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On the Structure of Contraction Operators. II

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1. INTRODUCTION

This paper is a continuation of [11] and uses the techniques of [8] in an essential way. We shall therefore assume that the reader is familiar with the notation and terminology of [11], which we continue to use below without extensive review. For the reader's convenience, however, we recall that $\mathscr{H}$ is a separable, infinite dimensional, complex Hilbert space, and $\mathscr{L}(\mathscr{H})$ is the algebra of all bounded linear operators on $\mathscr{H}$. Moreover, $\mathscr{C}_1(\mathscr{H}) \subset \mathscr{L}(\mathscr{H})$ is the Banach space of trace-class operators under the trace norm, $\mathbb{D}$ is the open unit disc in $\mathbb{C}$, $\mathbb{T} = \partial\mathbb{D}$, and $\mathbb{N}$ is the set of positive integers. The spaces $H^p(\mathbb{T})$ and $L^p(\mathbb{T})$, $1 \leqslant p \leqslant \infty$, are the usual Hardy and Lebesgue spaces with respect to normalized Lebesgue measure on $\mathbb{T}$. If $T \in \mathscr{L}(\mathscr{H})$ we write $\mathscr{A}_T$ for the dual algebra generated by T and Q_T for its predual $\mathscr{C}_1(\mathscr{H})/{}^\perp\mathscr{A}_T$, so $\mathscr{A}_T = Q_T^*$. Elements of Q_T are written as cosets $[L]_T$ or $[L]$, where $L \in \mathscr{C}_1(\mathscr{H})$. The class $\mathbb{A}(\mathscr{H})$ is defined to be the set of all absolutely continuous contractions T in $\mathscr{L}(\mathscr{H})$ for which the Sz.-Nagy–Foias functional calculus $\Phi_T: H^\infty(\mathbb{T}) \to \mathscr{A}_T$ is an isometry. If $T \in \mathbb{A}(\mathscr{H})$, then Φ_T is a weak* homeomorphism of $H^\infty(\mathbb{T})$ onto $\mathscr{A}_T$ and

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there exists a linear isometry φ_T of Q_T onto $L^1(\mathbb{T})/H_0^1(\mathbb{T})$, the predual of $H^\infty(\mathbb{T})$, such that $\varphi_T^* = \Phi_T$. The class $\mathbb{A}_1(\mathscr{H})$ is defined to be the set of all T in $\mathbb{A}(\mathscr{H})$ such that every element $[L]_T$ in Q_T has the form $[L]_T = [x \otimes y]_T$ for some rank-one operator $x \otimes y$ in $\mathscr{C}_1(\mathscr{H})$. With $r \geqslant 1$, the class $\mathbb{A}_1(r)$ consists of those T in $\mathbb{A}_1(\mathscr{H})$ such that for every $[L]_T$ in Q_T and every $r' > r$, $[L]_T$ can be written as $[L]_T = [x \otimes y]_T$ where the vectors x, y in $\mathscr{H}$ satisfy $\|x\| \, \|y\| \leqslant r' \|[L]_T\|$. One knows (cf. [5, Proposition 4.8]) that every T in $\mathbb{A}_1(\mathscr{H})$ has nontrivial invariant subspaces.

It is the purpose of this paper to combine the new techniques of [8] and the results of [11] to prove the following theorem.

THEOREM 1.1. *There exists $r \geqslant 6$ such that every contraction T in $\mathscr{L}(\mathscr{H})$ with $\sigma(T) \supset \mathbb{T}$ either has a nontrivial hyperinvariant subspace or belongs to the class $\mathbb{A}_1(r)$.*

COROLLARY 1.2. *Every contraction operator on Hilbert space whose spectrum contains the unit circle has a nontrivial invariant subspace.*

The results in this paper were announced in [10] and presented at the conference "Functional Analysis and Its Applications" in Nice, France, August 25–29, 1986.

2. Some Reductions

In this section we will consider a sequence of results which reduces the proof of Theorem 1.1 to more manageable proportions. For any T in $\mathscr{L}(\mathscr{H})$ we write $\sigma_l(T)$ for the left spectrum of T, $\sigma(T)$ for the spectrum of T, and

$$\zeta(T) = (\sigma_l(T) \cap \mathbb{D}) \cup \left\{ \lambda \in \mathbb{D} \setminus \sigma(T) \colon \|(T-\lambda)^{-1}\| > \frac{21}{1-|\lambda|} \right\}. \qquad (1)$$

A subset Λ of $\mathbb{D}$ is said to be *dominating* (for $\mathbb{T}$) if almost every point of $\mathbb{T}$ is a nontangential limit of a sequence of points from Λ. We consider first the following result.

THEOREM 2.1. *There exists θ satisfying $0 < \theta < 1$ such that every absolutely continuous contraction T in $\mathscr{L}(\mathscr{H})$ for which $\zeta(T)$ is dominating belongs to $\mathbb{A}_1(6/(1 - 2\theta^{1/2} + \theta))$.*

We note that Theorem 1.1 is a consequence of Theorem 2.1. Indeed, if T is a contraction in $\mathscr{L}(\mathscr{H})$ with $\sigma(T) \supset \mathbb{T}$, and T can be written as a direct sum $T = T_0 \oplus U$ where U is a unitary operator acting on some nonzero subspace, then either U is some scalar operator λ with $|\lambda| = 1$, in which case the eigenspace $\{x \in \mathscr{H}: Tx = \lambda x\}$ is a nontrivial hyperinvariant subspace for T, or else U has a nontrivial hyperinvariant subspace, in which case T also has a nontrivial hyperinvariant subspace (cf. [16, Theorem 1.4]). Thus we may suppose that T is a completely nonunitary contraction. Moreover, if $\lambda \in \sigma(T) \backslash \sigma_l(T)$, then it is easy to see that $\bar{\lambda}$ is an eigenvalue for T^*, and $\{x \in \mathscr{H}: T^*x = \bar{\lambda}x\}^\perp$ is a nontrivial hyperinvariant subspace for T, so we may further suppose that $\sigma_l(T) = \sigma(T)$. Finally, if $\zeta(T)$ is not dominating for $\mathbb{T}$, then one can apply a well-known construction (cf. [1, 7]) to obtain a function of T of the form

$$f(T) = \frac{1}{2\pi i} \int_\Gamma (T - \lambda)^{-1}(\lambda - \lambda_1)(\lambda - \lambda_2)\, d\lambda, \tag{2}$$

where Γ is a simple rectifiable closed path intersecting the unit circle at λ_1 and λ_2, with the property that the kernel of $f(T)$ is a proper nonzero subspace of $\mathscr{H}$, and one trivially verifies that this kernel is hyperinvariant for T. Since, as we have noted already, all operators in $\mathbb{A}_1$ have nontrivial invariant subspaces, and since completely nonunitary contractions are trivially absolutely continuous, this completes the proof that Theorem 2.1 implies Theorem 1.1.

Now let us see what goes into the proof of Theorem 2.1. Recall that if $T \in \mathbb{A}(\mathscr{H})$ and $\lambda \in \mathbb{D}$, then there is an element $[C_\lambda]_T$ in Q_T with the property that

$$\langle h(T), [C_\lambda]_T \rangle = h(\lambda), \qquad h \in H^\infty(\mathbb{T}). \tag{3}$$

The first ingredient that we need is a result from [11].

THEOREM 2.2 ([11, Corollary 4.8]). *Suppose* $T \in \mathbb{A}(\mathscr{H})$, $0 \leq \theta < 1$, *and* $\Lambda \subset \mathbb{D}$ *is dominating for* $\mathbb{T}$. *If for each* $\lambda \in \Lambda$ *there exists a sequence* $\{x_n\}$ *in the unit ball of* $\mathscr{H}$ *such that*

$$\overline{\lim_n} \|[C_\lambda]_T - [x_n \otimes x_n]_T\| \leq \theta \tag{4}$$

and

$$\|[w \otimes x_n]_T\| \to 0, \qquad w \in \mathscr{H}, \tag{5}$$

then $T \in \mathbb{A}_1(6/(1 - 2\theta^{1/2} + \theta))$.

We also recall from [12] that if T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ for which $\zeta(T)$ is dominating, then $T \in \mathbb{A}(\mathscr{H})$.

Given these two results, it is obvious that Theorem 2.1 is an immediate consequence of the following:

THEOREM 2.3. *There exists* θ *satisfying* $0<\theta<1$ *such that if* T *is any absolutely continuous contraction in* $\mathscr{L}(\mathscr{H})$ *with* $\zeta(T)$ *dominating, then for every* μ *in* $\mathbb{D}$ *there exists a sequence* $\{x_n\}_{n=1}^{\infty}$ *in the unit ball of* $\mathscr{H}$ *such that*

$$\|[C_\mu]_T-[x_n\otimes x_n]_T\|\leqslant\theta,\qquad n\in\mathbb{N},\tag{6}$$

and

$$\|[w\otimes x_n]_T\|\to 0,\qquad \forall w\in\mathscr{H}.\tag{7}$$

Thus, in order to prove Theorem 1.1 it suffices to prove Theorem 2.3. Our next reduction consists of establishing that it suffices to do business with the element $[C_0]_T$ of Q_T.

THEOREM 2.4. *There exists* θ *satisfying* $0<\theta<1$ *such that if* T *is any element of* $\mathbb{A}(\mathscr{H})$ *and* $\Lambda\subset\mathbb{D}$ *is any dominating set with* T *and* Λ *related by*

$$\forall\lambda\in\Lambda,\ \exists x_\lambda\in\mathscr{H}:\quad \|x_\lambda\|=1\qquad \textit{and}\qquad \|(T-\lambda)\,x_\lambda\|<\tfrac{1}{10}(1-|\lambda|),\tag{8}$$

then there exists a sequence $\{x_n\}_{n=1}^{\infty}$ *in the unit ball of* $\mathscr{H}$ *satisfying*

$$\|[C_0]_T-[x_n\otimes x_n]_T\|\leqslant\theta,\qquad \forall n\in\mathbb{N},\tag{9}$$

and

$$\|[w\otimes x_n]_T\|\to 0,\qquad \forall w\in\mathscr{H}.\tag{10}$$

Proof that Theorem 2.4 *implies Theorem* 2.3. Let θ be as in the statement of Theorem 2.4 and suppose T satisfies the hypotheses of Theorem 2.3. Then, as noted earlier, $T\in\mathbb{A}(\mathscr{H})$. Let μ be arbitrary in $\mathbb{D}$. It suffices to show that there exists a sequence $\{x_n\}_{n=1}^{\infty}$ in the unit ball of $\mathscr{H}$ satisfying (6) and (7). For this purpose, consider the Möbius transformation f_μ in $H^\infty(\mathbb{T})$ defined by

$$f_\mu(e^{it})=(e^{it}-\mu)/(1-\bar{\mu}e^{it}),\qquad e^{it}\in\mathbb{T},\tag{11}$$

and set $T_\mu=f_\mu(T)$. Since $f_{-\mu}(f_\mu(e^{it}))\equiv e^{it}$, we have $T=f_{-\mu}(T_\mu)$, so $\mathscr{A}_T=\mathscr{A}_{T_\mu}$ and $Q_T=Q_{T_\mu}$, from which it follows trivially that $[x\otimes y]_T=[x\otimes y]_{T_\mu}$ for all vectors x and y. We will continue to write f_μ for the analytic extension of f_μ to $\mathbb{D}^-$, and since this function is a homeomorphism of $\mathbb{D}^-$, we have

$$\|h(T_\mu)\|=\|(h\circ f_\mu)(T)\|=\|h\circ f_\mu\|_\infty=\|h\|_\infty,\qquad h\in H^\infty(\mathbb{T}).$$

Since T_μ is also an absolutely continuous contraction, we have $T_\mu \in \mathbb{A}(\mathscr{H})$ along with T. Moreover, it follows easily from (3) that

$$[C_\lambda]_T = [C_{f_\mu(\lambda)}]_{f_\mu(T)}, \qquad \forall \lambda \in \mathbb{D},$$

and hence, in particular, that

$$[C_\mu]_T = [C_0]_{T_\mu}.$$

Thus, (6) and (7) are equivalent to

$$\|[C_0]_{T_\mu} - [x_n \otimes x_n]_{T_\mu}\| \equiv\leqslant \theta, \qquad \forall n \in \mathbb{N}, \tag{12}$$

and

$$\|[w \otimes x_n]_{T_\mu}\| \to 0, \qquad \forall w \in \mathscr{H}. \tag{13}$$

We next define $\Lambda_\mu = \overline{f_\mu(\zeta(T))} \subset \mathbb{D}$, where the bar denotes complex conjugation, and observe that Λ_μ is dominating along with $\zeta(T)$. Thus, in order to apply Theorem 2.4 to conclude that (12) and (13) are valid, it suffices to show that the pair (T_μ, Λ_μ) satisfies (8). For this purpose, recall that $\lambda \in \sigma_l(T)$ if and only if there exists a sequence $\{y_n\}$ of unit vectors in $\mathscr{H}$ such that

$$\|(T - \lambda)\, y_n\| \to 0.$$

Using this fact and the definition of T_μ, it follows easily that

$$f_\mu(\mathbb{D} \cap \sigma_l(T)) = \mathbb{D} \cap \sigma_l(T_\mu), \tag{14}$$

and we know from the Riesz functional calculus that

$$\sigma(T_\mu) = f_\mu(\sigma(T)). \tag{15}$$

To verify (8), let $\bar{\lambda} \in \Lambda_\mu$. Then $\bar{\lambda} = f_\mu(\alpha)$ for a unique α in $\zeta(T)$, and if $\alpha \in \sigma_l(T)$, then $\bar{\lambda} \in \sigma_l(T_\mu)$ from (14), and thus there exists a sequence of unit vectors $\{z_n\}$ in $\mathscr{H}$ such that

$$\|(T_\mu - \bar{\lambda})\, z_n\| \to 0.$$

Hence one can choose x_λ in (8) equal to any z_n with n sufficiently large. If $\alpha \notin \sigma_l(T)$, then $\alpha \in \mathbb{D} \setminus \sigma(T)$ and

$$\|(T - \alpha)^{-1}\| > \frac{21}{1 - |\alpha|}. \tag{16}$$

Thus from (15) we see that $\bar{\lambda} \in \mathbb{D} \backslash \sigma(T_\mu)$, and we want to find a unit vector x_λ in $\mathscr{H}$ such that

$$\|(T_\mu - \bar{\lambda}) x_\lambda\| < \tfrac{1}{10}(1 - |\lambda|). \tag{17}$$

It is elementary to verify (cf. [20, p. 263]) that for any a in $\mathbb{D}$,

$$(1 - |a|)\|(T - a)^{-1}\| \leqslant \|(T_a)^{-1}\| \tag{18}$$

and

$$\|(T_a)^{-1}\| \leqslant 1 + 2(1 - |a|)\|(T - a)^{-1})\|, \tag{19}$$

where, of course, $T_a = f_a(T)$. From (16) and (18) (with $a = \alpha$) we obtain

$$21 < \|(T_\alpha)^{-1}\| \tag{20}$$

and from (19) (with $T = T_\mu$ and $a = \bar{\lambda}$) we obtain

$$\|((T_\mu)_{\bar{\lambda}})^{-1}\| \leqslant 1 + 2(1 - |\lambda|)\|(T_\mu - \bar{\lambda})^{-1}\|. \tag{21}$$

Therefore if we can establish that

$$\|(T_\alpha)^{-1}\| \leqslant \|((T_\mu)_{\bar{\lambda}})^{-1}\|, \tag{22}$$

then from (20) and (21) we will have that

$$21 < 1 + 2(1 - |\lambda|)\|(T_\mu - \bar{\lambda})^{-1}\|,$$

from which it will follow trivially that there exists a unit vector x_λ in $\mathscr{H}$ satisfying (17), and the proof will be complete.

To establish (22) we recall from the generalized Schwarz lemma (which is itself an easy consequence of Schwarz' lemma) that

$$\left|\frac{f_\mu(\zeta) - f_\mu(\alpha)}{1 - \overline{f_\mu(\alpha)} f_\mu(\zeta)}\right| \leqslant \left|\frac{\zeta - \alpha}{1 - \bar{\alpha}\zeta}\right|, \qquad \zeta \in \mathbb{D},$$

and therefore there exists g in $H^\infty(\mathbb{T})$ with $\|g\|_\infty \leqslant 1$ such that

$$\frac{f_\mu(\zeta) - f_\mu(\alpha)}{1 - \overline{f_\mu(\alpha)} f_\mu(\zeta)} = g(\zeta)\left(\frac{\zeta - \alpha}{1 - \bar{\alpha}\zeta}\right), \qquad \zeta \in \mathbb{D}. \tag{23}$$

If the H^∞-function on each side of (23) is applied to T, and the resulting equation is multiplied by $(T_\alpha)^{-1}$ and $((T_\mu)_{\bar{\lambda}})^{-1}$, one has

$$(T_\alpha)^{-1} = g(T)((T_\mu)_{\bar{\lambda}})^{-1},$$

from which (22) follows at once, and the proof is complete.

3. The Minimal Coisometric Extension

We saw in the preceding section that in order to establish Theorem 1.1 it suffices to prove Theorem 2.4. To accomplish this, we will employ the minimal coisometric extension B of a contraction T in $\mathbb{A}(\mathscr{H})$, as described in [20] or [15]. For the reader's convenience, we briefly summarize some properties of B. There is a separable Hilbert space $\mathscr{K} \supset \mathscr{H}$ with B in $\mathscr{L}(\mathscr{K})$ such that B^* is an isometry, $B\mathscr{H} \subset \mathscr{H}$, and $B|\mathscr{H} = T$. Moreover $\mathscr{K}$ can be decomposed as $\mathscr{K} = \mathscr{S} \oplus \mathscr{R}$ corresponding to a decomposition of B^* as $B^* = S \oplus R^*$, where S is a unilateral shift operator (of some multiplicity not exceeding $\aleph_0$) and R^* is a unitary operator. (Of course, either direct summand $\mathscr{S}$ or $\mathscr{R}$ may be the subspace (0).) That B is minimal means that

$$\{(\mathscr{H} \subset \mathscr{M} \subset \mathscr{K}) \wedge (B\mathscr{M} \subset \mathscr{M}) \wedge (B^*\mathscr{M} \subset \mathscr{M})\} \Rightarrow \mathscr{M} = \mathscr{K}. \tag{24}$$

Since $T \in \mathbb{A}(\mathscr{H})$ it follows easily that R is an absolutely continuous unitary operator (cf. [19] or [20, p. 84]) and that $B \in \mathbb{A}(\mathscr{K})$. In what follows, the projections of $\mathscr{K}$ onto $\mathscr{S}$, $\mathscr{R}$, and $\mathscr{H}$ will be denoted, respectively, by Q, A, and P. (This terminology is suggested by [15], where pretty geometric constructions of the minimal coisometric extension and minimal unitary dilation of a contraction are given.) Thus, for every x in $\mathscr{H}$ and every h in $H^\infty(\mathbb{T})$, we have

$$h(T)x = h(B)x = h(S^*)(Qx) \oplus h(R)(Ax) = Q(h(T)x) \oplus A(h(T)x). \tag{25}$$

The first result that we shall need on the way to proving Theorem 2.4 is

Proposition 3.1. *Suppose T belongs to $\mathbb{A}(\mathscr{H})$ and has minimal coisometric extension $B = S^* \oplus R$ in $\mathscr{L}(\mathscr{S} \oplus \mathscr{R})$. Suppose also that $\Lambda \subset \mathbb{D}$ is a dominating set such that the pair (T, Λ) satisfies (8). Then $\mathscr{S} \neq (0)$ and for every λ in Λ there exists a unit vector e_λ in $\ker(S-\lambda)^*$ such that*

$$\|x_\lambda - e_\lambda\| < 0.101. \tag{26}$$

Moreover, for every n in $\mathbb{N}$ the family $\{e_\lambda\}_{\lambda \in \Lambda}$ of these unit vectors satisfies

$$\varliminf_{\substack{|\lambda| \to 1 \\ \lambda \in \Lambda}} \|PS^n e_\lambda\|^2 > 1 - (0.101)^2. \tag{27}$$

Proof. If $\mathscr{S} = (0)$, then T is an isometry, which is incompatible with an equation of the form

$$\|(T-\lambda)x_\lambda\| < \tfrac{1}{10}(1-|\lambda|),$$

where x_λ is a unit vector; thus $\mathscr{S} \neq (0)$. Now fix λ in Λ, define $\beta = \frac{1}{10}$, and let x_λ be a unit vector in $\mathscr{H}$ satisfying

$$\|(T-\bar{\lambda})\, x_\lambda\| < \beta(1-|\lambda|). \tag{28}$$

Then we have

$$x_\lambda = Qx_\lambda \oplus Ax_\lambda, \qquad 1 = \|Qx_\lambda\|^2 + \|Ax_\lambda\|^2. \tag{29}$$

Moreover, using (28) we see that the Möbius transform $T_\lambda = f_\lambda(T)$ satisfies

$$\|f_\lambda(T)\, x_\lambda\| = \|(1-\lambda T)^{-1}(T-\bar{\lambda})\, x_\lambda\| \leqslant (1-|\lambda|)^{-1}\, \|(T-\bar{\lambda})\, x_\lambda\| < \beta,$$

and, using (25), we obtain

$$f_\lambda(T)\, x_\lambda = f_\lambda(B)\, x_\lambda = f_\lambda(S^*)(Qx_\lambda) \oplus f_\lambda(R)(Ax_\lambda)$$

and

$$\|f_\lambda(S^*)(Qx_\lambda)\|^2 + \|Ax_\lambda\|^2 < \beta^2, \tag{30}$$

since $f_\lambda(R)$ is a unitary operator. Thus, in particular,

$$\|Ax_\lambda\|^2 < \beta^2, \qquad f_\lambda(S^*)(Qx_\lambda)\|^2 < \beta^2, \qquad 1-\beta^2 < \|(Qx_\lambda)\|^2. \tag{31}$$

Let y_λ in $\mathscr{S}$ be defined as the projection of Qx_λ onto $\ker(f_\lambda(S))^* = \ker(S-\lambda)^*$, so $Qx_\lambda - y_\lambda \in \mathscr{S} \ominus \ker(S-\lambda)^*$ and

$$\|y_\lambda\|^2 + \|Qx_\lambda - y_\lambda\|^2 = \|Qx_\lambda\|^2. \tag{32}$$

Since $f_\lambda(S^*) = (f_\lambda(S))^*$ and (as is easily seen) $f_\lambda(S)$ is an isometry with S, one deduces that

$$\operatorname{dist}(\ker(f_\lambda(S))^*, Qx_\lambda) = \|f_\lambda(S^*)\, Qx_\lambda\|.$$

Thus

$$\|Qx_\lambda - y_\lambda\| = \|f_\lambda(S^*)\, Qx_\lambda\|, \tag{33}$$

and in view of (29), (33), and (30), we have

$$\|x_\lambda - y_\lambda\|^2 = \|Qx_\lambda - y_\lambda\|^2 + \|Ax_\lambda\|^2 < \beta^2. \tag{34}$$

In particular, (34) shows that y_λ cannot be zero, so we define the sought for unit vector e_λ in $\ker(S-\lambda)^*$ as $e_\lambda = y_\lambda / \|y_\lambda\|$. Since $e_\lambda \in \ker(S-\lambda)^*$, we obtain from the obvious orthogonality relations the equations

$$\|x_\lambda - e_\lambda\|^2 = \|Qx_\lambda - e_\lambda\|^2 + \|Ax_\lambda\|^2 \tag{35}$$

and

$$\|Qx_\lambda - e_\lambda\|^2 = \|Qx_\lambda - y_\lambda\|^2 + \|y_\lambda - e_\lambda\|^2. \tag{36}$$

Combining these equations with (34) gives

$$\|x_\lambda - e_\lambda\|^2 = \|y_\lambda - e_\lambda\|^2 + \|Qx_\lambda - y_\lambda\|^2 + \|Ax_\lambda\|^2 < \|y_\lambda - e_\lambda\|^2 + \beta^2. \tag{37}$$

Furthermore, since $y_\lambda = \|y_\lambda\| e_\lambda$ and $\|y_\lambda\| \leqslant 1$, we have

$$\|y_\lambda - e_\lambda\| = 1 - \|y_\lambda\|.$$

Therefore, from (37) and (32), we obtain

$$\begin{aligned}
\|x_\lambda - e_\lambda\|^2 &= 1 - 2\|y_\lambda\| + \|y_\lambda\|^2 + \|Ax_\lambda\|^2 + \|Qx_\lambda - y_\lambda\|^2 \\
&= 1 - 2(\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2)^{1/2} + (\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2) \\
&\quad + \|Ax_\lambda\|^2 + \|Qx_\lambda - y_\lambda\|^2 \\
&= \|Ax_\lambda\|^2 + \|Qx_\lambda\|^2 + 1 - 2(\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2)^{1/2} \\
&= 2(1 - (\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2)^{1/2}) \\
&= \frac{2(1 - (\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2))}{1 + (\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2)^{1/2}} \\
&= \frac{2(\|Ax_\lambda\|^2 + \|Qx_\lambda - y_\lambda\|^2)}{1 + (\|Qx_\lambda\|^2 - \|Qx_\lambda - y_\lambda\|^2)^{1/2}}.
\end{aligned}$$

We next apply (34) twice and (31) to the right-hand side of this last equation to yield

$$\|x_\lambda - e_\lambda\|^2 < \frac{2\beta^2}{1 + ((1-\beta^2) - \beta^2)^{1/2}} = \frac{2\beta^2}{1 + (1 - 2\beta^2)^{1/2}},$$

and that (26) is valid now follows from an arithmetical calculation which shows (with $\beta = 0.1$) that

$$\gamma_0 = \left(\frac{2\beta^2}{1 + (1 - 2\beta^2)^{1/2}}\right)^{1/2} < 0.101. \tag{38}$$

In other words we have now constructed, for each λ in Λ, a unit vector e_λ in $\ker(S-\lambda)^*$ such that, with x_λ as in (8),

$$\|x_\lambda - e_\lambda\| < \gamma_0 < 0.101.$$

To establish (27), we consider, for any fixed n in $\mathbb{N}$ and arbitrary λ in Λ, the inequality

$$\begin{aligned}\|(1-P)(\bar{\lambda}^n S^n e_\lambda)\| &= \|P\bar{\lambda}^n S^n e_\lambda - \bar{\lambda}^n S^n e_\lambda\| \\ &\leqslant \|P(\bar{\lambda}^n S^n e_\lambda - e_\lambda)\| + \|Pe_\lambda - e_\lambda\| + \|e_\lambda - \bar{\lambda}^n S^n e_\lambda\| \\ &\leqslant 2\|\bar{\lambda}^n S^n e_\lambda - e_\lambda\| + \|Pe_\lambda - e_\lambda\|. \qquad (39)\end{aligned}$$

Since $x_\lambda \in \mathscr{H}$ and P is the projection onto $\mathscr{H}$,

$$\|Pe_\lambda - e_\lambda\| \leqslant \|x_\lambda - e_\lambda\| < \gamma_0. \qquad (40)$$

Furthermore, since S is a unilateral shift (of some multiplicity), the eigenvectors of S^* corresponding to the eigenvalue $\bar{\lambda}$ can be computed explicitly (cf. the proof of Proposition 5.1), and an easy calculation shows that

$$\|\bar{\lambda}^n S^n e_\lambda - e_\lambda\| = (1-|\lambda|^2)^{1/2}\left(\sum_{i=0}^{n-1} |\lambda|^{2i}\right)^{1/2}. \qquad (41)$$

Therefore, for n fixed and an arbitrary λ in Λ, we obtain from (39), (40), and (41),

$$\|(1-P)\,\bar{\lambda}^n S^n e_\lambda\| < \gamma_0 + 2(1-|\lambda|^2)^{1/2}\left(\sum_{i=0}^{n-1} |\lambda|^{2i}\right)^{1/2},$$

from which it follows immediately that

$$\overline{\lim_{\substack{|\lambda|\to 1\\ \lambda\in\Lambda}}}\ \|(1-P)\,S^n e_\lambda\| \leqslant \gamma_0 < 0.101.$$

Since

$$1 = \|S^n e_\lambda\|^2 = \|PS^n e_\lambda\|^2 + \|(1-P)\,S^n e_\lambda\|^2, \qquad \forall \lambda \in \Lambda,$$

we obtain, finally,

$$\begin{aligned}\underline{\lim_{\substack{|\lambda|\to 1\\ \lambda\in\Lambda}}}\ \|PS^n e_\lambda\|^2 &\geqslant 1 - \overline{\lim_{\substack{|\lambda|\to 1\\ \lambda\in\Lambda}}}\ \|(1-P)\,S^n e_\lambda\|^2 \\ &\geqslant 1-\gamma_0^2 > 1-(0.101)^2, \qquad (42)\end{aligned}$$

and the proof of Proposition 3.1 is complete.

4. The minimal Unitary Dilation

In our program to prove Theorem 2.4, and therefore Theorem 1.1, we have gone as far as we can go using only the minimal coisometric extension of an absolutely continuous contraction. From here on, we shall need the minimal unitary dilation of such a contraction, and thus we now make some remarks about this concept.

If $\mathscr{D}$ is a (separable, complex) Hilbert space, then the Hilbert space $L^2(\mathscr{D})$ consists of all those (equivalence classes of) measurable functions $g: \mathbb{T} \to \mathscr{D}$ that are square integrable with respect to normalized Lebesgue measure on $\mathbb{T}$. The inner product on $L^2(\mathscr{D})$ is, of course, given by

$$(g_1, g_2) = \frac{1}{2\pi} \int_0^{2\pi} (g_1(e^{it}), g_2(e^{it}))_{\mathscr{D}}\, dt.$$

The Hardy space $H^2(\mathscr{D})$ is the subspace of $L^2(\mathscr{D})$ consisting of those functions g whose Fourier coefficients

$$c_n(g) = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} g(e^{it})\, dt$$

vanish for all negative integers n. The operator $U = M_{e^{it}}$ of multiplication by the position function on $L^2(\mathscr{D})$ is a unitary operator which is a bilateral shift, and the restriction $\tilde{S}$ of U to the invariant subspace $H^2(\mathscr{D})$ is a unilateral shift operator of multiplicity $\dim(\mathscr{D})$. Moreover, every unilateral shift operator S is unitarily equivalent to such a multiplication operator $\tilde{S}$. Therefore, if T is an absolutely continuous contraction in $\mathscr{L}(\mathscr{H})$ and $B = S^* \oplus R$ is its minimal coisometric extension in $\mathscr{L}(\mathscr{K})$, where $\mathscr{K} = \mathscr{S} \oplus \mathscr{R}$ with $\mathscr{S} \neq (0)$, then we may identify $\mathscr{S}$ with $H^2(\mathscr{D})$ for some Hilbert space $\mathscr{D} \neq (0)$, and S with the restriction $\tilde{S} = U | H^2(\mathscr{D})$. This means that $\mathscr{K}$ becomes identified with $H^2(\mathscr{D}) \oplus \mathscr{R}$ and S^* becomes identified with the compression

$$(U^*)_{H^2(\mathscr{D})} = \mathrm{Proj}_{H^2(\mathscr{D})} U^* | H^2(\mathscr{D})$$

of the unitary operator $U^* = M_{e^{-it}}$ to the semi-invariant subspace $H^2(\mathscr{D})$. Thus $\mathscr{H}$ becomes identified with a subspace of $H^2(\mathscr{D}) \oplus \mathscr{R}$ that is invariant under $B = (U^*)_{H^2(\mathscr{D})} \oplus R$. The operator $W = U^* \oplus R$ acting on $\mathscr{W} = L^2(\mathscr{D}) \oplus \mathscr{R} \supset \mathscr{H}$ is the *minimal unitary dilation* of T (cf. [20]), and is, of course, an absolutely continuous unitary operator. It is easy to see that $\mathscr{H} \subset \mathscr{W}$ is the difference of the invariant subspaces $(L^2(\mathscr{D}) \ominus H^2(\mathscr{D})) \oplus \mathscr{H}$ and $L^2(\mathscr{D}) \ominus H^2(\mathscr{D})$ for W, and thus $\mathscr{H}$ is a semi-invariant subspace for W. Henceforth we shall write Q, A, and P for the projections of $\mathscr{W}$ onto the subspaces $H^2(\mathscr{D})$, $\mathscr{R}$, and $\mathscr{H}$, respectively. Thus for any h in $H^\infty(\mathbb{T})$ we have

$$h(T) = Ph(W)|\mathscr{H}, \qquad h(B) = (Q + A)\, h(W)|\mathscr{K}. \tag{43}$$

We will need the following lemma.

LEMMA 4.1. *Suppose T belongs to $\mathbb{A}(\mathscr{H})$ and has minimal unitary dilation $W = U^* \oplus R$ acting on $\mathscr{W} = L^2(\mathscr{D}) \oplus \mathscr{R}$, where $\mathscr{D} \neq (0)$. Then $W \in \mathbb{A}(\mathscr{W})$ and for each pair of vectors x, y in $\mathscr{H}$,*

$$\|[C_0]_T - [x \otimes y]_T\| = \|[C_0]_W - [x \otimes y]_W\|. \tag{44}$$

Proof. Since $T \in \mathbb{A}(\mathscr{H})$, one has $\|h(T)\| = \|h\|_\infty$ for all h in $H^\infty(\mathbb{T})$, and since $\|h(W)\| \geq \|h(T)\|$ from (43), it is immediate that $W \in \mathbb{A}(\mathscr{W})$. The validity of (44) follows from the calculation

$$\begin{aligned}
\|[C_0]_T - [x \otimes y]_T\| &= \sup_{\substack{h \in H^\infty \\ \|h\| = 1}} |\langle h(T), [C_0]_T - [x \otimes y]_T \rangle| \\
&= \sup_{\substack{h \in H^\infty \\ \|h\| = 1}} |h(0) - (h(T)\, x, y)| \\
&= \sup_{\substack{h \in H^\infty \\ \|h\| = 1}} |h(0) - (h(W)\, x, y)| \\
&= \sup_{\substack{h \in H^\infty \\ \|h\| = 1}} |\langle h(W), [C_0]_W - [x \otimes y]_W \rangle| \\
&= \|[C_0]_W - [x \otimes y]_W\|.
\end{aligned}$$

Before we can state the next proposition, we need an additional definition. Quantifying expressions such as "$e^{it} \in \mathbb{T}$" are frequently to be interpreted as "almost everywhere on $\mathbb{T}$."

DEFINITION 4.2. Let $F_0 = 1 - (0.15)^2$, a slightly smaller number than that appearing on the right-hand side of (27). If $T \in \mathbb{A}(\mathscr{H})$ and has minimal coisometric extension $B = S^* \oplus R$ on $H^2(\mathscr{D}) \oplus \mathscr{R}$ and minimal unitary dilation $W = U^* \oplus R$ on $\mathscr{W} = L^2(\mathscr{D}) \oplus \mathscr{R}$, we write, for each positive integer n, $\Sigma_n(T)$ for the set of all vectors x in the unit ball of $H^2(\mathscr{D})$ such that

$$\|PS^n x\|^2 \geq F_0 \|x\|^2 \tag{45}$$

and such that there exists a decomposition of x as $x = x_l + x_h$, where x_l, x_h belong to $L^2(\mathscr{D})$ and satisfy

$$\|x_l(e^{it})\| \leqslant 1, \qquad e^{it} \in \mathbb{T}, \tag{46}$$

and

$$\|x_h\|^2 \leqslant (1 - F_0)\|x_l\|^2. \tag{47}$$

PROPOSITION 4.3. *If T is any operator in $\mathbb{A}(\mathscr{H})$ (with minimal coisometric extension $B = S^* \oplus R$ in $\mathscr{L}(H^2(\mathscr{D}) \oplus \mathscr{R})$ and minimal unitary dilation $W = U^* \oplus R$ in $\mathscr{L}(H^2(\mathscr{D}) \oplus \mathscr{R})$) and $\Lambda \subset \mathbb{D}$ is any dominating set with T and Λ related by* (8), *then*

$$\|[C_0]_T - [PS^n x \otimes PS^n x]_T\| \leqslant 1 - (0.02)\|x\|^2 \tag{48}$$

for every positive integer n and every x in $\Sigma_n(T)$.

Proof. Since T and Λ are related by (8), Proposition 3.1 applies, and, in particular, $\mathscr{D} \neq (0)$. Let n be an arbitrary positive integer, which will remain fixed for the duration of the proof. Suppose now that $x \in \Sigma_n(T)$. Then we may write

$$PS^n x = \alpha S^n x + z, \tag{49}$$

where z is orthogonal to $S^n x$. Using (45), we obtain easily that

$$F_0 \leqslant \alpha \leqslant 1 \tag{50}$$

and that

$$\|z\| = (\|PS^n x\|^2 - \alpha^2 \|x\|^2)^{1/2} \leqslant (1 - F_0^2)^{1/2} \|x\|. \tag{51}$$

Moreover, since $x \in \Sigma_n(T)$, we may write $x = x_l + x_h$, where $x_l, x_h \in L^2(\mathscr{D})$ and satisfy (46) and (47). Therefore

$$\|x_h\| \leqslant (1 - F_0)^{1/2}\|x_l\|, \qquad \|x\| \leqslant (1 + (1 - F_0)^{1/2})\|x_l\|. \tag{52}$$

We introduce the vectors $y_1 = \alpha U^n x_l$ and $y_2 = \alpha U^n x_h + z$. Then, upon recalling that $U|H^2(\mathscr{D}) = S$, we obtain

$$y_1 + y_2 = \alpha U^n (x_l + x_h) + z = \alpha S^n x + z = PS^n x, \tag{53}$$

$$\|y_2\| \leqslant \|z\| + \alpha\|x_h\| \leqslant \|z\| + (1 - F_0)^{1/2} \alpha \|x_l\| = \|z\| + (1 - F_0)^{1/2} \|y_1\|, \tag{54}$$

and (from (51) and (52))

$$\begin{aligned}\|z\| &\leqslant (1 - F_0^2)^{1/2}\|x\| \leqslant (1 - F_0^2)^{1/2}\{1 + (1 - F_0)^{1/2}\}\|x_l\| \\ &\leqslant \frac{1}{F_0}(1 - F_0)^{1/2}\{1 + (1 - F_0)^{1/2}\}\|y_1\|.\end{aligned} \tag{55}$$

Furthermore, upon combining (54) and (55), we have

$$\|y_2\| \leqslant (1-F_0)^{1/2}\left\{1+\frac{(1+F_0)^{1/2}\{1+(1-F_0)^{1/2}\}}{F_0}\right\}\|y_1\|,$$

so we define

$$\rho_0=(1-F_0)^{1/2}\left\{1+\frac{(1+F_0)^{1/2}\{1+(1-F_0)^{1/2}\}}{F_0}\right\}\approx 0.39815931 \quad (56)$$

and obtain

$$\|y_2\| \leqslant \rho_0\|y_1\|, \qquad \|PS^n x\| \leqslant \|y_1\|+\|y_2\| \leqslant (1+\rho_0)\|y_1\|. \quad (57)$$

Since $PS^n x \in \mathscr{H}$, we obtain from Lemma 4.1, (53), and (57) that

$$\begin{aligned}\|[C_0]_T-[PS^n x\otimes PS^n x]_T\| &= \|[C_0]_W-[(y_1+y_2)\otimes(y_1+y_2)]_W\| \\ &\leqslant \|[C_0]_W-[y_1\otimes y_1]_W\| \\ &\quad +2\|y_1\|\,\|y_2\|+\|y_2\|^2 \\ &\leqslant \|[C_0]_W-[y_1\otimes y_1]_W\|+(2\rho_0+\rho_0^2)\|y_1\|^2.\end{aligned} \quad (58)$$

Now let us estimate $\|[C_0]_W-[y_1\otimes y_1]_W\|$, keeping in mind that $W\in\mathbb{A}(\mathscr{W})$, $y_1\in L^2(\mathscr{D})$, and

$$\|y_1(e^{it})\| = \alpha\|e^{int}x_l(e^{it})\| \leqslant \|x_l(e^{it})\| \leqslant 1, \qquad e^{it}\in\mathbb{T}:$$

$$\begin{aligned}\|[C_0]_W-[y_1\otimes y_1]_W\| &= \sup_{\substack{h\in H^\infty\\ \|h\|=1}} |\langle h(W), [C_0]_W-[y_1\otimes y_1]_W\rangle| \\ &= \sup_{\substack{h\in H^\infty\\ \|h\|=1}} |h(0)-(h(W)y_1, y_1)| \\ &= \sup_{\substack{h\in H^\infty\\ \|h\|=1}} |h(0)-(h(U^*)y_1, y_1)| \\ &= \sup_{\substack{h\in H^\infty\\ \|h\|=1}} \left|\frac{1}{2\pi}\int_0^{2\pi}\{h(e^{-it})-h(e^{-it})\|y_1(e^{it})\|^2\}\,dt\right| \\ &\leqslant \frac{1}{2\pi}\int_0^{2\pi}(1-\|y_1(e^{it})\|^2)\,dt = 1-\|y_1\|^2.\end{aligned} \quad (59)$$

Therefore, noting that $1-2\rho_0-\rho_0^2$ (≈ 0.04515058) >0, and combining (58), (59), (57), and (45), we obtain

$$\begin{aligned}\|[C_0]_T-[PS^nx\otimes PS^nx]_T\| &\leqslant 1-(1-2\rho_0-\rho_0^2)\|y_1\|^2\\ &\leqslant 1-\left(\frac{1-2\rho_0-\rho_0^2}{(1+\rho_0)^2}\right)\|PS^nx\|^2\\ &\leqslant 1-\left(\frac{1-2\rho_0-\rho_0^2}{(1+\rho_0)^2}\right)F_0\|x\|^2.\end{aligned}$$

The proof can now be completed by doing the arithmetical calculation which shows that

$$0.02<\frac{1-2\rho_0-\rho_0^2}{(1+\rho_0)^2}F_0.$$

5. An Estimate on the Numbers $\sigma_n(T)$

In this section we shall finally complete the proof of Theorem 2.4 (and therefore also of Theorem 1.1). If T is contraction in $\mathbb{A}(\mathscr{H})$ with minimal unitary dilation $W=U^*\oplus R$ acting on $\mathscr{W}=L^2(\mathscr{D})\oplus\mathscr{R}\supset\mathscr{H}$, then associated with T is the sequence $\{\Sigma_n(T)\}_{n=1}^{\infty}$ of subsets of $H^2(\mathscr{D})$ given by Definition 4.2. For such T, we now define

$$\sigma_n(T)=\sup\{\|x\|^2: x\in\Sigma_n(T)\},\qquad n\in\mathbb{N}.$$

It is obvious from the definition that $0\in\Sigma_n(T)$ for every n in $\mathbb{N}$, so $\sigma_n(T)\geqslant 0$. Our first proposition shows that under the hypotheses on T with which we have been working, these numbers $\sigma_n(T)$ are all positive.

Proposition 5.1. *Suppose $T\in\mathbb{A}(\mathscr{H})$ and satisfies the hypotheses of Proposition 4.3. Then $\sigma_n(T)>0$ for every positive integer n.*

Proof. Fix an arbitrary positive integer n. Since T satisfies the hypotheses of Proposition 3.1, we know that

$$\varliminf_{\substack{|\lambda|\to 1\\ \lambda\in\Lambda}}\|PS^ne_\lambda\|^2>F_1=1-(0.101)^2,$$

where $\{e_\lambda\}_{\lambda\in\Lambda}$ is a family of unit vectors satisfying $e_\lambda\in\ker(S-\lambda)^*$. Choose $r_n\in(.9,1)$ such that if $\lambda\in\Lambda$ and $|\lambda|>r_n$, then $\|PS^ne_\lambda\|^2>F_1$, and write

$$\Omega_n=\Lambda\cap\{\zeta\in\mathbb{D}: |\zeta|>r_n\}. \tag{60}$$

Then clearly Ω_n is a dominating set and

$$\|PS^n e_\lambda\|^2 > F_1 \|e_\lambda\|^2, \qquad \forall \lambda \in \Omega_n. \tag{61}$$

Since $S^* e_\lambda = \bar{\lambda} e_\lambda$ for each λ in Λ, the vector e_λ in $\mathscr{S} = H^2(\mathscr{D})$ can be calculated explicitly to be

$$e_\lambda(e^{it}) = (1 - |\lambda|^2)^{1/2}(d + \bar{\lambda} e^{it} d + \bar{\lambda}^2 e^{i2t} d + \cdots), \qquad e^{it} \in \mathbb{T},$$

where d is some unit vector in $\mathscr{D}$, and therefore

$$\|e_\lambda(e^{it})\|^2 = \frac{1 - |\lambda|^2}{|1 - \bar{\lambda} e^{it}|^2} = P_{\bar{\lambda}}(e^{it}) \leqslant \frac{1 + |\lambda|}{1 - |\lambda|}, \qquad e^{it} \in \mathbb{T},$$

where $P_{\bar{\lambda}}$ is the Poisson kernel at the point $\bar{\lambda}$. Thus

$$\left\| \left(\frac{1 - |\lambda|}{1 + |\lambda|} \right)^{1/2} e_\lambda(e^{it}) \right\| \leqslant 1, \qquad e^{it} \in \mathbb{T}, \tag{62}$$

and using (61) we see that $((1 - |\lambda|)/(1 + |\lambda|))^{1/2} e_\lambda \in \Sigma_n(T)$ for every λ in Ω_n. In particular,

$$\sigma_n(T) \geqslant \left(\frac{1 - |\lambda|}{1 + |\lambda|} \right), \qquad \forall \lambda \in \Omega_n,$$

so the proof is complete.

The next proposition will be needed in what is to follow.

PROPOSITION 5.2. *Consider the topological spaces $L^\infty(\mathbb{T})$ with its weak* topology and $L^1(\mathbb{T})$ with its weak topology. Let $\mathscr{B}$ be the closed unit ball in $L^\infty(\mathbb{T})$ and let j be the natural embedding of $L^\infty(\mathbb{T})$ into $L^1(\mathbb{T})$. Then j is a homeomorphism between $\mathscr{B}$ and $j(\mathscr{B})$ (where $\mathscr{B}$ and $j(\mathscr{B})$ are given the relative topologies as subspaces of $L^\infty(\mathbb{T})$ and $L^1(\mathbb{T})$), and, in particular, $j(\mathscr{B})$ is compact and metrizable.*

Proof. If $\{g_\alpha\}$ is a net in $\mathscr{B}$ converging weak* to an element g_0 in $\mathscr{B}$, then it is trivial to verify that $j(g_\alpha)$ converges weakly to $j(g_0)$ in $L^1(\mathbb{T})$. Thus j is a continuous, one-to-one map from the compact space $\mathscr{B}$ onto the Hausdorff space $j(\mathscr{B})$, and all such maps are homeomorphisms. Since $L^\infty(\mathbb{T})$ is the dual space of a separable space, $\mathscr{B}$ is (weak*) metrizable, and thus $j(\mathscr{B})$ is also compact and (weakly) metrizable.

We are finally ready to establish the result from which Theorem 2.4 follows easily.

THEOREM 5.3. *Suppose T is any operator in $\mathbb{A}(\mathscr{H})$ (with minimal coisometric extension $B = S^* \oplus R$ in $\mathscr{L}(H^2(\mathscr{D}) \oplus \mathscr{R})$ and minimal unitary dilation $W = U^* \oplus R$ in $\mathscr{L}(L^2(\mathscr{D}) \oplus \mathscr{R})$), and $\Lambda \subset \mathbb{D}$ is any dominating set with T and Λ related by* (8). *Then*

$$\sigma_n(T) > \frac{1}{(10)^8} \qquad \forall n \in \mathbb{N}.$$

Proof. We fix an arbitrary positive integer n, write, for brevity, $\sigma_n = \sigma_n(T)$, and choose a sequence $\{x_j\}_{j=1}^{\infty}$ of nonzero vectors in $\Sigma_n(T)$ such that

$$\|x_j\|^2 \nearrow \sigma_n. \tag{63}$$

By definition of $\Sigma_n(T)$, we may write

$$x_j = x_j^l + x_j^h, \qquad j \in \mathbb{N},$$

where x_j^l and x_j^h belong to $L^2(\mathscr{D})$ and satisfy (46) and (47). We next define

$$G_j = \{e^{it} \in \mathbb{T} : \|x_j^l(e^{it})\| < 1/2\}, \qquad j \in \mathbb{N}. \tag{64}$$

(Of course, the sets G_j are only determined up to a set of measure zero, but this will cause no difficulties.)

We wish now to drop down to a subsequence of positive integers along which five sequences are converging weakly in $L^2(\mathscr{D})$ or $L^1(\mathbb{T})$. To do this, we note from (46) that all of the vector-valued functions x_j^l belong to the unit ball of $L^2(\mathscr{D})$, as do all of the functions $\chi_{G_j} x_j^l : e^{it} \to \chi_{G_j}(e^{it})\, x_j^l(e^{it})$. Moreover, by virtue of (47), all of the functions x_j^h, as well as the $\chi_{G_j} x_j^h$, lie in the ball of $L^2(\mathscr{D})$ of radius $(1 - F_0)^{1/2}$. Furthermore, if we denote by $\|x_j^l(\cdot)\|^2$ the function

$$e^{it} \to \|x_j^l(e^{it})\|_{\mathscr{D}}^2, \qquad j \in \mathbb{N},$$

then it follows from (46) that these functions are in the unit ball of $L^{\infty}(\mathbb{T})$, and so Proposition 5.2 applies. Therefore, by dropping down to five successive subsequences, and then changing the notation, we may suppose that there exist vectors x_b, y, x_a, w in $L^2(\mathscr{D})$ and a function g in $L^1(\mathbb{T})$ such that

$$\{x_j^l\} \quad \text{converges weakly to } x_b \quad \text{in } L^2(\mathscr{D}), \tag{65}$$

$$\{\chi_{G_j} x_j^l\} \quad \text{converges weakly to } y \quad \text{in } L^2(\mathscr{D}), \tag{66}$$

$$\{x_j^h\} \quad \text{converges weakly to } x_a \quad \text{in } L^2(\mathscr{D}), \tag{67}$$

$$\{\chi_{G_j} x_j^h\} \quad \text{converges weakly to } w \quad \text{in } L^2(\mathscr{D}), \tag{68}$$

and

$$\{\|x_j^l(\cdot)\|^2\} \qquad \text{converges weakly to } g \qquad \text{in } L^1(\mathbb{T}). \tag{69}$$

We next define $x = x_b + x_a$ and note that since x is the weak limit of the sequence $\{x_j\}$ of vectors in $H^2(\mathscr{D})$, x also belongs to $H^2(\mathscr{D})$. With Q and P the projections of $\mathscr{W} = L^2(\mathscr{D}) \oplus \mathscr{R}$ onto $H^2(\mathscr{D})$ and $\mathscr{H}$, respectively, we introduce five nonnegative functions in $L^1(\mathbb{T})$ defined, for $e^{it} \in \mathbb{T}$, by

$$\begin{aligned}
f_1(e^{it}) &= \|(Qx_b)(e^{it})\|_{\mathscr{S}}^2,\\
f_2(e^{it}) &= \|(Qy)(e^{it})\|_{\mathscr{S}}^2,\\
f_3(e^{it}) &= \|(S^{*n}QPS^n x)(e^{it})\|_{\mathscr{S}}^2,\\
s_1(e^{it}) &= \|(Qw)(e^{it})\|_{\mathscr{S}}^2,\\
s_2(e^{it}) &= \|(Qx_a)(e^{it})\|_{\mathscr{S}}^2.
\end{aligned} \tag{70}$$

We wish to estimate $\|k\|_1$ where $k = f_1 + f_2 + f_3 + s_1 + s_2 + g$, and we first observe that

$$\begin{aligned}
\|k\|_1 &= \frac{1}{2\pi}\int_0^{2\pi} k(e^{it})\,dt\\
&= \|f_1\|_1 + \cdots + \|g\|_1.
\end{aligned} \tag{71}$$

From (69) and a variation of (52) we get immediately

$$\begin{aligned}
\|g\|_1 &= \frac{1}{2\pi}\int_0^{2\pi} 1\cdot g(e^{it})\,dt = \lim_j \frac{1}{2\pi}\int_0^{2\pi} 1\cdot \|x_j^l(e^{it})\|_{\mathscr{S}}^2\,dt\\
&= \lim_j \|x_j^l\|^2 \leqslant \lim_j \left(\frac{\|x_j\|}{1-(1-F_0)^{1/2}}\right)^2 \leqslant \frac{\sigma_n}{(1-(1-F_0)^{1/2})^2}.
\end{aligned} \tag{72}$$

Also, using (72), (52), and the fact that closed balls in $L^2(\mathscr{D})$ are weakly closed, we obtain

$$\|f_1\|_1 = \|Qx_b\|^2 \leqslant \|x_b\|^2 \leqslant \lim_j \|x_j^l\|^2 \leqslant \frac{\sigma_n}{(1-(1-F_0)^{1/2})^2},$$

$$\|f_2\|_1 = \|Qy\|^2 \leqslant \|y\|^2 \leqslant \varliminf_j \|\chi_{G_j} x_j^l\|^2 \leqslant \lim_j \|x_j^l\|^2 \leqslant \frac{\sigma_n}{(1-(1-F_0)^{1/2})^2},$$

$$\|s_1\|_1 = \|Qw\|^2 \leqslant \|w\|^2 \leqslant \varliminf_j \|\chi_{G_j} x_j^h\|^2 \leqslant \varliminf_j \|x_j^h\|^2 \leqslant \frac{(1-F_0)\,\sigma_n}{(1-(1-F_0)^{1/2})^2},$$

and

$$\|s_2\|_1 = \|Qx_a\|^2 \leqslant \|x_a\|^2 \leqslant \varliminf_j \|x_j^h\|^2 \leqslant \frac{(1-F_0)\,\sigma_n}{(1-(1-F_0)^{1/2})^2}.$$

Finally, since $\{x_j\}$ converges weakly to x, we get

$$\|f_3\|_1 = \|S^{*n}QPS^n x\|^2 \leqslant \|x\|^2 \leqslant \lim_j \|x_j\|^2 = \sigma_n.$$

From these inequalities we obtain

$$\|k\|_1 \leqslant \delta\sigma_n, \tag{73}$$

where δ is defined to be the positive number

$$\delta = 1 + \frac{5-2F_0}{(1-(1-F_0)^{1/2})^2} = 1 + \frac{3.045}{0.7225}.$$

Note that $\delta < 6$, so (by Proposition 5.1) $\delta\sigma_n < 6\sigma_n$, and set

$$E = \{e^{it} \in \mathbb{T} : k(e^{it}) < 6\sigma_n\}.$$

It follows immediately from (73) that E has positive measure, and, of course, each of the six functions $f_1, f_2, f_3, s_1, s_2, g$ is strictly less than $6\sigma_n$ on E. Since the vector-valued functions Qx_a, Qx_b, Qw, Qy, and $S^{*n}QPS^n x$ all belong to $H^2(\mathscr{D})$, one knows (cf. [20, p. 186]) that there exist holomorphic functions $\widetilde{Qx_a}$, $\widetilde{Qx_b}$, $\widetilde{Qw}$, $\widetilde{Qy}$, and $\widetilde{S^{*n}QPS^n x}$ on $\mathbb{D}$ taking values in $\mathscr{D}$ with the property that each of the five holomorphic functions has a nontangential (strong) limit at almost every point of $\mathbb{T}$ and this limit coincides (almost everywhere) with the value of the corresponding boundary function there. Moreover, the L^1-function g has a harmonic extension $\tilde{g}$ to $\mathbb{D}$, and $\tilde{g}$ has a nontangential limit almost everywhere on $\mathbb{T}$ that coincides with the value of g there (cf. [17, p. 38]).

Consider now the dominating set Ω_n defined in (60) and having the property that (61) is valid. Since E has positive measure, there exists a subset E_1 of E having the same positive measure such that every point of E_1 is a nontangential limit of a sequence of points from Ω_n. It follows that there exists a point e^{it} in E_1 at which all six of the functions $\widetilde{Qx_a}, ..., \tilde{g}$ have nontangential limits equal to the correponding numbers $(Qx_a)(e^{it}), ..., g(e^{it})$. Thus we may choose λ_0 in Ω_n sufficiently close to e^{it} that

$$\|PS^n e_{\lambda_0}\|^2 > F_1 = 1 - (0.101)^2, \tag{74}$$

$$\tilde{g}(\lambda_0) < 6\sigma_n, \tag{75}$$

$$\|(\widetilde{Qx_a})(\lambda_0)\| < (6\sigma_n)^{1/2}, \tag{76}$$

$$\|(\widetilde{Qx_b})(\lambda_0)\| < (6\sigma_n)^{1/2}, \tag{77}$$

$$\|(\widetilde{Qw})(\lambda_0)\| < (6\sigma_n)^{1/2}, \tag{78}$$

$$\|(\widetilde{Qy})(\lambda_0)\| < (6\sigma_n)^{1/2}, \tag{79}$$

and

$$\|(\widetilde{S^{*n}QPS^n x})(\lambda_0)\| < (6\sigma_n)^{1/2}. \tag{80}$$

Next let us define

$$\hat{e} = \frac{1}{2}\left(\frac{1-|\lambda_0|}{1+|\lambda_0|}\right)^{1/2} e_{\lambda_0}. \tag{81}$$

As was observed in the proof of Proposition 5.1., there exists a unit vector d in $\mathscr{D}$ such that

$$e_{\lambda_0}(e^{it}) = (1-|\lambda_0|^2)^{1/2}(d + \bar{\lambda}_0 e^{it} d + \bar{\lambda}_0^2 e^{i2t} d + \cdots), \qquad e^{it} \in \mathbb{T}. \tag{82}$$

Therefore if k is any function in $H^2(\mathscr{D})$, say

$$k(e^{it}) \sim \sum_{j=0}^{\infty} e^{it} d_j \tag{83}$$

where each $d_j \in \mathscr{D}$ and $\sum_{j=0}^{\infty} \|d_j\|^2 = \|k\|^2$, and $\tilde{k}$ is the holomorphic extension of k to $\mathbb{D}$, then, of course,

$$\tilde{k}(\lambda_0) = \sum_{j=0}^{\infty} (\lambda_0)^j d_j,$$

and we may calculate

$$\begin{aligned} |(\hat{e}, k)| &= \frac{1}{2}\left(\frac{1-|\lambda_0|}{1+|\lambda_0|}\right)^{1/2} |(e_{\lambda_0}, k)| \\ &= \frac{1}{2}(1-|\lambda_0|)\left|\sum_{j=0}^{\infty} \lambda_0^j (d, d_j)_{\mathscr{D}}\right| \\ &= \frac{1}{2}(1-|\lambda_0|)|(d, \tilde{k}(\lambda_0))| \leqslant \frac{1}{2}(1-|\lambda_0|)\|\tilde{k}(\lambda_0)\|. \end{aligned} \tag{84}$$

If we apply the inequality (84) with k equal to Qx_a, Qx_b, Qw, Qy, and $S^{*n}QPS^nx$ in turn, and take into account (76)–(80), we obtain, upon setting $r = |\lambda_0|$,

$$|(x_a, \hat{e})| = |(Qx_a, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r),$$
$$|(x_b, \hat{e})| = |(Qx_b, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r),$$
$$|(w, \hat{e})| = |(Qw, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r),$$
$$|(y, \hat{e})| = |(Qy, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r),$$

and

$$|(PS^nx, PS^n\hat{e})| = |(S^{*n}QPS^nx, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r).$$

Furthermore, if we write P_{λ_0} for the Poisson kernel

$$P_{\lambda_0}(e^{it}) = \frac{1-|\lambda_0|^2}{|1-\bar{\lambda}_0 e^{it}|^2}, \qquad e^{it} \in \mathbb{T},$$

then, of course, $P_{\lambda_0} \in L^\infty(\mathbb{T})$, and by virtue of (69) and (75), we have

$$\lim_j \frac{1}{2\pi}\int_0^{2\pi} P_{\lambda_0}(e^{it})\|x_j^l(e^{it})\|^2\,dt = \tilde{g}(\lambda_0) < 6\sigma_n. \tag{85}$$

By virtue of (63), (85), and the weak convergence in (65)–(69), we may obviously choose a positive integer j_0 so large that

$$|(x_{j_0}^h, \hat{e})| < \tfrac{1}{2}(6\sigma_0)^{1/2}(1-r), \tag{86}$$
$$|(x_{j_0}^l, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r), \tag{87}$$
$$|(\chi_{G_{j_0}} x_{j_0}^h, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r), \tag{88}$$
$$|(\chi_{G_{j_0}} x_{j_0}^l, \hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r), \tag{89}$$
$$|(PS^nx_{j_0}, PS^n\hat{e})| < \tfrac{1}{2}(6\sigma_n)^{1/2}(1-r), \tag{90}$$
$$\frac{1}{2\pi}\int_0^{2\pi} P_{\lambda_0}(e^{it})\|x_{j_0}^l(e^{it})\|^2\,dt < 6\sigma_n, \tag{91}$$

and

$$\|x_{j_0}\|^2 + \|\hat{e}\|^2 > \sigma_n + \tfrac{1}{4}\|\hat{e}\|^2. \tag{92}$$

Consideration of (64) and (91) yields

$$\frac{1}{2\pi}\int_{\mathbb{T}\setminus G_{j_0}} P_{\lambda_0}(e^{it})\,dt < 24\sigma_n,$$

which, in turn, gives

$$\frac{1}{2\pi}\int_{G_{j0}} P_{\lambda_0}(e^{it})\,dt > 1 - 24\sigma_n,$$

$$\|\chi_{\mathbb{T}\setminus G_{j0}}\hat{e}\|^2 = \frac{1}{2\pi}\int_0^{2\pi} \chi_{\mathbb{T}\setminus G_{j0}}(e^{it})\|\hat{e}(e^{it})\|^2\,dt \tag{93}$$

$$= \frac{1}{4}\left(\frac{1-r}{1+r}\right)\frac{1}{2\pi}\int_{\mathbb{T}\setminus G_{j0}} P_{\lambda_0}(e^{it})\,dt < \left(\frac{1-r}{1+r}\right)6\sigma_n,$$

and

$$\|\chi_{G_{j0}}\hat{e}\|^2 = \frac{1}{4}\left(\frac{1-r}{1+r}\right)\frac{1}{2\pi}\int_{G_{j0}} P_{\lambda_0}(e^{it})\,dt > \frac{1}{4}\left(\frac{1-r}{1+r}\right)(1-24\sigma_n). \tag{94}$$

Moreover, from (86) and (88) we deduce that

$$\begin{aligned} |(x_{j0}^h, \chi_{\mathbb{T}\setminus G_{j0}}\hat{e})| &= |(\chi_{\mathbb{T}\setminus G_{j0}}x_{j0}^h, \hat{e})| \\ &= |(x_{j0}^h, \hat{e}) - (\chi_{G_{j0}}x_{j0}^h, \hat{e})| \\ &< (6\sigma_n)^{1/2}(1-r), \end{aligned} \tag{95}$$

and, similarly, from (87) and (89) that

$$|(x_{j0}^l, \chi_{\mathbb{T}\setminus G_{j0}}\hat{e})| < (6\sigma_n)^{1/2}(1-r). \tag{96}$$

We are finally prepared to make some definitions that will give information about the number σ_n. Set

$$l = x_{j0}^l + \chi_{G_{j0}}\hat{e}, \qquad h = x_{j0}^h + \chi_{\mathbb{T}\setminus G_{j0}}\hat{e}, \tag{97}$$

and define

$$\bar{x} = l + h = x_{j0} + \hat{e}.$$

Clearly $\bar{x} \in H^2(\mathscr{D})$, and we investigate conditions under which $\bar{x} \in \Sigma_n(T)$. In the first place, we see from (62), (64), and (81) that $l \in L^2(\mathscr{D})$ and satisfies

$$\|l(e^{it})\| \leqslant 1, \qquad e^{it} \in \mathbb{T}.$$

Secondly, we note from (89) and (94) that

$$\begin{aligned} \|l\|^2 &= \|x_{j0}^l\|^2 + 2Re(x_{j0}^l, \chi_{G_{j0}}\hat{e}) + \|\chi_{G_{j0}}\hat{e}\|^2 \\ &\geqslant \|x_{j0}^l\|^2 - (6\sigma_n)^{1/2}(1-r) + \frac{1}{4}\left(\frac{1-r}{1+r}\right)(1-24\sigma_n), \end{aligned} \tag{98}$$

and a similar computation using (93) and (95) gives

$$\|h\|^2 \leqslant \|x_{j0}^h\|^2 + 2(6\sigma_n)^{1/2}(1-r) + \left(\frac{1-r}{1+r}\right) 6\sigma_n. \tag{99}$$

Since

$$|(x_{j_0}, \hat{e})| < (6\sigma_n)^{1/2}(1-r)$$

from (86) and (87), we get the following upper and lower bounds on $\|\bar{x}\|^2$:

$$\begin{aligned}\|x_{j_0}\|^2 + \|\hat{e}\|^2 - 2(6\sigma_n)^{1/2}(1-r) &\leqslant \|\bar{x}\|^2 \\ &\leqslant \|x_{j_0}\|^2 + \|\hat{e}\|^2 + 2(6\sigma_n)^{1/2}(1-r).\end{aligned} \tag{100}$$

Therefore a sufficient condition in order that $\|\bar{x}\| \leqslant 1$ is

$$\sigma_n + \frac{1}{4}\left(\frac{1-r}{1+r}\right) + 2(6\sigma_n)^{1/2}(1-r) \leqslant 1, \tag{101}$$

and, in view of (47), (98), and (99), a sufficient condition in order that $\|h\|^2 \leqslant (1-F_0)\|l\|^2$ is

$$2(6\sigma_n)^{1/2} + \frac{6\sigma_n}{1+r} \leqslant (1-F_0)\left\{\frac{1-24\sigma_n}{4(1+r)} - (6\sigma_n)^{1/2}\right\}. \tag{102}$$

What about a sufficient condition in order that

$$\|PS^n\bar{x}\|^2 \geqslant F_0\|\bar{x}\|^2? \tag{103}$$

Easy calculations using (45), (61), and (90) show that

$$\|PS^n\bar{x}\|^2 \geqslant F_0\|x_{j_0}\|^2 - (6\sigma_n)^{1/2}(1-r) + \frac{1}{4}\left(\frac{1-r}{1+r}\right) F_1$$

and

$$F_0\left(\|x_{j_0}\|^2 + \frac{1}{4}\left(\frac{1-r}{1+r}\right) + 2(6\sigma_n)^{1/2}(1-r)\right) \geqslant F_0\|\bar{x}\|^2,$$

so a sufficient condition for the validity of (103) is

$$\frac{(F_1-F_0)}{4(1+r)} \geqslant (2F_0+1)(6\sigma_n)^{1/2}. \tag{104}$$

In other words, if σ_n is such that (101), (102), and (104) are satisfied, then $\bar{x} \in \Sigma_n(T)$ and consequently (from (100))

$$\sigma_n \geqslant \|\bar{x}\|^2 \geqslant \|x_{j_0}\|^2 + \|\hat{e}\|^2 - 2(6\sigma_n)^{1/2}(1-r).$$

Thus, applying (92), we would have

$$\sigma_n \geqslant \sigma_n + \frac{1}{16}\left(\frac{1-r}{1+r}\right) - 2(6\sigma_n)^{1/2}(1-r),$$

which yields

$$(6\sigma_n)^{1/2} \geqslant \frac{1}{32(1+r)} \geqslant \frac{1}{64} \tag{105}$$

and

$$\sigma_n \geqslant \frac{1}{6}\left(\frac{1}{64}\right)^2 = \frac{1}{24576}. \tag{106}$$

Now recall from Proposition 5.1 that $r = |\lambda_0| > 0.9$, so (101) will certainly be satisfied if

$$\sigma_n + 0.2\sqrt{6}\,\sigma_n^{1/2} + 0.025 \leqslant 1, \tag{107}$$

while (102) is equivalent to

$$4(1+r)(3-F_0)(6\sigma_n)^{1/2} \leqslant (1-F_0)(1-24\sigma_n) - 24\sigma_n, \tag{108}$$

and since $1+r<2$ and $F_0 = 0.9775$, (108) will be valid if

$$24(1.0225)\,\sigma_n + (16.18)\sqrt{6}\,\sigma_n^{1/2} \leqslant 0.0225. \tag{109}$$

Finally, (104) is equivalent to

$$\left(\frac{(F_1 - F_0)}{4(1+r)(2F_0+1)\sqrt{6}}\right)^2 \geqslant \sigma_n$$

and since $F_1 - F_0 = 0.012299$, (104) will certainly be valid if

$$\left(\frac{0.012299}{8\sqrt{6}\,(2.955)}\right)^2 \geqslant \sigma_n. \tag{110}$$

Suppose now that

$$\sigma_n = \sigma_n(T) \leqslant \frac{1}{(10)^8}. \tag{111}$$

Then arithmetical calculations show that (107), (109), and (110) are satisfied, so $\bar{x}$ constructed above belongs to $\Sigma_n(T)$, and (106) must be valid. But (106) and (111) are incompatible, so for every T satisfying the hypotheses of the proposition, we must have

$$\sigma_n(T) > \frac{1}{(10)^8},$$

and since n was arbitrary in $\mathbb{N}$, this completes the proof of Theorem 5.3.

We are finally in a position to prove Theorem 2.4, and with it, of course, Theorem 1.1.

Proof of Theorem 2.4. Set $\theta = 1 - 2/(10)^{10}$, and suppose T belongs to $\mathbb{A}(\mathscr{H})$ and $\Lambda \subset \mathbb{D}$ is a dominating set such that the pair (T, Λ) satisfies (8). Then, according to Theorem 5.3, for every n in $\mathbb{N}$, $\sigma_n(T) > 1/(10)^8$, which means, by definition, that for each n in $\mathbb{N}$ there exists a vector $y_n \in \Sigma_n(T)$ such that $\|y_n\|^2 \geqslant 1/(10)^8$. Define now

$$x_n = PS^n y_n, \qquad n \in \mathbb{N}.$$

Then $x_n \in \mathscr{H}$, $\|x_n\| \leqslant \|y_n\| \leqslant 1$, and by Proposition 4.3 we have

$$\begin{aligned} \|[C_0]_T - [x_n \otimes x_n]_T\| &= \|[C_0]_T - [PS^n y_n \otimes PS^n y_n]_T\| \\ &\leqslant 1 - (0.02)\|y_n\|^2 \leqslant 1 - \frac{2}{(10)^{10}} = \theta, \qquad n \in \mathbb{N}. \end{aligned}$$

Therefore it suffices to verify that the sequence $\{x_n\}$ satisfies (10). For this purpose, let w be an arbitrary vector in $\mathscr{H}$, and let $B = S^* \oplus R$ in $\mathscr{L}(H^2(\mathscr{D}) \oplus \mathscr{R})$ be the minimal coisometric extension of T. It follows from Proposition 3.1 that $\mathscr{D} \neq (0)$ and, utilizing [11, Lemma 3.6], we obtain

$$\begin{aligned} \|[w \otimes x_n]_T\| &= \|[w \otimes x_n]_B\| = \|[w \otimes PS^n y_n]_B\| \\ &= \|[w \otimes S^n y_n]_B\| = \|[Qw \otimes S^n y_n]_B\| \\ &= \|[S^{*n}(Qw) \otimes y_n]_B\| \leqslant \|S^{*n}(Qw)\| \, \|y_n\| \to 0, \end{aligned}$$

since the sequence $\{S^{*n}\}$ converges to zero in the strong operator topology. Thus the proof of Theorems 2.4 and 1.1 are complete.

There are several consequences of Theorem 1.1 concerning reflexivity, invariant subspace lattices, and the coincidence of the weak operator and weak* topologies that will be taken up in a later paper (cf. *On the structure of contraction operators.* III.).

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