}_1(r)$. For this purpose we need some definitions.

Definition 4.1. Suppose $\mathscr{M}$ is a weak* closed subspace of $\mathscr{L}(\mathscr{H})$ and $0 \leqslant \theta < 1$. The set of all those $[L]$ in $Q_{\mathscr{M}}$ such that there exist sequences $\{x_n\}$ and $\{y_n\}$ in the closed unit ball of $\mathscr{H}$ satisfying

(a) $\overline{\lim} \, \|[L] - [x_n \otimes y_n]\| \leqslant \theta$,

and

(b′) $\|[z \otimes y_n]\| \to 0, \quad z \in \mathscr{H}$,

(c′) $\{x_n\}$ converges weakly to zero, will be denoted by $\mathscr{E}^l_\theta(\mathscr{M})$. The corresponding subset of $Q_{\mathscr{M}}$ obtained by replacing conditions (b′) and (c′) by the conditions

(b^r) $\|[x_n \otimes z]\| \to 0, \quad z \in \mathscr{H}$,

(c^r) $\{y_n\}$ converges weakly to zero,

will be denoted by $\mathscr{E}^r_\theta(\mathscr{M})$.

The reader will observe that the sets $\mathscr{E}^l_\theta(\mathscr{M})$ and $\mathscr{E}^r_\theta(\mathscr{M})$ are unilateral analogs of the set $\mathscr{X}_\theta(\mathscr{M})$ appearing in [5, Definition 2.7]. Likewise the properties $E^l_{\theta,\gamma}$ and $E^r_{\theta,\gamma}$ to be defined next are the unilateral analogs of the property $X_{\theta,\gamma}$ of [5, Definition 2.8].

DEFINITION 4.2. If $0 \leqslant \theta < \gamma \leqslant 1$, a weak* closed subspace $\mathscr{M}$ of $\mathscr{L}(\mathscr{H})$ will be said to have property $E^l_{\theta,\gamma}$ [resp. $E^r_{\theta,\gamma}$] if the closed absolutely convex hull (notation: $\overline{\text{aco}}$) of the set $\mathscr{E}^l_\theta(\mathscr{M})$ [resp. $\mathscr{E}^r_\theta(\mathscr{M})$] contains the closed ball in $Q_{\mathscr{M}}$ centered at 0 with radius γ.

The following proposition gives an easy sufficient condition for a dual algebra $\mathscr{A}_T$ to have property $E^l_{\theta,1}$ (resp. $E^r_{\theta,1}$).

PROPOSITION 4.3. *Suppose* $T \in \mathbb{A}(\mathscr{H})$, $0 \leqslant \theta < 1$, *and* $\Lambda \subset \mathbb{D}$ *is dominating for* $\mathbb{T}$. *If for each* $\lambda \in \Lambda$ *there exists a sequence* $\{x_n = x_n(\lambda)\}$ *in the closed unit ball of* $\mathscr{H}$ *such that*

(1) $\overline{\lim}\, \|[C_\lambda] - [x_n \otimes x_n]\| \leqslant \theta$, *and*

(2′) $\|[z \otimes x_n]\| \to 0$ [*resp.* (2^r) $\|[x_n \otimes z]\| \to 0$], $z \in \mathscr{H}$,

then $\mathscr{A}_T$ *has property* $E^l_{\theta,1}$ [*resp.* $E^r_{\theta,1}$].

Proof. If $\|[z \otimes x_n\| \to 0$ for each z in $\mathscr{H}$, then $\{x_n\}$ converges weakly to zero, and that $\mathscr{A}_T$ has property $E^l_{\theta,1}$ follows from the definitions and the fact (cf. [5, Proposition 1.21]) that $\overline{\text{aco}}\,\{[C_\lambda]: \lambda \in \Lambda\}$ is the closed unit ball in Q_T. The other case is just as easy.

The following theorem, which is a unilateral analog of [5, Proposition 6.1], may fairly be called the principal result of this paper.

THEOREM 4.4. *Suppose* $T \in \mathbb{A}(\mathscr{H})$ *and for some* $0 \leqslant \theta < \gamma \leqslant 1$, $\mathscr{A}_T$ *has either property* $E^l_{\theta,\gamma}$ *or property* $E^r_{\theta,\gamma}$. *Then* $T \in \mathbb{A}_1(r)$ *where* $r = (6/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2$. *In particular, if* $\mathscr{A}_T$ *has property* $E^l_{0,1}$ *or* $E^r_{0,1}$, *then* $T \in \mathbb{A}_1(6)$.

The proof of Theorem 4.4 will use Section 3, especially Theorem 3.11, and some additional tools. The following proposition gives a good idea of the game plan.

PROPOSITION 4.5. *Suppose $T \in \mathbb{A}(\mathscr{H})$ and has minimal coisometric extension B in $\mathscr{L}(\mathscr{S} \oplus \mathscr{R})$ as in Section 3, and suppose that for every $[L]$ in Q_T there exists a Cauchy sequence $\{x_n\}$ in $\mathscr{H}$ and sequences $\{w_n\}$ in $\mathscr{S}$ and $\{b_n\}$ in $\mathscr{R}$ such that $\{\|w_n + b_n\|\}$ is bounded and*

$$\|(\varphi_B^{-1} \circ \varphi_T)([L]_T) - [x_n \otimes (w_n + b_n)]_B\| \to 0.$$

Then $T \in \mathbb{A}_1$.

Proof. We employ the notation connecting T and B from Section 3. With $[L]$ in Q_T and $\{w_n\}$ and $\{b_n\}$ as above, set $v_n = P(w_n + b_n)$, $n \in \mathbb{N}$. Since $\{v_n\}$ is bounded, we may suppose, without loss of generality, that $\{v_n\}$ is weakly convergent to v. Moreover, since the sequence $\{x_n\}$ is Cauchy, it converges strongly–say to x. Since $\{v_n\}$ is bounded, we have

$$\|[x \otimes v_n] - [x_n \otimes v_n]\| \leqslant \|x - x_n\| \, \|v_n\| \to 0.$$

Also, from (15) and (17), with $j = \varphi_B^{-1} \circ \varphi_T$, we have

$$\begin{aligned}\|[L]_T - [x_n \otimes v_n]_T\| &= \|j([L]_T) - [x_n \otimes v_n]_B\| \\ &= \|j([L]_T) - [x_n \otimes (w_n + b_n)]_B\| \to 0,\end{aligned}$$

so

$$\|[L]_T - [x \otimes v_n]_T\| \to 0.$$

We now compute to show that $[L]_T = [x \otimes v]_T$, and thus complete the proof; for h in $H^\infty(\mathbb{T})$, we have

$$\begin{aligned}\langle h(T), [L]_T\rangle &= \lim_n \langle h(T), [x \otimes v_n]_T\rangle \\ &= \lim_n (h(T)\,x, v_n) = (h(T)\,x, v) \\ &= \langle h(T), [x \otimes v]_T\rangle,\end{aligned}$$

so the proposition is established.

The following proposition contains the heart of the approximation process needed to prove Theorem 4.4.

PROPOSITION 4.6. *Suppose $T \in \mathbb{A}(\mathscr{H})$ with minimal coisometric extension $B \in \mathscr{L}(\mathscr{S} \oplus \mathscr{R})$ as in Section 3, and for some $0 < \theta < \gamma \leqslant 1$, $\mathscr{A}_T$ has property $E^r_{\theta,\gamma}$. Suppose also that $0 < \rho < 1$, $[L] \in Q_B$, $a \in \mathscr{H}$, $w \in \mathscr{S}$, $b \in \mathscr{R}$, and $\delta > 0$ are given such that*

$$\|[L]_B - [a \otimes (w + b)]_B\| < \delta. \tag{53}$$

Then there exist $a' \in \mathscr{H}$, $w' \in \mathscr{S}$, $b' \in \mathscr{R}$ such that

$$\|[L]_B - [a' \otimes (w' + b')]_B\| < \frac{\theta}{\gamma}\delta \tag{54}$$

and

$$\begin{gathered} \|a' - a\| < 3(\delta/\gamma)^{1/2}, \qquad \|w' - w\| < (\delta/\gamma)^{1/2}, \\ \|b'\| < \frac{1}{\rho}\{\|b\| + (\delta/\gamma)^{1/2}\}. \end{gathered} \tag{55}$$

Proof. Of course, either of the spaces $\mathscr{S}$ or $\mathscr{R}$ may be zero, but the proof is unchanged in these special cases. Let

$$[L_1]_B = [L]_B - [a \otimes (w + b)]_B \tag{56}$$

and set $d = \|[L_1]\|$, so $0 \leqslant d < \delta$. If $d = 0$ just set $a' = a$, $w' = w$, and $b' = b$. Thus we may suppose that $d > 0$. Choose $\varepsilon > 0$ such that

$$\left(\frac{\theta}{\gamma}\right) d + \varepsilon < \left(\frac{\theta}{\gamma}\right) \delta. \tag{57}$$

With j as in Lemma 3.5, note that $\|(\gamma/d)\, j^{-1}([L_1]_B)\| = \gamma$, and thus, by hypothesis, there exist $N \in \mathbb{N}$, elements $[K_1], ..., [K_N]$ from $\mathscr{E}_0^r(\mathscr{A}_T)$, and scalars $\alpha_1', ..., \alpha_N'$ such that

$$\left\| \frac{\gamma}{d} j^{-1}([L_1]_B) - \sum_{i=1}^{N} \alpha_1' [K_i]_T \right\| < \frac{\varepsilon}{2}\frac{\gamma}{d} \tag{58}$$

and

$$\sum_{i=1}^{N} |\alpha_i'| < 1.$$

Upon setting $\alpha_i = (d/\gamma)\, \alpha_i'$, $i = 1, ..., N$, we obtain, by multiplying (58) by d/γ,

$$\left\| j^{-1}([L_1]_B) - \sum_{i=1}^{N} \alpha_i [K_i]_T \right\| < \varepsilon/2 \tag{59}$$

and

$$\sum_{i=1}^{N} |\alpha_i| < d/\gamma. \tag{60}$$

For each $i = 1, \ldots, N$, by definition of $\mathscr{E}_\theta^r(T)$, there exist sequences $\{x_{n_i}^i\}_{n_i=1}^\infty$ and $\{y_{n_i}^i\}_{n_i=1}^\infty$ in the unit ball of $\mathscr{H}$ such that

$$\|[K_i]_T - [x_{n_i}^i \otimes y_{n_i}^i]_T\| < \theta + \frac{\varepsilon}{2}\frac{\gamma}{d}, \qquad n_i \in \mathbb{N}, \tag{61}$$

$$\lim_{n_i \to \infty} \|[x_{n_i}^i \otimes z]_T\| = 0, \qquad z \in \mathscr{H}, \tag{62}$$

and

$$\{y_{n_i}^i\}_{n_i=1}^\infty \qquad \text{converges weakly to zero.} \tag{63}$$

Upon adding (59) and (61), we get, for any choice of the N-tuple $v = (n_1, \ldots, n_N)$,

$$\left\| j^{-1}([L_1]_B) - \sum_{i=1}^N \alpha_i [x_{n_i}^i \otimes y_{n_i}^i]_T \right\| < \frac{\varepsilon}{2} + \frac{d}{\gamma}\left(\theta + \frac{\varepsilon}{2}\frac{\gamma}{d}\right) = \varepsilon + \frac{d\theta}{\gamma}, \tag{64}$$

and, passing to the predual Q_B by applying j to (64), we obtain, using (15),

$$\left\| [L_1]_B - \sum_{i=1}^N \alpha_i [x_{n_i}^i \otimes y_{n_i}^i]_B \right\| < \varepsilon + \frac{d\theta}{\gamma} \tag{65}$$

for every choice of v. Let us denote the difference

$$\frac{\theta\delta}{\gamma} - \left\{\frac{d\theta}{\gamma} + \varepsilon\right\} = 5\tau \tag{66}$$

for some $\tau > 0$. Using (18) and (56) we may combine (65) and (66) to yield

$$\left\| [L]_B - [a \otimes (w+b)]_B - \sum_{i=1}^N \alpha_i \{[Qx_{n_i}^i \otimes Qy_{n_i}^i]_B + [Ax_{n_i}^i \otimes Ay_{n_i}^i]_B\} \right\| < \frac{\theta\delta}{\gamma} - 5\tau \tag{67}$$

for every choice of v, and using (18) once more, we rewrite (67) as

$$\left\| [L]_B - [Qa \otimes w]_B - \sum_{i=1}^N \alpha_i [Qx_{n_i}^i \otimes Qy_{n_i}^i]_B - [M(v)]_B \right\| < \frac{\theta\delta}{\gamma} - 5\tau, \tag{68}$$

where $[M(v)]_B$ is defined as

$$[M(v)]_B = [Aa \otimes b]_B + \sum_{i=1}^{N} \alpha_i [Ax^i_{n_i} \otimes Ay^i_{n_i}]_B. \tag{69}$$

We shall "work on" the term

$$[Qa \otimes w]_B + \sum_{i=1}^{N} \alpha_i [Qx^i_{n_i} \otimes Qy^i_{n_i}]_B$$

from (68). Let us define, for arbitrary $v = (n_1, ..., n_N)$,

$$u_v = \sum_{i=1}^{N} \beta_i x^i_{n_i}, \qquad v_v = \sum_{i=1}^{N} \bar{\beta}_i y^i_{n_i}, \tag{70}$$

where $\beta_i^2 = \alpha_i$ for $i = 1, ..., N$. Then, for every choice of v,

$$\begin{aligned}[Q(a+u_v) \otimes (w + Qv_v)]_B = {} & [Qa \otimes w]_B + [Qu_v \otimes w]_B \\ & + [Qa \otimes Qv_v]_B + [Qu_v \otimes Qv_v]_B\end{aligned} \tag{71}$$

and

$$\begin{aligned}[Qu_v \otimes Qv_v]_B = {} & \sum_{i=1}^{N} \alpha_i [Qx^i_{n_i} \otimes Qy^i_{n_i}]_B \\ & + \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \beta_i \beta_j [Qx^i_{n_i} \otimes Qy^i_{n_j}]_B.\end{aligned} \tag{72}$$

We assert that the indices $n_1^0, ..., n_N^0$ can be chosen (one at a time, in the indicated order) sufficiently large that for $v_0 = (n_1^0, ..., n_N^0)$ we have (simultaneously)

$$\|[Qa \otimes Qv_{v_0}]_B\| < \tau/3, \tag{73}$$

$$\|[Qu_{v_0} \otimes w]_B\| < \tau/3, \tag{74}$$

$$\left\| \sum_{\substack{i,j=1 \\ i \neq j}}^{N} \beta_i \beta_j [Qx^i_{n_i^0} \otimes Qy^i_{n_i^0}]_B \right\| < \tau/3, \tag{75}$$

$$\|[Au_{v_0} \otimes b]_B\| < \tau, \tag{76}$$

and

$$\|u_{v_0}\|^2 \approx \sum_{i=1}^{N} |\alpha_i| \, \|x^i_{n_i^0}\|^2 < \frac{\delta}{\gamma}, \qquad \|v_{v_0}\|^2 \approx \sum_{i=1}^{N} |\alpha_i| \, \|y^i_{n_i^0}\|^2 < \frac{\delta}{\gamma}. \tag{77}$$

This is by now a standard argument; we are content to cite the needed facts and leave the epsilontics to the reader. To establish (73) (which must be done simultaneously with (74) – (77)) one uses the expansion

$$\|[Qa \otimes Qv_\nu]_B\| \leqslant \sum_{i=1}^{N} |\beta_i| \, [Qa \otimes Qy^i_{n_i}]_B\|,$$

(63), and Lemma 3.8. To establish (74), one uses (62) and (21) from Lemma 3.7. To establish (75), recall that n_1^0 is chosen first, then n_2^0, etc., and use (63), Lemma 3.8, (62), and Lemma 3.7. To establish (76), use (62) and (22) from Lemma 3.7. Finally, to establish (77), it suffices to make all cross terms such as $(x^i_{n_i^0}, x^j_{n_j^0})$, $i \neq j$, small, and this can be done using (63) and the elementary conclusion from (62) that each sequence $\{x^i_{n_i}\}_{n_i=1}^{\infty}$, $i = 1, ..., N$, converges weakly to zero.

Thus we now suppose that $\nu_0 = (n_1^0, ..., n_N^0)$ has been chosen so that (73)–(77) are satisfied. Therefore, by combining (71)–(75), we obtain

$$\left\| [Qa \otimes w]_B + \sum_{i=1}^{N} \alpha_i [Qx^i_{n_{i0}} \otimes Qy^i_{n_{i0}}]_B - [Q(a + u_{\nu_0}) \otimes (w + Qv_{\nu_0})]_B \right\| < \tau. \tag{78}$$

We next define

$$a_1 = a + u_{\nu_0}, \qquad w' = w + Qv_{\nu_0}, \tag{79}$$

and conclude from (79), (78), and (68) that

$$\|[L]_B - [Qa_1 \otimes w']_B - [M(\nu_0)]_B\| < \frac{\theta\delta}{\gamma} - 4\tau. \tag{80}$$

Moreover, if in $[M(\nu_0)]_B$ we replace a by a_1, and so define

$$[M_1(\nu_0)]_B = [Aa_1 \otimes b]_B + \sum_{i=1}^{N} \alpha_i [Ax^i_{n_{i0}} \otimes Ay^i_{n_{i0}}]_B, \tag{81}$$

then by virtue of (69), (76), (79), and (80), we have

$$\|[L]_B - [Qa_1 \otimes w']_B - [M_1(\nu_0)]_B\| < \frac{\theta\delta}{\gamma} - 3\tau. \tag{82}$$

Now suppose that $\mathscr{R} = (0)$. Then $b = 0$, $[M_1(\nu_0)]_B = 0$, $Qa_1 = a_1$, and

$$\|[L]_B - [a_1 \otimes w']_B\| < \frac{\theta\delta}{\gamma} - 3\tau.$$

Thus, by virtue of (79) and (77), we have $\|a - a_1\| < (\delta/\gamma)^{1/2}$ and

$\|w-w'\|<(\delta/\gamma)^{1/2}$, so (with $b'=0$) the proof in this case is complete. Hence we may suppose that $\mathscr{R}\neq(0)$, we let $\Sigma\subset\mathbb{T}$ be as in Proposition 3.10, and we prepare to apply Theorem 3.11 to deal with the term $[M_1(v_0)]_B$ in (82). By (81) and Lemma 3.9 we have

$$\varphi_B([M_1(v_0)]_B)=[Aa_1 \overset{R}{\cdot} b]+\sum_{i=1}^{N}\alpha_i[Ax^i_{n_{i0}} \overset{R}{\cdot} Ay^i_{n_{i0}}]. \tag{83}$$

Thus we define the function h in $L^1(\Sigma)$ to be

$$h=\sum_{i=1}^{N}\alpha_i(Ax^i_{n_{i0}} \overset{R}{\cdot} Ay^i_{n_{i0}}),$$

we note from (25) and (60) that $\|h\|_1\leqslant\sum_{i=1}^{N}|\alpha_i|<\delta/\gamma$, and we set $\varepsilon'=\tau/(\|w'\|+1)$. With a_1 and b as in (83), an application of Theorem 3.11 yields the existence of $\tilde{u}\in\mathscr{H}$ and $c\in\mathscr{R}$ such that

$$\left\|Aa_1 \overset{R}{\cdot} b+\sum_{i=1}^{N}\alpha_i(Ax^i_{n_{i0}} \overset{R}{\cdot} Ay^i_{n_{i0}})-A(a_1+\tilde{u}) \overset{R}{\cdot} c\right\|_1<\varepsilon'<\tau, \tag{84}$$

$$\|Q\tilde{u}\|<\tau/(\|w'\|+1), \tag{85}$$

$$\|\tilde{u}\|\leqslant 2\,\|h\|_1^{1/2}\leqslant 2\left(\sum_{i=1}^{N}|\alpha_i|\right)^{1/2}<2(\delta/\gamma)^{1/2}, \tag{86}$$

and

$$\|c\|\leqslant\frac{1}{\rho}\{\|b\|+\|h\|_1^{1/2}\}<\frac{1}{\rho}\{\|b\|+(\delta/\gamma)^{1/2}\}. \tag{87}$$

Since $L^1(\Sigma)\subset L^1(\mathbb{T})$ and the norm in $L^1(\mathbb{T})$ dominates the norm in $(L^1/H^1_0)(\mathbb{T})$, we obtain using (81), (26), and (84),

$$\begin{aligned}&\|[M_1(v_0)]_B-[A(a_1+\tilde{u})\otimes c]_B\|\\&\quad=\left\|[Aa_1\otimes b]_B+\sum_{i=1}^{N}\alpha_i[Ax^i_{n_{i0}}\otimes Ay^i_{n_{i0}}]_B-[A(a_1+\tilde{u})\otimes c]_B\right\|<\tau.\end{aligned} \tag{88}$$

Thus from (82) and (88) we get

$$\|[L]_B-[Qa_1\otimes w']_B-[A(a_1+\tilde{u})\otimes c]_B\|<\frac{\theta\delta}{\gamma}-2\tau, \tag{89}$$

and since, via (85), we have

$$\|[Q\tilde{u}\otimes w']_B\| \leqslant \|Q\tilde{u}\|\ \|w'\| < \tau, \tag{90}$$

the inequality (89) yields

$$\|[L]_B - [Q(a_1+\tilde{u})\otimes w']_B - [A(a_1+\tilde{u})\otimes c]_B\| < \frac{\theta\delta}{\gamma} - \tau. \tag{91}$$

Since $w' \in \mathscr{S}$ and $c \in \mathscr{R}$, by using (18) one can rewrite (91) as

$$\|[L]_B - [(a_1+\tilde{u})\otimes(w'+c)]_B\| < \frac{\theta\delta}{\gamma} - \tau,$$

so if we define

$$a' = a_1 + \tilde{u} = a + u_{v0} + \tilde{u}, \qquad b' = c, \tag{92}$$

then (54) is satisfied. Moreover,

$$\|a' - a\| \leqslant \|u_{v0}\| + \|\tilde{u}\| < (\delta/\gamma)^{1/2} + 2(\delta/\gamma)^{1/2}$$

from (77) and (86), so the first inequality in (55) is satisfied. Furthermore, from (79) and (77) we have

$$\|w' - w\| = \|Qv_{v0}\| \leqslant \|v_{v0}\| < (\delta/\gamma)^{1/2},$$

which takes care of the second inequality in (55). Finally,

$$\|b'\| = \|c\| < \frac{1}{\rho}\{\|b\| + (\delta/\gamma)^{1/2}\}$$

from (87), so the proposition is proved.

We are now prepared to prove Theorem 4.4, and in fact, we shall prove the following stronger version of Theorem 4.4 that will be useful for later applications.

THEOREM 4.7. *Suppose* $T \in \mathbb{A}(\mathscr{H})$ *with minimal coisometric extension* B *in* $\mathscr{L}(\mathscr{S}\oplus\mathscr{R})$, *and for some* $0 \leqslant \theta < \gamma \leqslant 1$, $\mathscr{A}_T$ *has property* $E^r_{\theta,\gamma}$. *Suppose, moreover, that* $\delta > 0$, $[L] \in Q_T$, $a \in \mathscr{H}$, $w \in \mathscr{S}$, *and* $b \in \mathscr{R}$ *are given such that*

$$\|[L]_T - [a\otimes P(w+b)]_T\| < \delta. \tag{93}$$

Then there exist $\hat{a} \in \mathscr{H}$, $\hat{w} \in \mathscr{S}$, *and* $\hat{b} \in \mathscr{R}$ *such that*

$$[L]_T = [\hat{a}\otimes P(\hat{w}+\hat{b})]_T, \tag{94}$$

$$\|\hat{a}-a\| < 3\left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}), \tag{95}$$

$$\|\hat{w}-w\| < \left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}), \tag{96}$$

and

$$\|\hat{b}\| \leqslant \frac{4}{3}\|b\| + \frac{4}{3}\left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}). \tag{97}$$

In particular, if for some $0 \leqslant \theta < \gamma \leqslant 1$, $\mathscr{A}_T$ *has either property* $E^l_{\theta,\gamma}$ *or* $E^r_{\theta,\gamma}$, *then* $T \in \mathbb{A}_1(r(\theta,\gamma))$, *where*

$$r(\theta,\gamma) = (6/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2. \tag{98}$$

Proof. Since $T \in \mathbb{A}$, it is easy to see that the mapping $j_1: Q_T \to Q_{T^*}$ defined by

$$j_1([L]_T) = \varphi_{T^*}^{-1}(\widetilde{\varphi_T([L]_T)}), \tag{99}$$

where $\tilde{f}(e^{it}) = \overline{f(e^{-it})}$, is a conjugate linear isometry of Q_T onto Q_{T^*} which possesses the further property that

$$j_1([x \otimes y]_T) = [y \otimes x]_{T^*}. \tag{100}$$

From these facts it follows easily that $\mathscr{A}_T$ has property $E^l_{\theta,\gamma}$ if and only if $\mathscr{A}_{T^*}$ has property $E^r_{\theta,\gamma}$, and since each set $\mathbb{A}_1(r)$ is self-adjoint, it suffices to treat the case in which $\mathscr{A}_T$ has property $E^r_{\theta,\gamma}$. Moreover, since if $\mathscr{A}_T$ has property $E^r_{0,\gamma}$, it also has property $E^r_{\theta,\gamma}$ for all $0 < \theta < \gamma$, and the right-hand sides of (95), (96), and (97) are continuous functions of θ and δ, it suffices to treat the case $0 < \theta < \gamma$.

Suppose now that (93) holds, let $\{s_n\}$ be a sequence of positive numbers strictly decreasing to $\frac{3}{4}$ such that $s_1 = 1$, and define $\rho_n = s_{n+1}/s_n$, $n \in \mathbb{N}$. Upon setting $[\hat{L}]_B = \varphi_B^{-1} \circ \varphi_T([L]_T)$, we have, by virtue of (93), (16), and (17),

$$\|[\hat{L}]_B - [a \otimes (w+b)]_B\| < \delta. \tag{101}$$

We now set $a = a_1$, $w = w_1$, $b = b_1$, and apply Proposition 4.6 to obtain $a_2 \in \mathscr{H}$, $w_2 \in \mathscr{S}$, and $b_2 \in \mathscr{R}$ such that

$$\|[\hat{L}]_B - [a_2 \otimes (w_2+b_2)]_B\| < \frac{\theta}{\gamma}\delta, \tag{102}$$

$$\begin{gathered} \|a_2 - a_1\| < 3(\delta/\gamma)^{1/2}, \qquad \|w_2 - w_1\| < (\delta/\gamma)^{1/2}, \\ \|b_2\| < \frac{1}{\rho_1}\left\{\|b_1\| + \left(\frac{\delta}{\gamma}\right)^{1/2}\right\}. \end{gathered} \tag{103}$$

Suppose now that vectors $\{a_k\}_{k=1}^n$ in $\mathscr{H}$, $\{w_k\}_{k=1}^n$ in $\mathscr{S}$, and $\{b_k\}_{k=1}^n$ in $\mathscr{R}$ have been chosen so that for $k=2, ..., n$,

$$\|[\hat{L}]_B - [a_k \otimes (w_k + b_k)]_B\| < \left(\frac{\theta}{\gamma}\right)^{k-1} \delta, \tag{104$_k$}$$

$$\|a_k - a_{k-1}\| < 3\left(\frac{\delta}{\gamma}\right)^{1/2} \left(\frac{\theta}{\gamma}\right)^{(k-2)/2}, \tag{105$_k$}$$

$$\|w_k - w_{k-1}\| < \left(\frac{\delta}{\gamma}\right)^{1/2} \left(\frac{\theta}{\gamma}\right)^{(k-2)/2}, \tag{106$_k$}$$

and

$$\|b_k\| < \frac{1}{\rho_{k-1}} \left\{\|b_{k-1}\| + \left(\frac{\delta}{\gamma}\right)^{1/2} \left(\frac{\theta}{\gamma}\right)^{(k-2)/2}\right\}. \tag{107$_k$}$$

Then, applying Proposition 4.6 once again, we deduce the existence of vectors a_{n+1} in $\mathscr{H}$, w_{n+1} in $\mathscr{S}$, and b_{n+1} in $\mathscr{R}$ such that the inequalities $(104)_{n+1}$, $(105)_{n+1}$, $(106)_{n+1}$, and $(107)_{n+1}$ are valid. Therefore, by induction, there exist sequences $\{a_n\}_{n=1}^\infty$ in $\mathscr{H}$, $\{w_n\}_{n=1}^\infty$ in $\mathscr{S}$, and $\{b_n\}_{n=1}^\infty$ in $\mathscr{R}$ satisfying the appropriate inequalities for all n in $\mathbb{N}$, and it is clear from $(105)_k$ and $(106)_k$ that $\{a_n\}$ and $\{w_n\}$ are Cauchy. Define

$$\hat{a} = \lim a_n, \qquad \hat{w} = \lim w_n,$$

and observe that since

$$\begin{aligned}\|\hat{a} - a\| &= \left\| \sum_{k=2}^{\infty} (a_k - a_{k-1}) \right\| \leqslant \sum_{n=2}^{\infty} \|(a_k - a_{k-1})\| \\ &< \sum_{n=2}^{\infty} 3\left(\frac{\delta}{\gamma}\right)^{1/2} \left(\frac{\theta}{\gamma}\right)^{(k-2)/2} \\ &= 3\left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}),\end{aligned}$$

and, similarly,

$$\|\hat{w} - w\| < \left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}),$$

inequalities (95) and (96) are satisfied. Furthermore, by iterating $(107)_k$, one sees easily that

$$\frac{3}{4}\|b_n\| < s_n \|b_n\| \leqslant \|b\| + \left(\frac{\delta}{\gamma}\right)^{1/2} \sum_{k=1}^{n-1} s_k \left(\frac{\theta}{\gamma}\right)^{(k-1)/2},$$

and therefore that

$$\|b_n\| \leqslant \frac{4}{3}\|b\| + \frac{4}{3}\left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}), \qquad n \in \mathbb{N}.$$

Thus the sequence $\{b_n\}$ is bounded, and without loss of generality we may suppose that $\{b_n\}$ converges weakly to $\hat{b}$. Hence

$$\|\hat{b}\| \leqslant \frac{4}{3}\|b\| + \frac{4}{3}\left(\frac{\delta}{\gamma}\right)^{1/2} (1/\{1-(\theta/\gamma)^{1/2}\}),$$

which establishes (97). That (94) is valid now follows from $(104)_k$ as in the proof of Proposition 4.5, so T certainly belongs to some $\mathbb{A}_1(r)$. To see that r may be taken to be as in (98), let $\varepsilon > 0$, and set $a = 0$, $w = 0$, $b = 0$, and $\delta = \|[L]_T\| + \varepsilon$ in (93). Then from (95), (96), and (97), we see that

$$\begin{aligned} \|\hat{a}\| \; \|P(\hat{w}+\hat{b})\| &\leqslant \|\hat{a}\|(\|\hat{w}\|^2 + \|\hat{b}\|^2)^{1/2} \\ &< (6/\gamma)(1/\{1-(\theta/\gamma)^{1/2}\})^2(\|[L_T]\| + \varepsilon), \end{aligned}$$

so $T \in \mathbb{A}_1(r(\theta, \gamma))$, where $r(\theta, \gamma)$ is in (98), by definition, and Theorems 4.4 and 4.7 are proved.

The following corollary is immediate from Proposition 4.3 and Theorem 4.4.

COROLLARY 4.8. *Suppose $T \in \mathbb{A}(\mathscr{H})$, $0 \leqslant \theta < 1$, and $\Lambda \subset \mathbb{D}$ is dominating for $\mathbb{T}$. If for each $\lambda \in \Lambda$ there exists a sequence $\{x_n^\lambda\}$ in the unit ball of $\mathscr{H}$ such that*

$$\overline{\lim_n} \; \|[C_\lambda]_T - [x_n^\lambda \otimes x_n^\lambda]_T\| \leqslant \theta \tag{108$_\theta$}$$

and

$$\lim_n \|[x_n^\lambda \otimes y]_T\| = 0 \qquad [\textit{resp.} \lim_n \|[y \otimes x_n^\lambda]_T\| = 0], \quad y \in \mathscr{H}, \tag{109}$$

then $T \in \mathbb{A}_1(6/\{1-\theta^{1/2}\}^2)$.

5. SPECTRAL CRITERIA FOR MEMBERSHIP IN $\mathbb{A}_1(r)$

In this section we use Theorem 4.4 to improve the previously known spectral sufficient conditions for membership in some $\mathbb{A}_1(r)$ from [5, 12, 13, 18].

Recall that if $T \in \mathscr{L}(\mathscr{H})$ and H is a hole in $\sigma_e(T)$ (i.e., H is a bounded component of $\mathbb{C}\backslash\sigma_e(T)$), then H is associated with a unique finite Fredholm index $i(H)$, defined by choosing any λ in H and setting $i(H)=i(T-\lambda)$. If H is a hole in $\sigma_e(T)$ such that $i(H)\neq 0$, then, of course, $H\subset\sigma(T)$. On the other hand, if H is a hole in $\sigma_e(T)$ with $i(H)=0$, then either $H\subset\sigma(T)$ or $H\cap\sigma(T)$ consists of a countable (possibly empty) set of isolated points. We recall the following notation from [12].

Notation 5.1. For each T in $\mathscr{L}(\mathscr{H})$ we write $\mathscr{F}_-(T)$ [resp. $\mathscr{F}_+(T)$] for the (possibly empty) union of all holes H in $\sigma_e(T)$ such that $i(H)\leqslant 0$ [resp. $i(H)\geqslant 0$] and $H\subset\sigma(T)$. Moreover, we write $\mathscr{F}'_-(T)$ [resp. $\mathscr{F}'_+(T)$] for the union of all holes H in $\sigma_e(T)$ such that $i(H)<0$ [resp. $i(H)>0$], and $\mathscr{F}(T)=\mathscr{F}_-(T)\cup\mathscr{F}_+(T)$.

The following lemma will be needed.

LEMMA 5.2. *Suppose $T\in\mathbb{A}(\mathscr{H})$, $\mathscr{N}\in\operatorname{Lat}(T^*)$, and we write $T_{\mathscr{N}}$ for the compression of T to the semi-invariant subspace $\mathscr{N}$. If $\lambda\in\sigma(T_{\mathscr{N}})\cap\mathbb{D}$, and*

$$x_n\in\ker(T_{\mathscr{N}}-\lambda)^{n+1}\ominus\ker(T_{\mathscr{N}}-\lambda)^n,\qquad n\in\mathbb{N},$$

with $\|x_n\|=1$ for all n, then $\{x_n\}$ is an orthonormal sequence satisfying

$$[C_\lambda]_T=[x_n\otimes x_n]_T,\qquad n\in\mathbb{N},\tag{110}$$

and

$$\|[y\otimes x_n]_T\|\to 0,\qquad y\in\mathscr{H}.\tag{111}$$

Proof. It is obvious from the definition that $\{x_n\}$ is orthonormal, and that (110) is valid follows from [12, Lemma 2.3] applied to T^*. To establish (111), fix y in $\mathscr{H}$. Then, since $T\in\mathbb{A}$, for each $n\in\mathbb{N}$ there exists h_n in $H^\infty(\mathbb{T})$ with $\|h_n\|=1$ such that

$$\|[y\otimes x_n]_T\|=\langle h_n(T),[y\otimes x_n]_T\rangle=(h_n(T)\,y,x_n).$$

Since each $x_n\in\mathscr{N}$ and $\mathscr{N}\in\operatorname{Lat}(T^*)$, we have, upon denoting the projection onto $\mathscr{N}$ by $P_{\mathscr{N}}$,

$$\begin{aligned}\|[y\otimes x_n]_T\|&=(y,\tilde{h}_n(T^*)\,x_n)=(P_{\mathscr{N}}\,y,\tilde{h}_n(T^*)\,x_n)\\&=(P_{\mathscr{N}}\,y,\tilde{h}_n(T^*\mid\mathscr{N})\,x_n)\\&=(h_n(T_{\mathscr{N}})\,P_{\mathscr{N}}\,y,x_n).\end{aligned}\tag{112}$$

Employing momentarily the (slightly abusive) notation of $B=S^*\oplus R$ on $\mathscr{K}=\mathscr{S}\oplus\mathscr{R}$ for the minimal coisometric extension of $T_{\mathscr{N}}$ (not T!), we see that since

$$0=(T_{\mathscr{N}}-\lambda)^{n+1}\,x_n=(S^*-\lambda)^{n+1}\,Qx_n\oplus(R-\lambda)^{n+1}\,Ax_n,$$

we have $(R-\lambda)^{n+1} Ax_n=0$, and since R is unitary and $\lambda \in \mathbb{D}$, this forces $Ax_n=0$, $n \in \mathbb{N}$, and hence $x_n = Qx_n$. Thus from (112) we get

$$\begin{aligned}\|[y \otimes x_n]_T\| &= (h_n(T_{\mathcal{N}})\, P_{\mathcal{N}}\, y, x_n)\\ &= (h_n(S^*)\, QP_{\mathcal{N}}\, y, Qx_n)\\ &= \langle h_n(S^*), [QP_{\mathcal{N}}\, y \otimes Qx_n]_{S^*}\rangle\\ &\leqslant \|[QP_{\mathcal{N}}\, y \otimes Qx_n]_{S^*}\|,\end{aligned}$$

and this last quantity tends to 0 as n gets large by [12, Lemma 2.7], since $\{Qx_n\}$ tends weakly to zero.

We now present a spectral sufficient condition for membership in $\mathbb{A}_1(6)$.

THEOREM 5.3. *Suppose T is an absolutely continuous contraction in $\mathcal{L}(\mathcal{H})$ and $\Lambda = (\sigma_e(T) \cap \mathbb{D}) \cup \mathcal{F}_-(T)$ is dominating for $\mathbb{T}$. Then $T \in \mathbb{A}_1(6)$.*

Proof. That $T \in \mathbb{A}$ is clear from [5, Proposition 4.6], and we will prove the theorem by showing that Corollary 4.8 can be applied with $\theta = 0$. If $\lambda \in \sigma_e(T) \cap \mathbb{D}$, then there is a standard argument (cf., e.g., [12, Theorem 3.1]) showing that there exists an orthonormal sequence $\{x_n^\lambda\}$ in $\mathcal{H}$ satisfying $(108)_0$ and

$$\|[x_n^\lambda \otimes y]_T\| \to 0, \qquad y \in \mathcal{H}. \tag{113}$$

On the other hand, if $\lambda \in \mathcal{F}_-(T)$, then by [12, Lemma 2.2] (in the non-trivial case $i(T-\lambda)=0$), there exists an orthonormal sequence $\{x_n^\lambda\}$ in $\mathcal{H}$ satisfying

$$x_n^\lambda \in \ker(T-\lambda)^{*n+1} \ominus \ker(T-\lambda)^{*n}, \qquad n \in \mathbb{N}.$$

It follows from [12, Lemma 2.3] that

$$[x_n^\lambda \otimes x_n^\lambda]_T = [C_\lambda]_T, \qquad n \in \mathbb{N}, \tag{114}$$

and from Lemma 5.2 (applied to T^* with $\mathcal{N} = \mathcal{H}$) that

$$\|[y \otimes x_n^\lambda]_{T^*}\| \to 0, \qquad y \in \mathcal{H}. \tag{115}$$

Since $\|[y \otimes x_n^\lambda]_{T^*}\| = \|[x_n^\lambda \otimes y]_T\|$ as in (99) and (100), the proof is complete.

An immediate corollary of Theorem 5.3 is an improvement of [18, Theorem 4.1].

COROLLARY 5.4. *Suppose T is an absolutely continuous contraction in*

$\mathscr{L}(\mathscr{H})$ and there exists a dominating set $\Lambda \subset \mathbb{D}$ such that for all $\lambda \in \Lambda$, $T-\lambda$ is a Fredholm operator such that $i(T-\lambda)<0$. Then $T \in \mathbb{A}_1(6)$.

We also recapture, as a corollary of Theorem 5.3 and elementary Fredholm theory, the following modest improvement of the main result of [13] (namely, Theorem 3.3).

COROLLARY 5.5. *Suppose T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ such that $\sigma(T)=\mathbb{D}^-$ and for every $\lambda \in \mathbb{D}$, $T-\lambda$ is a semi-Fredholm operator. Then $T \in \mathbb{A}_1(6)$.*

As mentioned in the Introduction, we show, in the sequel [10] to this paper, that Theorem 4.4, together with the recent new techniques of [8], enables us to prove that every contraction T in $\mathscr{L}(\mathscr{H})$ such that $\sigma(T) \supset \mathbb{T}$ has nontrivial invariant subspaces.

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