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Jack Symmetric Functions and Some Combinatorial Properties of Young Symmetrizers

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DEDICATED TO THE MEMORY OF HERBERT J. RYSER

1. INTRODUCTION

The Jack symmetric functions, $J_\lambda(\alpha; \mathbf{x})$, are symmetric functions indexed by partitions λ which depend on an indeterminate α. These symmetric functions were introduced by the statistician H. Jack [2] who defined them with respect to a certain inner product $\langle\ ,\ \rangle_\alpha$ on the space of symmetric functions. This inner product is most easily expressed in terms of the power sum symmetric functions, p_λ, by

$$\langle p_\lambda, p_\mu \rangle_\alpha = \begin{cases} \alpha^{l(\lambda)}/z_\lambda & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

(here z_λ is the order of the centralizer in the symmetric group of a permutation whose cycle type is λ). The Jack symmetric functions are defined to be the unique symmetric functions satisfying the following three conditions:

(C1) (Orthogonality) $\langle J_\lambda(\alpha; \mathbf{x}), J_\mu(\alpha; \mathbf{x}) \rangle_\alpha = 0$ unless $\lambda = \mu$.

(C2) (Triangularity) If we write $J_\lambda(\alpha; \mathbf{x})$ in terms of the monomial symmetric functions $J_\lambda(\alpha; \mathbf{x}) = \sum_\mu c_{\lambda\mu}(\alpha)\, m_\mu(\mathbf{x})$ then $c_{\lambda\mu}(\alpha) = 0$ unless λ dominates μ.

(C3) For all $\lambda \vdash f$, $c_{\lambda, 1^f}(\alpha) = f!$

(throughout this paper notation and terminology follow Macdonald [5]).

* During my seven years at Caltech, I had the pleasure of knowing Herb Ryser as a teacher and a colleague. It was difficult not to be inspired by Herb. He held such a deep understanding of combinatorics but was still honestly fascinated by the subject. Through his research, his books, and his students he added immeasurably to the wealth of combinatorial mathematics.

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The Jack symmetric functions are of recent interest because they are common generalizations of two sets of spherical functions, the Schur functions $s_\lambda(\mathbf{x})$ and the zonal polynomials $Z_\lambda(x)$ (see James [3]). The Schur function $s_\lambda(x)$ is the Jack symmetric function $J_\lambda(\alpha; \mathbf{x})$ evaluated at $\alpha = 1$ and the zonal polynomial $Z_\lambda(\mathrm{x})$ is the Jack symmetric function $J_\lambda(\alpha; \mathbf{x})$ evaluated at $\alpha = 2$. In the last few years I. G. Macdonald began to discover more of the remarkable properties of the Jack symmetric functions [6]. In a very elegant piece of work [7], R. P. Stanley proved many of Macdonald's conjectures about the Jack polynomials (as well as many conjectures of his own).

The present work stems from an effort to settle the still open question of whether the coefficients $c_{\lambda\mu}(\alpha)$ are polynomials in α (the definition only assures that they are rational functions of α). Our starting point is a well-known formula for the Schur functions:

$$s_\lambda(\mathbf{x}) = \left(\prod h_{ij}\right)^{-1} \sum_{\gamma \in C_t} \sum_{\sigma \in R_t} \operatorname{sgn}(\gamma)\, p_{\gamma\sigma}(\mathbf{x}). \tag{1.1}$$

In Eq. (1.1) the product $\prod h_{ij}$ is over the squares (i, j) in λ and h_{ij} denotes the hook-length of the square at (i, j). Also C_t and R_t are the column stabilizer and row stabilizer of a standard Young tableau t of shape λ and $p_{\gamma\sigma}(\mathbf{x})$ denotes the power sum symmetric function associated with the permutation $\gamma\sigma$. The hope was to find an α-analog of Eq. (1.1), i.e., an equation

$$J_\lambda(\alpha; \mathbf{x}) = \left(\prod h_{ij}\right)^{-1} \sum_{\gamma \in C_t} \sum_{\sigma \in R_t} \alpha^{f(\gamma, \sigma)} \operatorname{sgn}(\gamma)\, p_{\gamma\sigma}(\mathbf{x}), \tag{1.2}$$

where $f(\gamma, \sigma)$ is some function of the pair (γ, σ).

In this paper we study properties of such a function $f(\gamma, \sigma)$ by looking at certain combinatorial properties related to the permutations which have nonzero coefficient in

$$e_t = \sum_{\gamma \in C_t} \sum_{\sigma \in R_t} \operatorname{sgn}(\gamma)(\gamma\sigma). \tag{1.3}$$

The sum in (1.3) is called the *Young symmetrizer* defined by t. In the course of this work we will uncover a great deal of combinatorial structure enjoyed by these Young symmetrizers.

2. R–C Digraphs

Let X be a set of points. A digraph D on X is called *bijective* if every point of X has indegree and outdegree 1 in D. A bijective digraph corresponds in an obvious way to a permutation of X. The facts we prove in this paper will really be facts about permutations. However, we choose to represent them as bijective digraphs to make the arguments seem more natural.

Let λ be a partition. We denote the length of λ by $l(\lambda)$, the ith part of λ by λ_i and the ith part of its conjugate by λ'_i. In what follows we will work with bijective digraphs on $V_\lambda = \{(i, j): 1 \leqslant i \leqslant l(\lambda),\ 1 \leqslant j \leqslant \lambda_i\}$. Let R_λ and C_λ denote the row stabilizer and column stabilizer subgroups of λ consider as acting on V_λ.

DEFINITION 2.1. For $\sigma \in R_\lambda$ and $\gamma \in C_\lambda$ let $D(\gamma, \sigma)$ be the bijective digraph with vertex set V_λ having a directed edge from v to $(\gamma\sigma)\, v$ for all $v \in V_\lambda$. Digraphs of the form $D(\gamma, \sigma)$ are called *r–c digraphs on* V_λ. The *sign* of $D(\gamma, \sigma)$, denoted *sgn*$(D(\gamma, \sigma))$ is defined to be $\operatorname{sgn}(\gamma)$.

Not every bijective digraph is an r–c digraph. The next result gives a simple necessary and sufficient condition for a bijective digraph to be an r–c digraph. This result is well known (see, for example, James and Kerber [4]) but we give a proof here because the ideas in the proof will be used repeatedly in what follows.

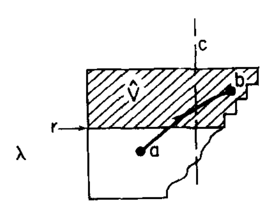
PROPOSITION 2.2. *Let D be a bijective digraph on V_λ. Then the following are equivalent*:

(i) *D is an r–c digraph on V_λ.*

(ii) *If (u, v) and (x, y) are directed edges of D with u and x in the same row and v and y in the same column then $u = x$ and $v = y$.*

Proof. Assume (i) holds so $D = D(\gamma, \sigma)$. Since v and y are in the same column then $\gamma^{-1} v = \sigma u$ and $\gamma^{-1} y = \sigma x$ are in the same column. But u and x are in the same row so σu and σx are in the same row. Thus $\sigma u = \sigma x$ so $u = x$ and $v = y$.

Now assume that condition (ii) holds for D. Let (a, b) be any edge of D with a in row r and b in column c. We will now show that $c \leqslant \lambda_r$. Suppose to the contrary that $c > \lambda_r$. Let $r = \lambda'_c$, and let $\hat{V}$ be the set of squares of V_λ in rows $1, 2, \ldots, r$. We will count the set of edges directed out of a point of $\hat{V}$ in two different ways.



On the one hand, we can add these up by columns. For every column, the number of edges directed into that column from a point in $\hat{V}$ is no more than r (by the pigeonhole principle). For columns $c+1, ..., \lambda_1$ a better bound of $\lambda_{c+1}, ..., \lambda_{\lambda_1}$ can be given. For column c the number is at most $r-1$ since the edge into the point b is not from a point in $\hat{V}$. So the total number of edges coming from a point in $\hat{V}$ is at most

$$(c-1)\, r(r-1) + \sum_{j=c+1}^{\lambda_1} \lambda_j = |\hat{V}| - 1.$$

On the other hand, the total number of edges directed out of a point in $\hat{V}$ is $|\hat{V}|$. This is a contradiction which shows that if (a, b) is an edge of D with a row r and b in column c then $c \leqslant \lambda_r$.

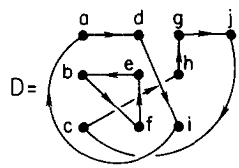
Define γ: $V_\lambda \to V_\lambda$ as follows:

(1) For each b, find $a \in V_\lambda$ such that (a, b) is an edge of D.

(2) Let a be in row r, b in column c. Define $\gamma(b)$ to be the square (r, c).

We proved above that $\gamma(b) \in V_\lambda$. Also the condition (ii) shows that if $\gamma(b) = \gamma(c)$ then $b = c$. Let $\hat{D}$ be the bijective digraph with an edge from a to $\gamma(b)$ for every edge (a, b) in D. Obviously $\hat{D}$ is a bijective digraph with all edges joining points in the same row. So $\hat{D}$ is of the form $D(e, \sigma)$ for some $\sigma \in R_\lambda$. By definition $\gamma \in C_\lambda$ and $D = D(\gamma^{-1}, \sigma)$.

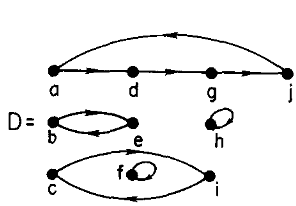
EXAMPLE. Let $\lambda = 433$. The following digraph satisfies condition (ii) of Proposition 2.



Applying the algorithm given above we obtain

$$\gamma = (a, c)\,(b)\,(d)\,(e, f)\,(g, h, i)\,(j).$$

So $\hat{D}$ is



which is $D(e, \sigma)$ with $\sigma = (a, d, g, j)\ (b, e)\ (h)\ (c, i)\ (f)$. Hence $D = D(\gamma^{-1}, \sigma)$.

The method of proof used in Proposition 2 shows that if D satisfies condition (ii) then there is a unique $\gamma \in C_\lambda$ and $\sigma \in R_\lambda$ with $D = D(\gamma, \sigma)$. Again this result is well known but is worth mentioning because it will be used throughout the rest of this work.

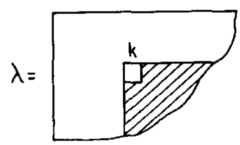
We end this section with one more piece of terminology. If D is a bijective digraph and $x \in V_\lambda$ we say y is the *predecessor of x in D* if there is an edge in D from y to x. Likewise *y is the successor to x in D* if there is an edge in D from x to y.

3. Some Constructions with R–C Digraphs

A. *Reflections around Corners*

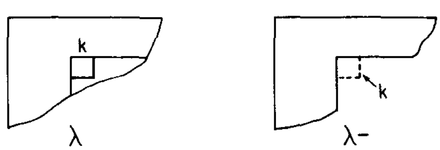
Definition 3.1. A *southeast corner* in an *r–c* digraph $D(\gamma, \sigma)$ is a square k with the property that $\gamma\sigma x = x$ for every square x in V_λ distinct from k that lies weakly to the southeast of k.

In the diagram below, the square k is a corner if and only if all the squares in the shaded portion of λ are fixed by $\gamma\sigma$.



We will also use the notion of a southwest corner. These corners are defined in exactly the same way that southeast corners are defined. More importantly, the obvious analogy of the technical lemma proved in this part holds for southwest corners. To simplify the exposition we will deal only with the case of southeast corners and we will leave the other case to the reader.

Fix a square $k = (u, v)$ in V_λ. By λ_- we mean the partition obtained by removing all boxes which are weakly southeast of k in the Ferrer's diagram of λ.



We will think of R_{λ_-} and C_{λ_-} as being contained in R_λ and C_λ.

For $1 \leqslant i < v$ let γ_i be the transposition in C_λ which interchanges the squares (i, v) and $(u, v) = k$ and leaves every other square fixed. For

$1 \leqslant j < u$ let σ_j denote the permutation in R_λ which interchanges (j, v) and $k = (u, v)$ and leaves every other square fixed. Let $\gamma_0 = \sigma_0 = e$.

We will be interested in the set of all r–c digraphs $D(\gamma, \sigma)$ on λ which have k as a corner. The purpose for introducing the γ_j and σ_i is that they are a set of coset representatives for the subgroups $R_{\lambda\_}$ and $C_{\lambda\_}$ in the subgroups of R_λ and C_λ which fix every square southeast of k. To be more precise, $D(\gamma, \sigma)$ has k as a southeast corner if and only if $\gamma = \gamma_i \gamma_-$ $(i = 0, 1, ..., u-1)$ and $\sigma = \sigma_j \sigma_-$ $(j = 0, 1, ..., v-1)$, where $\gamma_- \in C_{\lambda\_}$ and $\sigma_- \in R_{\lambda\_}$.

We say $D = D(\gamma_i \gamma_-, \sigma_j \sigma_-)$ is *degenerate* if $i = 0$ or $j = 0$ (or both). Otherwise we say that D is nondegenerate. The next definition will turn out to give a sign-reversing pairing of the nondegenerate r–c digraphs having corner k.

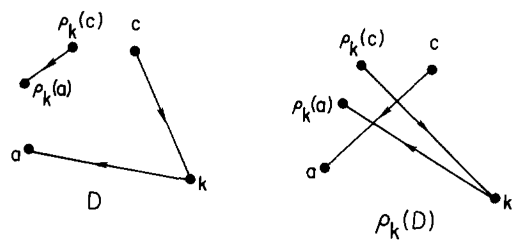
DEFINITION 3.2. Let D be a nondegenerate r–c digraph with (southeast) corner k. Let a and c be the predecessor and successor of k in D. Define $\rho_k(c)$ and $\rho_k(a)$ to be the unique squares in V_λ such that

(1) $\rho_k(c)$ is in the same row as c.

(2) $\rho_k(a)$ is in the same column as a.

(3) There is an edge in D from $\rho_k(c)$ to $\rho_k(a)$.

Lastly, define $\rho_k(D)$ to be the bijective digraph obtained from D by removing the edges $\langle c, k\rangle$, $\langle k, a\rangle$, $\langle \rho_k(c), \rho_k(a)\rangle$ and replacing them with edges $\langle \rho_k(c), k\rangle$, $\langle k, \rho_k(a)\rangle$, $\langle c, a\rangle$. We call $\rho_k(c)$, $\rho_k(a)$, and $\rho_k(D)$ *the reflections of* c, a, *and* D *around the corner* k. A picture will make this clearer.



Two things should be mentioned about this construction. First, it needs to be shown that there is an edge in D from a square in the same row as c to a square in the same column as a. This follows from the fact that k is a corner so the row containing c is above the row containing k and the column containing a is left of the column containing k. So if c is in row i and a is in column j then $j \leqslant \lambda_i$.

Second, we need to observe that $\rho_k(a)$ is not equal to a. For if $\rho_k(a) = a$

then $\rho_k(c)=k$. Hence c and k are in the same row so $\gamma_i=c$. But this violates the nondegeneracy of D.

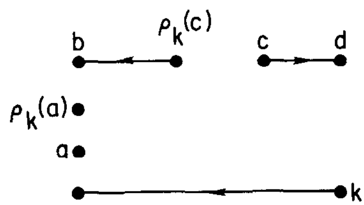
LEMMA 3.3. *Let D be a nondegenerate r–c digraph with (southeast) corner k:*

(A) *$\rho_k(D)$ is a nondegenerate r–c digraph with corner k*

(B) $\operatorname{sgn}(\rho_k(D))=-\operatorname{sgn}(D)$

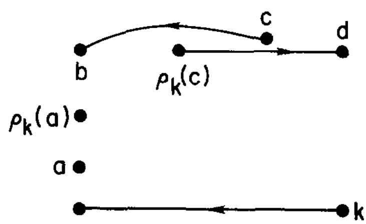
(C) $\rho_k(\rho_k(D))=D$.

Proof. Write $D=D(\gamma_i\gamma_-, \sigma_j\sigma_-)$ and let b be the square which is in the same column as a and the same row as c. Let d be the square which is in the same column as k and the same row as c. Observe that b and d are in the same row and that in $D(e, \sigma_j\sigma_-)$ we have the following three edges:



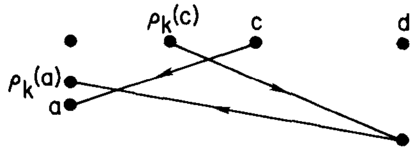
$D(e,\sigma_j\sigma_-)$

The edges of $D(e, (c, \rho_k(c))\,\sigma_j c_-)$ are identical to those of $D(e, \sigma_j\sigma_-)$ except for those pictured below:



$D(e,(c,\rho_k(c))\sigma_j\sigma_-)$

So $D((a, \rho_k(a))\,\gamma_j\gamma_-, (c, \rho_k(c))\,\sigma_j\sigma_-)$ has exactly the same edges as D except for the three below:

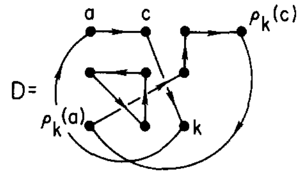


$D((a,\rho_k(a))\gamma_j\gamma_-, (c,\rho_k(c))\sigma_j\sigma_-)$

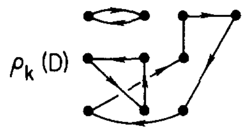
In other words, $\rho_k(D) = D((a, \rho_k(a))\ \gamma_j\gamma_-, \ (c, \rho_k(c))\ \sigma_j\sigma_-) = D(\gamma_j(a, \rho_k(a))\ \gamma_-, \sigma_j(c, \rho_k(c))\ \sigma_-)$.

From this, parts (A) and (B) follow immediately. Part (C) follows upon observing that $\rho_k(\rho_k(a)) = a$ and $\rho_k(\rho_k(c)) = c$.

We finish this part of Section 3 with an example. Let D be the r–c digraph of shape $\lambda = 433$:



The corner k is labelled above as are c, a, $\rho_k(c)$, and $\rho_k(a)$. The r–c digraph $\rho_k(D)$ is



B. *Consistent Labellings of r–c Digraphs*

In this section we introduce our fundamental objects of study.

DEFINITION 3.4. An *n-labelling of* V_λ is a function m from V_λ into $\{1, 2, ..., n\}$. Let $a_1, a_2, a_3, ...$ be indeterminates. For m an n-labelling of V_λ define a_m, the *weight of* m, by

$$a_m = \prod_{x \in V_\lambda} a_{m(x)} \in \mathbb{Z}[a_1, a_2, ..., a_n].$$

A labeling m is *column bijective* if m is one to one on each column of V_λ.

DEFINITION 3.5. Let D be a bijective digraph on V_λ and let m be an n-labeling of V_λ. We say m is *consistent with* D if $m(x) = m(y)$ whenever there is an edge from x to y in D.

Let $A_\lambda^{(n)}$ denote the set of pairs (D, m), where D is an r–c digraph on V_λ and m is a consistent n-labeling of D. Let $B_\lambda^{(n)}$ denote the subset of $A_\lambda^{(n)}$ consisting of those pairs (D, m), where m is column bijective.

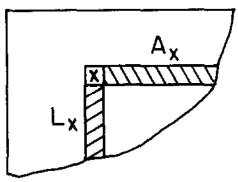
C. *First Column Heredity*

Let $\hat{\lambda}$ be the partition obtained from λ by removing the first column. We will think of $V_{\hat{\lambda}}$ as contained in V_λ and we will think of $R_{\hat{\lambda}}$ and $C_{\hat{\lambda}}$ as the subgroups of R_λ and C_λ which fix the first column of V_λ.

The purpose of this section is to establish some kind of relationship between the sets $B_{\hat{\lambda}}^{(l)}$ and $B_{\lambda}^{(l)}$, where $l = l(\lambda)$. Recall for $x \in V_\lambda$ the *hook of* x is the set of all squares in V_λ which are either in the same row as x and weakly to its right or in the same column as x and weakly below it. We let H_x denote the hook of x and we split H_x as

$$H_x = \{x\} \cup A_x \cup L_x,$$

where A_x, *the arm of* H_x, is the set of squares in the same row as x but strictly to its right, and L_x, the *leg of* H_x, is the set of squares in the same column as x but strictly below it.



DEFINITION 3.5. A *first-column hook choice* for V_λ is a function f which assigns to each square x in the first column of V_λ a box $f(x)$ from its hook H_x. The degree of f, denoted $\deg(f)$, is the number of x with $f(x) \in A_x$. Let C_1 denote the set of all first column hook choices for V_λ.

Note that $|C_1| = \prod_{i=1}^{l} (l + \lambda_i - i)$. We will prove the following result which relates $B_{\lambda}^{(l)}$ to $C_1 \times B_{\hat{\lambda}}^{(l)}$.

THEOREM 3.6. *There are natural maps* $\varphi\colon C_1 \times B_{\hat{\lambda}}^{(l)} \to B_{\lambda}^{(l)}$ *and* $\psi\colon (B_{\lambda}^{(l)} \backslash \operatorname{im} \varphi) \to (B_{\lambda}^{(l)} \backslash \operatorname{im} \varphi)$ *satisfying*:

(1) φ *is an injection.*

(2) *If* $(f_1(\hat{D}, \hat{m})) \in C_1 \times B_{\hat{\lambda}}^{(l)}$ *and* $\varphi(f_1(\hat{D}, \hat{m})) = (D, m)$ *then* $\operatorname{sgn}(D) = \operatorname{sgn}(\hat{D})$ *and* $a_m = (a_1 \dots a_l)\, a_{\hat{m}}$.

(3) *If* $(D, m) \in (B_{\lambda}^{(l)} \backslash \operatorname{im} \varphi)$ *and* $\psi(D, m) = (D, m)$ *then* $\operatorname{sgn}(D) = -\operatorname{sgn}(D')$ *and* $a_m = a_{m'}$.

(4) ψ *is a fixed-point free involution.*

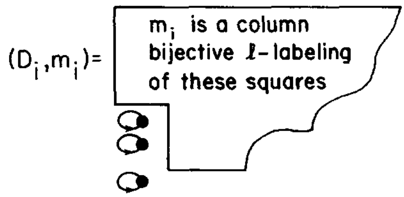
Proof. We will begin by describing algorithmically how to compute φ given as input a triple $(f, \hat{D}, \hat{m})$, where f is a hook choice for every square in the first column of V_λ, $\hat{D}$ is an r–c digraph on $V_{\hat{\lambda}}$, and $\hat{m}$ is a column-bijective l-labeling of $V_{\hat{\lambda}}$ which is consistent with $\hat{D}$. The computation of φ will produce $l+1$ pairs (D_0, m_0), (D_1, m_1), ..., (D_l, m_l), where

(1) $(D_0, m_0) = (\hat{D}, \hat{m})$

(2) $(D_l, m_l) = \varphi(f, (\hat{D}, \hat{m}))$

(3) D_i is an r–c digraph on V_λ with loops at the points $(i+1, 1)$, $(i+2, 1)$, ..., $(l, 1)$

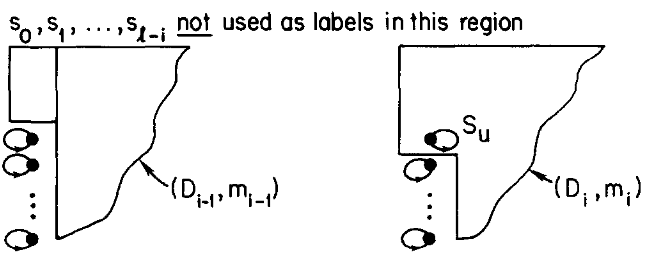
(4) m_i is a partial column-bijective l-labeling of V_λ which is consistent with D_i. To be precise, m_i labels all squares in V_λ except the squares $(i+1, 1)$, $(i+2, 1)$, ..., $(l, 1)$.



The condition $(D_0, m_0)=(\hat{D}, \hat{m})$ begins our definition of the (D_i, m_i). Suppose for $1 \leqslant i \leqslant l$ that we have (D_{i-1}, m_{i-1}). We will show how to get (D_i, m_i).

Case 1. $f(i, 1)=(i+u, 1)$ some u, $0 \leqslant u \leqslant l-i$. We know that m_{i-1} is column-bijective and has assigned labels to the $i-1$ points above $(i, 1)$. So there are $l-i+1$ labels $1 \leqslant s_0 < s_1 < \cdots < s_{l-i} \leqslant l$ which do not appear in m_{i-1} on a point in the first column.

Let $D_i = D_{i-1}$ and let m_i be m_{i-1} except that the label s_u is assigned to $(i, 1)$ by m_i.



Case 2. $f(i, 1)=(i, c)$ for $c>1$.

CLAIM. *There is a unique sequence of squares* $s_1, s_2, \ldots, s_r$; $t_1, t_2, \ldots, t_r$; $u_0, u_1, \ldots, u_{r-1}$; $v_0, v_1, \ldots, v_{r-1}$ *such that*

1. $u_0 = v_0 =$ *square in position* $(i, 1)$.
2. s_{j+1} *is in the same row as* u_j.
3. t_{j+1} *is in column* c.
4. v_i *is in column* 1.

5. *There are edges in D_{i-1} from s_{j+1} to t_{j+1} and from u_j to v_j.*

6. $m_{i-1}(v_j) = m_{i-1}(u_j) = m_{i-1}(s_j) = m_{i-1}(t_j)$.

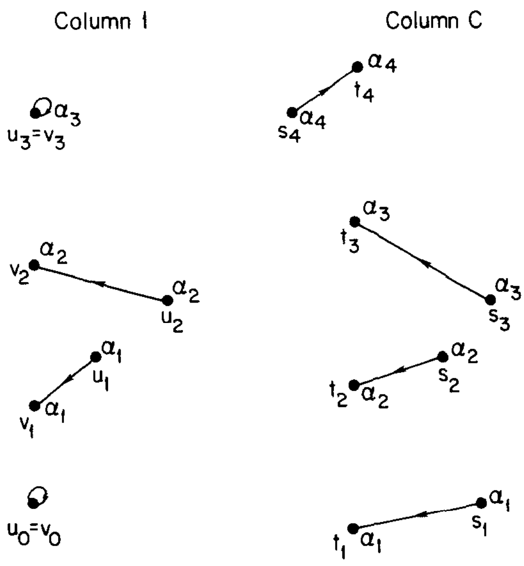
7. $m_{i-1}(s_r) = m_{i-1}(t_r)$ *is not the label of any square in column* 1 *of* (D_{i-1}, m_{i-1}).

Proof of Claim. Suppose that $s_1, ..., s_{j-1}$; $t_1, ..., t_{j-1}$; $u_0, ..., u_{j-1}$; and $v_0, ..., v_{j-1}$ have been chosen to satisfy 1–6 and that there has been a unique way to choose them. We show that there is a unique way to extend this choice one more step.

Let p be the row containing u_{j-1}. In D_{i-1}, there is an edge from u_{j-1} to a square (namely v_{j-1}) in column 1. Also there are loops at the squares in column 1 below row $i-1$. So p must be less than or equal to $i-1$. Since (i, c) is in the hook of $(i, 1)$ we have $c \leqslant \lambda_i \leqslant \lambda_p$. Hence there is a unique square s_j in row p which is the predecessor in D_{i-1} to a point t_j in column c. Let $\alpha_j = m_{i-1}(s_j)$.

If α_j is not the label of a square in column 1 of (D_{i-1}, m_{i-1}) then $j = r$ and conditions 1–7 are satisfied. Otherwise let v_j be the square in column 1 with $m_{i-1}(v_j) = \alpha_j$. Note that the choice of v_j is unique since m_{i-1} is column bijective. Let u_j be the predecessor to v_j in D_{i-1}.

This completes the choice of s_j, t_j, u_j, v_j. Note that the choice was unique subject to conditions 1–6 being satisfied. This proves the claim.



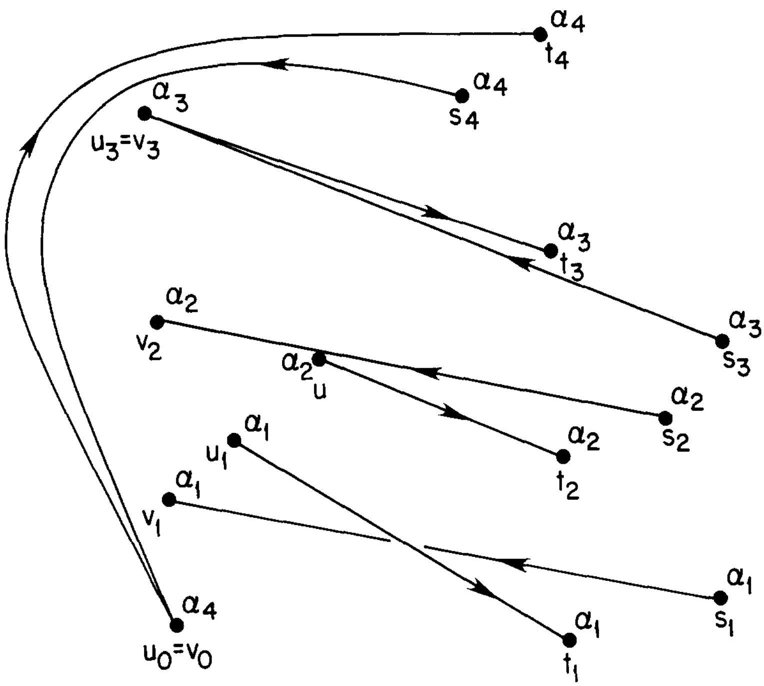
The Greek numbers above are the labels assigned the squares by m_{i-1}.

Now define m_i to be the extension of m_{i-1} which assigns the label α_r to $(i, 1) = v_0 = u_0$. Let D_i be identical to D_{i-1} except for the following edge removals and edge additions:

(Removals) The edges from u_j to v_j and from s_{j+1} to t_{j+1} are removed.

(Additions) Add edges from s_{j+1} to v_{j+1} and from u_{j+1} to v_{j+1} for $1 \leqslant j+1 \leqslant r-1$. Add an edge from s_r to u_0 and from u_0 to t_r.

This gives the following picture for D_i (see the previous figure for D_{i-1}).



Clearly D_i is a bijective digraph and m_i is consistent with D_i. It is not immediate that D_i is an r–c digraph. The problem is that the new edges we introduced might cause two edges from the same row to be in the same column. Obviously if that is the case, the column must be 1 or c and the row must be one of the rows containing an s_j and a u_{j-1}. But there is no such problem since we removed edges from s_j to column c and from u_{j-1} to column 1 and added edges from s_j to column 1 and from u_{j-1} to column c. So D_i is an r–c digraph and this completes Case 2.

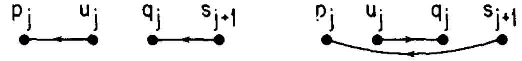
This explains how to get (D_i, m_i) from (D_{i-1}, m_{i-1}) and also how to compute $(D_l, {}_l) = \varphi(f, (\hat{D}, \hat{m}))$. We stop now for an observation.

Claim 2. *For each* $i = 1, 2, ..., l$ *we have* $\operatorname{sgn}(D_i) = \operatorname{sgn}(D_{i-1})$.

Proof of Claim 2. If D_i was obtained from D_{i-1} via Case 1 this is obvious. So assume D_i was obtained from D_{i-1} via Case 2. Let $s_1, ..., s_r$;

$t_1, ..., t_r$; $u_0, ..., u_{r-1}$; $v_0, ..., v_{r-1}$ be the sequence of squares used in the computation of D_i. Let $D_{i-1} = D(\gamma_{i-1}, \sigma_{i-1})$ and let $D_i = D(\gamma_i, \sigma_i)$.

For each $j = 0, 1, ..., r-1$ let ρ_j be the row containing u_j and s_{j+1}. Let p_j and q_j be the squares in positions $(\rho_j, 1)$ and (ρ_j, c), respectively. Note that $D(e, \sigma_{i-1})$ and $D(e, \sigma_i)$ are identical except for the following change made in each row ρ_j:



in $D(e, \sigma_{i-1})$ in $D(e, \sigma_i)$

So $\sigma_i = (p_0, q_0)(p_1, q_1) \cdots (p_{r-1}, q_{r-1})\, \sigma_{i-1}$.

Now in γ_{i-1} we map p_j to v_j and q_j to t_{j+1}. In γ_i we instead map p_j to v_{j+1} and q_j to t_j. So

$$\gamma_i = (v_0, v_1, ..., v_{r-1})(t_r, t_{r-1}, ..., t_2, t_1)\, \gamma_{i-1}.$$

Since we obtain γ_i from γ_{i-1} by multiplication with two cycles of the same length, we have $\operatorname{sgn}(\gamma_i) = \operatorname{sgn}(\gamma_{i-1})$ which completes the proof of Claim 2.

At this point we have defined φ and we have shown that $\operatorname{sgn}(D) = \operatorname{sgn}(\hat{D})$. Note that m_i is an extension of m_{i-1} so $m = m_l$ and $\hat{m} = m_0$ agree on $\hat{D}$. Also m_l is column bijective l-labeling where $l = l(\lambda)$ so the labels $1, 2, ..., l$ must each be used exactly once in the first column by m. Hence $a_m = (a_1 \cdots a_l)\, a_{\hat{m}}$.

We now consider how to reverse the map φ. In the process we will determine exactly which pairs (D, m) are not in the image of φ and we will see how to define ψ on these pairs.

Inverting φ: Assume now that you are given a pair (D, m). We will attempt to write (D, m) as $\varphi(f, (\hat{D}, \hat{m}))$ by reconstructing $(D_l, m_l) = (D, m)$, (D_{l-1}, m_{l-1}), ..., $(D_0, m_0) = (\hat{D}, \hat{m})$. If at some point we are unable to construct (D_{i-1}, m_{i-1}) from (D_i, m_i) then we will know that (D, m) was not in the image of φ and we will define $\psi(D, m)$.

So assume that there has been a unique choice of $(D_l, m_l) = (D, m)$, (D_{l-1}, m_{l-1}), ..., (D_i, m_i) and of hook choices for the squares $(i+1, 1)$, ..., $(l, 1)$ such that (D_j, m_j) and the hook choice for $(j+1, 1)$ give (D_{j+1}, m_{j+1}) when the algorithm φ is applied.

We are given (D_i, m_i), where D_i is an r–c digraph having loops at the squares $(i+1, 1)$, ..., $(l, 1)$, and m_i is a column-bijective labeling of the squares of V_λ other than $(i+1, 1)$, ..., $(l, 1)$ which is consistent with D_i. To construct (D_{i-1}, m_{i-1}) and a hook-choice at $(i, 1)$ we do the following:

Case 1. In D_i there is a loop at $(i, 1)$. In this case let $D_{i-1} = D_i$ and let m_{i-1} be m_i but with no label on the square $(i, 1)$. Let

$1 \leqslant s_0 < s_1 < \cdots < s_{l-i} \leqslant n$ be the numbers which do *not* appear as labels on the squares $(1, 1), ..., (i-1, 1)$ in m_i. The label given the square $(i, 1)$ by m_i will be s_u for some u between 0 and $l-i$. Let the hook-choice at $(i, 1)$ be the square $(i+u, 1)$.

Case 2. There is no loop at $(i, 1)$. Let $c > 1$ be the column containing the successor of $(i, 1)$ in D_i. Let $u_0 = v_0 = (i, 1)$. We will choose squares $s_1, ..., s_{j-1}; t_1, ..., t_{j-1}; u_0, ..., u_{j-1}; v_0, ..., v_{j-1}$ inductively to satisfy:

1. s_{p+1} is in the same row as u_p, $0 \leqslant p \leqslant j-2$.
2. t_p is in column c, $1 \leqslant p \leqslant j-1$.
3. v_p is in column 1, $0 \leqslant p \leqslant j-1$.
4. There are edges in D_i from s_p to v_p and from u_p to t_p, $1 \leqslant p \leqslant j-1$.
5. $m_i(s_p) = m_i(t_p) = m_i(u_p) = m_i(v_p)$, $1 \leqslant p \leqslant j-1$.

We will see that the choices of these $s_p, t_p, v_{p-1}, u_{p-1}$ are unique. Our choices will end in one of two ways.

Assume $s_1, ..., s_{j-1}; t_1, ..., t_{j-1}; u_0, ..., u_{j-1}; v_0, ..., v_{j-1}$ have been chosen. There is a unique square s_j in the same row as u_{j-1} such that s_j is the predecessor to a square v_j in column 1. One possibility is that u_{j-1} is in row $i+z$ for $z > 1$. In this case $s_j = v_j = (i+z, 1)$. Here we say that our choice *ended in failure at step j*. Otherwise v_j is labeled by m_i. Let $a_j = m_i(v_j)$. If a_j is not the label of some square of D_i in column c then we also say our choice *ended in failure at step j*. Otherwise there is a square t_j in column c with label a_j. Let u_j be the predecessor of t_j in D_i.

Suppose $v_j = (i, 1)$. We know that the successor of v_j is in column c and has label a_j. Since m_i is column bijective we have t_j is the successor of v_j so $v_j = u_j = u_0 = v_0 = (i, 1)$. When this happens we say our choice *ended in success at step j*.

If our choice ended in success at step j we define (D_{i-1}, m_{i-1}) in the following way. First m_{i-1} is identical to m_i except that the square $(i, 1)$ is not labeled. We obtain D_{i-1} from D_i by removing the edges from s_p to v_p and from u_p to t_p and replacing them with edges from s_p to t_p and u_p to v_p. In particular, we end up with a loop at $(i, 1)$. Also it is clear that m_{i-1} is consistent with D_{i-1} and it is easily checked that D_{i-1} is an r–c digraph.

Let the hook-choice at $(i, 1)$ be the square (i, c). It is easy to see that applying the procedure outlined in φ to the pair (D_{i-1}, m_{i-1}) with hook-choice (i, c) yields (D_i, m_i) and that this is the only pair and hook-choice which will yield (D_i, m_i).

We need to see what to do in the case that our choice ends in failure. The following observation will be crucial.

CLAIM 3. *Suppose our choice ends in failure. Then the predecessor of $(i, 1)$ is not in row i.*

Proof. If the predecessor of $(i, 1)$ is in row i then this predecessor would be our choice for s_1. So we would have $v_1 = (i, 1) = u_0 = v_0$ and t_1 would be the successor to $(i, 1)$. Hence our choice would end in success after one step.

Let k be the square $(i, 1)$ let c be the predecessor to k and let a be the successor to k. Suppose our choice of $\{s_p, t_p, u_p, v_p\}$ ends in failure after j steps. The claim above shows that c is in a different column than k. Also in D_i the squares below $(i, 1)$ have loops. So k is a nondegenerate corner of D_i. Let $\rho_k(c)$, $\rho_k(a)$, $\rho_k(D_i)$ be the reflections of c, a, and D_i around this corner. Define $\rho_k(m_i)$ to be identical to m_i except that the label on k is changed so that it matches the label of $\rho_k(c)$ and $\rho_k(a)$. Recall that $\operatorname{sgn}(D_i) = -\operatorname{sgn}(\rho_k(D_i))$.

Let $(D'_i, m'_i) = (\rho_k(D_i), \rho_k(m_i))$. Inductively define (D'_p, m'_p) to be the pair obtained from (D'_{p-1}, m'_{p-1}) using the procedure φ with the hook-choice at $(p, 1)$ being the hook-choice obtained above when computing $(D_l, m_l), ..., (D_i, m_i)$. Define $\psi(D, m)$ to be (D'_l, m'_l).

Observe that $\operatorname{sgn}(D'_l) = \operatorname{sgn}(\rho_k(D_i)) = -\operatorname{sgn}(D_i) = -\operatorname{sgn}(D)$. Also, if $\hat{m}$ is the restriction of m to $V_{\hat{\lambda}}$ then $\hat{m}$ also equals the restriction of m_l to $V_{\hat{\lambda}}$. The labelings m and m' differ only in the first column but in that first column they must each use the labels $1, 2, ..., l$ exactly once. So

$$a_m = (a_1 \cdots a_l)\, a_{\hat{m}} = a_{m'}.$$

CLAIM 4. $\psi(\psi(D, m)) = (D, m)$.

Proof of Claim. When we apply this inversion procedure to $\psi(D, m) = (D', m')$ we get the sequence (D'_l, m'_l), (D'_{l-1}, m'_{l-1}), ..., (D'_i, m'_i). Consider what happens when we attempt to choose the $\{s_p, t_p, v_p, u_p\}$ at this point. The successor of $(i, 1)$ is $\rho_k(a)$ which is in the same column c as a, so we proceed as in Case 2 of our inversion of φ.

We claim that we will in fact choose the same $s_1, ..., s_{j-1}$; $t_1, ..., t_{j-1}$; $u_0, ..., u_{j-1}$; $v_0, ..., v_{j-1}$ and that we will again fail at step j. Obviously we will choose the same u_0, v_0 so assume we have made the same choice up to s_p, t_p, u_p, v_p. At this point we chose s_{p+1} to be in the same row as u_p and to be a predecessor to a square in column 1. If we look at the sets of predecessors to squares in column 1 in D_i and D'_i we see that they are identical except that in D_i, c is a predecessor and $\rho_k(c)$ is not, whereas in D'_i, $\rho_k(c)$ is a predecessor and c is not. So if our choice of s_{p+1} is different it must be that before we chose c and this time we choose $\rho_k(c)$. But this is impossible because if before s_{p+1} was c then v_{p+1} would have been $(i, 1)$ and our choice would have succeeded after $p+1$ steps. So our choice of s_{p+1} and v_{p+1} is the same as before.

Since $c > 1$, the labellings m_i and m'_i agree on column c. So if $p+1$ is less than j our choice of t_{p+1} will be the same as before and if $p+1 = j$ we will again find that a_{p+1} is not a label in column c. So our choice will fail at step j as before.

This shows that we will obtain (D_i'', m_i'') by reflecting (D_i, m_i) around the corner $k=(i, 1)$. But we have already shown that reflecting twice is the identity so $(D_i'', m_i'')=(D_i, m_i)$ and $\psi^2(D, m)=(D, m)$.

What we have just done is to explain how to define ψ when the choice of $\{s_p, t_p, u_p, v_p\}$ ends in failure. If the choice never ends in failure then you obtain $(D_l, m_l), ..., (D_0, m_0)$ along with hook choices $f(i, 1)$, $i=1, ..., l$. It is clear that $\varphi_1(f, (D_0, m_0))=(D, m)$ and that these (D_i, m_i), $f(i, 1)$ were constructed uniquely. This finishes the proof of Theorem 3.6.

DEFINITION 3.7. If $(D, m) \in B_\lambda^{(l)}$ is in the image of φ then we will say that (D, m) is 1st-*column hereditary*. If $\varphi^{-1}(D, m)=(f, (\hat{D}, \hat{m}))$ then we say $(\hat{D}, \hat{m})$ is the 1st *ancestor of* (D, m) and we call f the *connector between* (D, m) *and its* 1st *ancestor*.

The correspondence given above answers a special case of an interesting bijection problem. Rewriting Eq. (1.1) we obtain

$$\left(\prod h_{ij}\right) s_\lambda(a_1, ..., a_n)=\sum_{(D, m) \in A_\lambda^{(n)}} \operatorname{sgn}(D)\, a_{\mathbf{m}}. \tag{3.1}$$

Let X be the set of pairs $(\mathbf{f}, t)$, where t is a semi-standard Young tableau of shape λ and $\mathbf{f}$ is a hook-choice for the shape λ. Define the *weight* of $(\mathbf{f}, t)$ to be the monomial m_t where the exponent of a_i is the number of times that i occurs in t.

Equation (3.1) above suggests the problem of finding an injection ϕ from X into $A_\lambda^{(n)}$ together with a fixed-point free involution ψ on $A_\lambda^{(n)} \backslash \operatorname{im} \phi$ which satisfy:

(1) If $\phi(\mathbf{f}, t)=(D, m)$ then $\operatorname{sgn}(D)=1$ and $a_t=a_m$.

(2) If $\psi(D, m)=(D', m')$ then $\operatorname{sgn}(D)=-\operatorname{sgn}(D')$ and $a_m=a_{m'}$.

In the next section we will define ψ on the set $A_\lambda^{(n)} \backslash B_\lambda^{(n)}$ so we will look for a map ψ with $\operatorname{im} \phi \subseteq B_\lambda^{(n)}$. The maps ϕ and ψ constructed in this section solve this problem for the subsets of $A_\lambda^{(n)}$ and X of weight $a_1^{\lambda_1} a_2^{\lambda_2} \cdots a_l^{\lambda_l}$ (where $l=l(\lambda)$).

The opposite extreme of this case are the subsets of $A_\lambda^{(n)}$ and X of weight $a_1 a_2 \cdots a_f$. Comparing the coefficients of those weights on both sides (3.1) gives the well-known identity

$$\left(\prod h_{ij}\right) f_\lambda = f!.$$

In this case only the map ϕ is needed since any pair (D, m) in $B_\lambda^{(n)}$ with $a_m=a_1 \cdots a_f$ has D consisting of a loop at each point. A map ϕ which handles this case was constructed by Franzblau and Zeilberger [1]. The general problem stated above is to interpolate between the Franzblau–Zeilberger case and the case answered in this section.

D. *Links and Simple R–C Digraphs*

In Section 3C we worked exclusively with the set $B_\lambda^{(l)}$, where the labelings m are column bijective. In later sections this will correspond to conjectures about the coefficients of the Jack symmetric functions in terms of the monomial symmetric functions. We will also be interested in how the Jack symmetric functions can be expanded in terms of power sum symmetric functions. In this case we will want to work with the larger set $A_\lambda^{(n)}$.

Note that if (D, m) is in $B_\lambda^{(n)}$ (so that m is column bijective) then it is impossible for there to be two squares in the some column which are the same cycle of D. This is no longer the case when we work with $(D, m) \in A_\lambda^{(n)}$. Our goal in this part is to set the stage for a cancellation argument which will be used to show that the contribution to the Jack symmetric function $J_\lambda(\alpha; \mathbf{x})$ made by the pairs in $A_\lambda^{(n)} \backslash B_\lambda^{(n)}$ is zero.

We need to set a total order on V_λ. Define $\lesssim$ to be the total order on V_λ which is lexicographic with respect to the second coordinate and then the first coordinate. So if $\lambda = 332$ and the squares of V_{332} are denoted by

$$\begin{array}{l} a \cdot b \cdot c \cdot \\ d \cdot e \cdot f \cdot \\ g \cdot h \cdot \end{array}$$

then $a \lesssim d \lesssim g \lesssim b \lesssim e \lesssim h \lesssim c \lesssim f$. In general, the squares of V_λ are ordered so that those in column i come before those in column $i+1$ and within column i the squares are ordered from top to bottom.

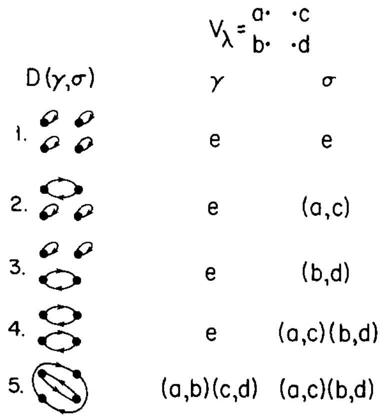
DEFINITION 3.8. *A link* in V_λ is a pair of squares from the same column.

We denote links by $\langle x, y \rangle$, where $x = (i, c)$ and $y = (j, c)$ with $i < j$. We let L_λ be the set of links of V_λ, so $|L_\lambda| = \sum_j \binom{\lambda_j + 1}{2}$. One more piece of notation will be important. If $\Lambda = \langle x, y \rangle$ is a link then $\gamma(\Lambda) = \gamma(\langle x, y \rangle)$ denotes the transposition in C_λ which exchanges x and y and leaves every other square fixed.

We extend the total order $\lesssim$ on V_λ to L lexicographically, and call this order $\lesssim$ as well. So, for $\langle x, y \rangle, \langle u, v \rangle \in L$ we have $\langle x, y \rangle \lesssim \langle u, v \rangle$ iff $x < u$ or $x = u$ and $y \lesssim v$.

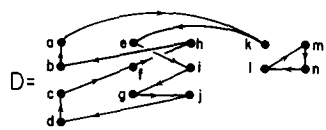
DEFINITION 3.9. Let D be an r–c digraph on V_λ, and let $\langle x, y \rangle$ be a link in V_λ. We say $\langle x, y \rangle$ *is inverted* by D if x and y lie in the same cycle of D. We say D is *simple* if it has no inverted links. Otherwise D is *complex*.

EXAMPLE 3.10. The five simple r–c digraphs of shape $\lambda = 2, 2$ appear below.



DEFINITION 3.11. Let $D = D(\gamma, \sigma)$ be a complex r–c digraph. Let C be a cycle of D which contains at least one inverted link. The r–c digraph D_C is the digraph $D((x, y)\gamma, \sigma)$, where $\langle x, y\rangle$ is the $\lesssim$-minimal inverted link contained in C. We say D_C is obtained from D by a *simple switch*.

For example, suppose $\lambda = 4431$ and

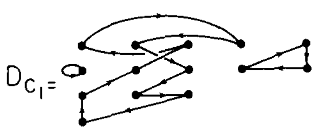


There are two cycles in D

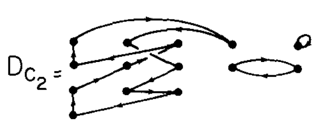
$$C_1 = (a, k, e, i, g, j, d, c, f, h, b)$$

$$C_2 = (l, m, n).$$

There are 12 links inverted by C_1 namely the set of links in the first 3 columns. Of these, $\langle a, b\rangle$ is the lexicographically smallest, so



There is only one link inverted by C_2, namely $\langle m, n\rangle$. So



DEFINITION 3.12. Let D be a complex r–c digraph. Let $P = C_1, C_2, ..., C_l$ be a sequence of cycles satisfying:

(1) C_i is a cycle of $D^{(i)} = (((D_{C_1})_{C_2} \cdots)_{C_{i-1}}$ which contains an inverted link.

(2) $D^{(l+1)} = (\cdots((D_{C_1})_{C_2})\cdots)_{C_l}$ is simple.

Then P is called a *simplification procedure* for D and the r–c digraph $D^{(l+1)}$ is called the *simplification of D along P* and is denoted D_p. The number l is called the *length* of P, denoted $l(P)$.

It is easy to see that the number of inverted links in D_C is less than the number of inverted links in D. So the length of any simplification procedure for D is less than or equal to the number of inverted links in D.

LEMMA 3.1. *Let P and Q be simplification procedures for D. Then $l(P) = l(Q)$ and $D_P = D_Q$.*

Proof. The proof is by induction on the number of inverted links of D. If that number is 1 then the result holds easily. Assume the result is true for simplification procedures of r–c digraphs with less than m inverted links. Suppose D has exactly m inverted links.

Write P and Q out as $P = (C_1, C_2, ..., C_l)$ and $Q = (E_1, E_2, ..., E_m)$. The proof splits into two cases:

Case 1. $C_1 = E_1$. Let $D' = D_{C_1} = D_{E_1}$. Then D' has fewer inverted links than D and $P' = (C_2, ..., C_l)$, $Q = (E_2, ..., E_m)$ are simplification procedures for D'. By our inductive hypothesis, $l - 1 = m - 1$ and

$$(D')_{P'} = (D')_{Q'}.$$

But note that $(D')_{P'} = D_P$ and $(D')_{Q'} = D_Q$ so the result follows.

Case 2. $C_1 \neq E_1$. Let I be the set of inverted links in D. Write I as a disjoint union

$$I = \bigcup_C I_C,$$

where I_C is the set of links inverted by C.

Let Γ, Ω be distinct cycles of D, both of which contain inverted links. Observe that

(i) Ω is a cycle in D_Γ.

(ii) I_Ω is contained in the set of inverted links in D_Γ.

In what follows, let Δ_j denote the jth step in the simplification of D along P,

$$\Delta_j = (\cdots((D)_{C_1})_{C_2} \cdots)_{C_j}.$$

Applying observations (i) and (ii) repeatedly we see that if $C_1, ..., C_i$ are each distinct from E_1 then E_1 is a cycle of Δ_i and I_{E_1} is contained in the set of links inverted by Δ_i. So at least one of the C_i's must be equal to E_1.

Choose i minimal such that $C_i = E_1$, and write $D = D(\gamma, \sigma)$. As the C_j with $j < l$ are distinct from E_1 we see that Δ_{i-1} is of the form

$$\Delta_{i-1} = D((x_{i-1}, y_{i-1}) \cdots (x_1, y_1)\gamma, \sigma),$$

where x_j and y_j are from C_j which is distinct from E_1. Let $\langle x, y\rangle$ be the minimal link inverted by E_1. So

$$\Delta_i = D((x, y)(x_{i-1}, y_{i-1}) \cdots (x_1, y_1)\gamma, \sigma).$$

As x, y are in E_1 and E_1 is disjoint from $\{x_1, y_1, ..., x_{i-1}, y_{i-1}\}$ we have

$$\begin{aligned}\Delta_i &= D((x_{i-1}, y_{i-1}) \cdots (x_1, y_1)(x, y)\gamma, \sigma)\\ &= (\cdots((D_{E_1})_{C_1})_{C_2} \cdots)_{C_{i-1}}.\end{aligned}$$

Thus $D_P = D_{P_1}$ and $l(P) = l(P_1)$, where P_1 is the simplification procedure $(E_1, C_1, ..., C_{i-1}, C_{i+1}, ..., C_l)$. But now by Case 1 we have

$$D_{P_1} = D_Q$$

and

$$l(P_1) = l(Q)$$

which completes the proof. ∎

DEFINITION 3.14. Let D be an r–c digraph. Define the *simplification of* D, $\Sigma(D)$, to be D_P for P any simplification procedure of D. Define the *level of* D, $l(D)$, to be the length of any simplification procedure of D.

By Lemma 11, $\Sigma(D)$ and $l(D)$ are well defined. The next lemma is one that we will need in Section 4. To state this result we need the notion of a minimal link.

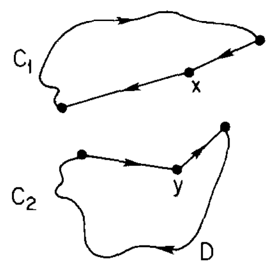
DEFINITION 3.15. Let D be an r–c digraph and let $\langle x, y\rangle$ be a link with x and y in cycles C_1 and C_2 of D. We say $\langle x, y\rangle$ is the *minimal link of* C_1 and C_2 if it is the lexicographically smallest link which contains one square from C_1 and one square from C_2.

Note that in the above definition C_1 and C_2 need not be distinct. In the case that $C_1 = C_2 = C$ we have used this minimal link of C before. In Definition 10 it was the minimal link of C which we switched to form D_C.

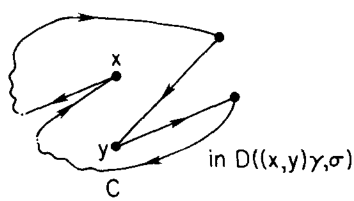
LEMMA 3.16. *Let $D = D(\gamma, \sigma)$ be an r–c digraph and let $\langle x, y\rangle$ be a minimal link of D. Then*

$$\Sigma(D((x, y)\gamma, \sigma)) = \Sigma(D).$$

Proof. If $\langle x, y\rangle$ is the minimal link of a single cycle C then $D((x, y)\gamma, \sigma) = D_C$ and the result is obvious. So assume x is in C_1 and y is in C_2, where $C_1 \neq C_2$.



In $D((x, y)\gamma, \sigma)$ the cycles C_1 and C_2 are joined into one long cycle C.



Also by the minimality of the link $\langle x, y\rangle$ in D_1 the link $\langle x, y\rangle$ is the minimal link of C in $D((x, y)\gamma, \sigma)$. So

$$D((x, y)\gamma, \sigma)_C = D$$

and the result follows. ∎

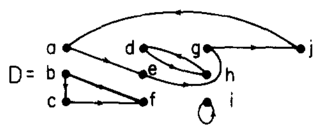
E. *The Link Digraph*

In this part we define something called the *link digraph* of an *r–c* digraph D. We believe that this link digraph has some connection to the weight conjectured to exist in the next section.

DEFINITION 3.17. Let D be an *r–c* digraph. Define a multidigraph $\mathbf{A}_P$ with edge set L (= set of links of V_λ) and vertex set equal to the set of cycles of D by putting an edge directed from C_1 to C_2 for every link $\langle x, y\rangle$ with x in C_1 and y in C_2.

Recall that in a link $\langle x, y\rangle$ we have x above y so the arrow directed from C_1 to C_2 means that C_1 and C_2 have points in the same column with the point of C_1 above the point of C_2. Note that A_D is simple if and only if A_D has no cycles.

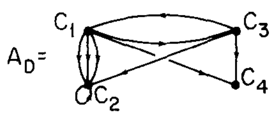
EXAMPLE 3.18. Let D be the r–c digraph



with cycles

$$C_1 = (a, e, g, i)$$
$$C_2 = (b, c, f)$$
$$C_3 = (d, h)$$
$$C_4 = (i).$$

Then A_D is given by



F. *Duality*

Let $(D, m) \in B_\lambda^{(n)}$ with $D = D(\gamma, \sigma)$ and let λ' be the diagram conjugate to λ. The set $V_{\lambda'}$ is the reflection of V_λ across the line $x = y$ in $\mathbb{R}^2$. Given $(i, j) = x$ in V_λ we let $x' = (j, i)$ be the reflected point in $V_{\lambda'}$. The permutations $\gamma \in C_\lambda$ and $\sigma \in R_\lambda$ correspond to permutations $\gamma' \in R_{\lambda'}$ and $\sigma' \in C_{\lambda'}$ by

$$\gamma'(x') = (\gamma x)'$$
$$\sigma'(x') = (\sigma x)'.$$

Also we define m' and D' by $D' = D(\sigma', \gamma')$ and $m'(x') = m(x)$. Note that D' is an r–c digraph on $V_{\lambda'}$ and m' is consistent with D'.

We call x', γ', σ', m', D' the *duals* of x, γ, σ, m, D, respectively. One thing to point out is that the signs of D and D' do not agree, in general. In fact,

$$\operatorname{sgn}(D') = \operatorname{sgn}(\gamma\sigma) \operatorname{sgn}(D)$$

for $D = D(\gamma, \sigma)$.

4. A Conjecture about the Jack Symmetric Functions

In this section we conjecture a combinatorial method for computing the Jack symmetric functions $J_\lambda(x_1, ..., x_n; \alpha)$. After stating the conjecture we go on to prove that any symmetric $g_\lambda(x_1, ..., x_n; \alpha)$ which is computed according to the rules given in the conjecture enjoys many of the same properties as the Jack function $g_\lambda(x_1, ..., x_n; \alpha)$.

Conjecture. There exists a weight function $w: A_\lambda^{(n)} \to \mathbb{N}$ satisfying the following conditions:

(C1) $w(D, m_1) = w(D, m_2)$ for any pair m_1, m_2 of consistent labellings of the r–c digraph D.

(2) If $D = D(\gamma, \sigma)$ and $c(\sigma)$ denotes the number of cycles of σ then

$$w(D(\gamma, \sigma), m) \leqslant |\lambda| - c(\sigma)$$

with equality if $A_{D(\gamma, \sigma)}$ is acyclic.

(C3) (Simplification rule) $w(D, m) = w(\Sigma(D), m)$.

(C4) (Corner rule) Let $k = (u, v)$ be a southeast corner of $D = D(\gamma, \sigma)$. Let $\sigma_0, \sigma_1, ..., \sigma_{v-1}, \gamma_0, \gamma_1, ..., \gamma_{u-1}$ be the coset representatives as chosen in Section 2A, and let λ_- be as in Section 2A. Then

1. $w(D(\gamma, \sigma)) = w(D(\gamma_-, \sigma_-))$ if $\gamma = \gamma_0 \gamma_-$, $\sigma = \sigma_0 \sigma_-$
2. $w(D(\gamma, \sigma)) = w(D(\gamma_-, \sigma_-))$ if $\gamma = \gamma_i \gamma_-$, $\sigma = \sigma_0 \sigma_-$ with $i \geqslant 1$.
3. $w(D(\gamma, \sigma)) = w(D(\gamma_-, \sigma_-)) + 1$ if $\gamma = \gamma_0 \gamma_-$, $\sigma = \sigma_j \sigma_-$ with $j \geqslant 1$
4. $w(D) = w(\rho_k(D))$ if $D = D(\gamma, \sigma)$ is nondegenerate.

(C5) (Duality) $w(D, m) + w(D', m') = |\lambda| - c(\gamma\sigma) = |\lambda| - c(\sigma'\gamma')$ for $D = D(\gamma, \sigma)$

(C6) (1st column heredity) Let $l = l(\lambda)$.

1. If $(D, m) \in B_\lambda^{(l)}$ is 1st column hereditary with ancestor $(\hat{D}, \hat{m})$ and connecting function f then

$$w(D, m) = w(\hat{D}, \hat{m}) + \deg(f).$$

2. If $(D, m) \in B_\lambda^{(l)}$ is not 1st column hereditary then

$$w(D, m) = w(\psi(D, m)).$$

(C7) (Jack condition)

$$\sum_{(D, m) \in A_\lambda^{(n)}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} x_m = J_\lambda(x_1, ..., x_n; \alpha).$$

Two properties of the sum $\Sigma_{(D,m)} \operatorname{sgn}(D)\, \alpha^{w(D,m)} x_m$ follow immediately from (C1). The first is that this sum is a symmetric function in $x_1, ..., x_n$. The second is that the sum has an expansion in terms of power sum symmetric functions given by

$$\sum_{(D,m)} \operatorname{sgn}(D)\, \alpha^{w(D,m)} x_m = \sum_{(\gamma,\sigma)} \operatorname{sgn}(\gamma)\, \alpha^{w(D(\gamma,\sigma))} p_{\gamma\sigma}(x_1, ..., x_n). \qquad (*)$$

Here $p_{\gamma\sigma}(x_1, ..., x_n)$ is the power sum symmetric function corresponding to the cycle type of $\gamma\sigma$, and $w(D(\gamma, \sigma))$ is the value of $w(D(\gamma, \sigma), m)$ for any labeling m consistent with $D(\gamma, \sigma)$. Equation $(*)$ follows immediately from the observation that $p_{\gamma\sigma}(x_1, ..., x_n)$ is equal to $\sum_m x_m$, where the sum is over all n-labelings of V_λ which are consistent with $D(\gamma, \sigma)$.

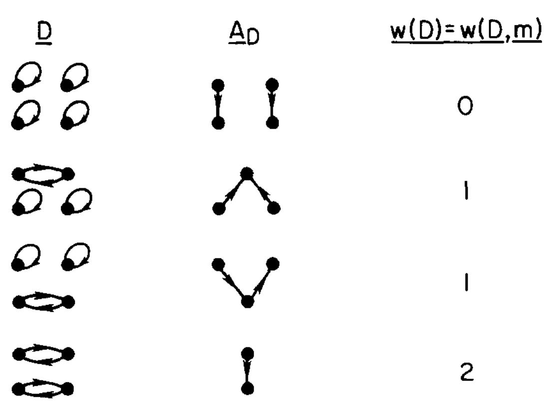
Given (C1) it would make more sense to speak of $w(D)$ and forget about the labeling m. However, we avoid this because it may be that condition (C1) is too strong and should be deleted. So there may be a weight function which depends on D and m and which satisfies (C2)–(C7) and the following weaker condition (C1′)

(C1′) For each $\pi \in \operatorname{Sym}(m)$ and each pair (D, m) we have

$$w(D, m) = w(D, \pi \circ m).$$

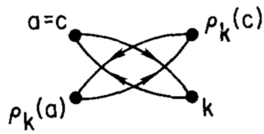
Before continuing we stop for an example. Let $\lambda = 2^2$. We will show that conditions (C1)–(C4) suffice in this case to determine the weight function w. We will check that this weight w does indeed give us the Jack symmetric function as in (C7).

By the simplification rule (C3), it is enough to determine w on simple r–c digraphs. The five simple r–c digraphs of shape $\lambda = 2^2$ were given in Example 3.10. The first four appear below along with their link digraphs A_D. The reader will observe that A_D is acyclic in all these cases so w is given by (C2):

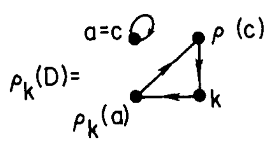


The fifth simple r–c digraph D does not have an acyclic link digraph A_D.

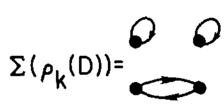
In this case (C2) only gives the information that $w(D) \leqslant 2$. But for this digraph D the corner rule applies with the corner k being the square in position (2, 2). This corner k is a nondegenerate southeast corner so part 4 of (C4) says that $w(D) = w(\rho_k(D))$. To find $\rho_k(D)$ first locate the predecessor c and the successor a to k in D. In this case $a = c =$ the square in position (1, 1). Next we find the unique squares $\rho_k(c)$ and $\rho_k(a)$ such that $\rho_k(c)$ is in the same row as c, $\rho_k(a)$ is in the same column as a and there is an edge from $\rho_k(c)$ to $\rho_k(a)$. In this case $\rho_k(a)$ and $\rho_k(c)$ are the squares in positions (2, 1) and (1, 2), respectively.



To obtain $\rho_k(D)$ we insert edges from $\rho_k(c)$ to k, from k to $\rho_k(a)$ and from c to a. We delete the edges from $\rho_k(c)$ to $\rho_k(a)$ from c to k and from k to a. Doing this we obtain



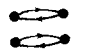
By (c4) we have $w(D) = w(\rho_k(D))$. The digraph $\rho_k(D)$ is not simple. Simplifying, we obtain



We see that $\sum(\rho_k(D))$ is one of our other four simple r–c digraphs and has weight 1. So we obtain

$$w(D) = w(\rho_k(D)) = w(\Sigma(\rho_k(D))) = 1.$$

Lastly we list, for each simple r–c digraph D, the sum of the $p_{\gamma\sigma}(x_1, ..., x_n)$ taken over all $\gamma \in C_\lambda$ and $\sigma \in R_\lambda$ with $\Sigma(D(\gamma, \sigma)) = D$.

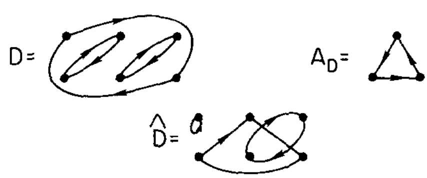
D	sum of $\{\alpha^{wt(D)}\mathrm{sgn}(\gamma)\, p_{\gamma\sigma}(x_1, \ldots, x_n)\}$
	$(p_1^4 - 2p_1^2 p_2 + p_2^2)$
	$\alpha(p_1^2 p_2 - 2p_1 p_3 + p_4)$
	$\alpha(p_1^2 p_2 - 2p_1 p_3 + p_4)$
	$\alpha^2(p_2^2 - p_4)$
	$\alpha(p_2^2 - p_4)$

According to (C7), adding, these will give the Jack symmetric function $J_{22}(\alpha; \mathbf{x})$. Summing we obtain

$$p_1^4 + 2(\alpha - 1)\, p_1^2 p_2 + (\alpha^2 + \alpha + 1)\, p_2^2 - 4\alpha p_1 p_3 + (-\alpha^2 + \alpha)\, p_4$$

which is indeed $J_{22}(\alpha; \mathbf{x})$.

In this case, rules (C1)–(C4) completely determine the weight function w. It is also true that $w(D)$ is determined for all r–c digraphs D on 5 or fewer points by (C1)–(C4). In all these cases (C7) holds, i.e., the weight function so determined gives the Jack symmetric function. However, the weight of the following r–c digraph on 6 points cannot be determined from (C1)–(C4) alone.



5. Properties that Follow from Conditions (C1)–(C6)

In this section we assume that w is a weight function defined on $A_\lambda^{(n)}$ for all shapes λ and all n, and we assume that w satisfies conditions (C1)–(C6). For each λ and each n we define $g_\lambda(x_1, \ldots, x_n; \alpha)$ by

$$g_\lambda(x_1, \ldots, x_n; \alpha) = \sum_{(D, m) \in A_\lambda^{(n)}} \mathrm{sgn}(D)\, \alpha^{w(D, m)} x_m.$$

Our goal in this section is to show that these $g_\lambda(x_1, \ldots, x_n; \alpha)$ satisfy many of the properties that are known to be true or conjectured to be true for the Jack symmetric functions $J_\lambda(x_1, \ldots, x_n; \alpha)$.

As mentioned in Section 4, it may be that condition (C1) is too strong and should be replaced with the weaker condition (C1′). There is only one property in the following list whose proof uses the full strength of (C1). We will clearly mark that result and prove all the other results using just (C1′) and (C2)–(C6).

We begin with some elementary properties of the $g_\lambda(x_1, ..., x_n; \alpha)$ which will be stated without proof.

PROPERTY 1. *The $g_\lambda(x_1, ..., x_n; \alpha)$ are symmetric functions in $x_1, ..., x_n$. Moreover, when $g_\lambda(x_1, ..., x_n; \alpha)$ is expanded in terms of any of the standard bases for the ring of symmetric functions, the coefficients are polynomials in α.*

Here the standard bases include power sum symmetric functions, monomial symmetric functions, Schur functions, and elementary symmetric functions. It has been conjectured that the coefficients of the Jack symmetric functions in terms of these standard bases are polynomials in α. At present it it only known that they are rational functions in α.

PROPERTY 2. *If λ is a hook-shape then*

$$g_\lambda(x_1, ..., x_n; \alpha) = \sum_{\substack{\gamma \in C_\lambda \\ \sigma \in R_\lambda}} \operatorname{sgn}(\gamma)\, \alpha^{n-c(\sigma)} p_{\gamma\sigma}(x_1, ..., x_n; \alpha).$$

Proof. First note that for any r–c digraph $D = D(\gamma, \sigma)$ the simplification of D is of the form $\Sigma(D) = D(\gamma, \sigma)$, i.e., the row stabilizers contributing to D and $\Sigma(D)$ are identical. So, by (C3), it is enough to show that $w(D(\gamma, \sigma), m) = n - c(\sigma)$ for $D(\gamma, \sigma)$ simple. This follows from (C2) and the observation that if D is a simple r–c digraph on V_λ, where λ is a hook-shape then A_D is acyclic.

I. G. Macdonald recently proved that Property 2 holds for Jack symmetric functions. His proof is as yet unpublished.

PROPERTY 3. *Let f_λ denote the number of standard Young tableaux of shape λ and let $s_\lambda(x_1, ..., x_n)$ denote the Schur function corresponding to shape λ. Then*

$$g_\lambda(x_1, ..., x_n; 1) = \frac{f!}{f_\lambda} s_\lambda(x_1, ..., x_n),$$

where $f = |\lambda|$.

Proof. Since $\alpha=1$, the weight function w is irrelevant. We have

$$\begin{aligned} g_\lambda(x_1, ..., x_n; 1) &= \sum_{(D,m)\in A_\lambda^{(n)}} \operatorname{sgn}(D)\, x_m \\ &= \sum_{\substack{\gamma\in C_\lambda \\ \sigma\in R_\lambda}} \operatorname{sgn}(\gamma)\, p_{\gamma\sigma}(x_1, ..., x_n) \\ &= \frac{1}{f_\lambda}\left\{ \sum_{\substack{t \text{ syt} \\ \text{shape } \lambda}} \sum_{\substack{\gamma\in C_\lambda \\ \sigma\in R_\lambda}} \operatorname{sgn}(\gamma)\, p_{\gamma\sigma}(x_1, ..., x_n)\right\} \end{aligned}$$

It is a well-known result due to Young that the equation

$$\sum_{\substack{t \text{ syt} \\ \text{shape } \lambda}} \sum_{\substack{\gamma\in R_t \\ \sigma\in R_t}} \operatorname{sgn}(\gamma)\, \gamma\sigma = f_\lambda\left(\sum_{\pi\in S_f} \chi^\lambda(\pi)\, \pi\right)$$

holds in the group algebra $\mathbb{C}S_f$ (here χ^λ is the irreducible character of S_f indexed by λ). So we have

$$\begin{aligned} g_\lambda(x_1, ..., x_n; 1) &= \left(\sum_{\pi\in S_f} \chi^\lambda(\pi)\, p_\pi(x_1, ..., x_n)\right) \\ &= \frac{f!}{f_\lambda} s_\lambda(x_1, ..., x_n) \end{aligned} \tag{5.1}$$

the last equality being another well-known result.

Property 3 is known to hold for the Jack symmetric functions.

PROPERTY 4 (Assuming (C1)). *The first few coefficients of $g_\lambda(x_1, ..., x_n; \alpha)$ expanded in terms of power sums are*

$$g_\lambda(x_1, ..., x_n; \alpha) = d_{1^f}(\alpha)\, p_1^f + d_{1^{f-2}2}(\alpha)\, p_1^{f-2} p_2 + d_{1^{f-3}3}(\alpha)\, p_1^{f-3} p_3 + \cdots,$$

where

$$d_{1^f}(\alpha) = 1$$

$$d_{1^{f-2}2}(\alpha) = \alpha\left(\sum_i \binom{\lambda_i}{2}\right) - \left(\sum_j \binom{\lambda'_j}{2}\right)$$

$$d_{1^{f-3}3}(\alpha) = \alpha^2\left(\sum_i 2\binom{\lambda_i}{3}\right) - \alpha\left(f^2 - f\lambda'_1 - \sum_i \lambda_i(\lambda_i - 1)\right) + \left(\sum_j 2\binom{\lambda'_j}{3}\right).$$

Proof. These are straightforward (though tedious) to prove by first identifying all pairs $(\gamma, \sigma)\in C_\lambda \times R_\lambda$ with $\gamma\sigma$ having cycle type 1^f, $1^{f-2}2$, and $1^{f-3}3$. For every such pair, it is easy to find the simplification of $D(\gamma, \sigma)$.

One finds in every case that $A_{\Sigma(D(\gamma,\sigma))}$ is acyclic and so the weight of $D(\gamma,\sigma)$ is given by (C2) and (C3). The details are left to the reader.

We now come to some of the more interesting properties of the $g_\lambda(x_1, ..., x_n; \alpha)$.

PROPERTY 5 (Assuming (C1)). *There is the following duality which relates the coefficients of $g_\lambda(x_1, ..., x_n; \alpha)$ expanded in terms of power sums to the coefficients of $g_{\lambda'}(x_1, ..., x_n; \alpha)$ expanded in terms of power sums: if $g_\lambda(x_1, ..., x_n; \alpha) = \sum_\mu d_{\lambda\mu}(\alpha)\, p_\mu(x_1, ..., x_n; \alpha)$ then*

$$d_{\lambda\mu}(\alpha) = (-\alpha)^{n-l(\mu)}\, d_{\lambda'\mu}(\alpha^{-1}).$$

Proof. Writing $g_\lambda(x_1, ..., x_n; \alpha)$ in terms of power sums we have

$$g_\lambda(x_1, ..., x_n; \alpha) = \sum_{\substack{\gamma \in C_\lambda \\ \sigma \in R_\lambda}} \operatorname{sgn}(\gamma)\, \alpha^{w(D(\gamma,\sigma))} p_{\gamma\sigma}(x_1, ..., x_n).$$

By the duality rule, $\alpha^{w(D(\gamma,\sigma))} = \alpha^{n-c(\gamma\sigma)}\alpha^{-w(D(\sigma',\gamma'))}$. Also $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\gamma)\operatorname{sgn}(\gamma\sigma) = \operatorname{sgn}(\gamma)(-1)^{n-c(\gamma\sigma)}$. So

$$\begin{aligned} g_\lambda(x_1, ..., x_n; \alpha) &= \sum_{\substack{\gamma \in R_{\lambda'} \\ \sigma \in C_{\lambda'}}} (-\alpha)^{n-c(\gamma\sigma)}\, \alpha^{-w(D(\sigma',\gamma'))} p_{\sigma'\gamma'}(x_1, ..., x_n) \\ &= \sum_\mu (-\alpha)^{n-c(\mu)}\, d_{\lambda'\mu}(\alpha^{-1})\, p_\mu(x_1, ..., x_n). \end{aligned}$$

Property 5 was shown to be true for Jack symmetric functions by Macdonald. The next result says that the transition matrix from the $g_\lambda(x_1, ..., x_n; \alpha)$ to the monomial symmetric functions $m_\mu(x_1, ..., x_n)$ is triangular.

PROPERTY 6. *Let $c_{\lambda,\mu}(\alpha)$ be the coefficient of $m_\mu(x_1, ..., x_n)$ in the expansion of $g_\lambda(x_1, ..., x_n; \alpha)$ in terms of monomials. Then $c_{\lambda,\mu}(\alpha) = 0$ unless λ dominates μ.*

Proof. Let m be an n-labeling of V_λ which is not column bijective. Let $\langle x, y\rangle$ be the lexicographically smallest link with $m(x) = m(y)$. Let $D = D(\gamma,\sigma)$ be an r–c digraph consistent with m. Note that m is also consistent with $D((x, y)\gamma, \sigma)$. Also one of these two r–c digraphs has x and y in the same cycle. Without loss of generality we may assume x and y are in the same cycle of $D(\gamma,\sigma)$.

Since $\langle x, y\rangle$ is the minimal link in $D(\gamma,\sigma)$ with $m(x) = m(y)$ then $\langle x, y\rangle$ is the minimal link inverted in $D(\gamma,\sigma)$. So if C denotes the cycle in $D(\gamma,\sigma)$ containing x and y then $D((x, y)\gamma, \sigma) = D(\gamma,\sigma)_C$. In particular,

$D(\gamma, \sigma)$ and $D((x, y)\gamma, \sigma)$ have the same simplification so $w(D(\gamma, \sigma), m) = w(D((x, y)\gamma, \sigma), m)$. Also they have opposite signs so the contributions to $g_\lambda(x_1, ..., x_n; \alpha)$ made by $D(\gamma, \sigma), m$ and $D((x, y)\gamma, \sigma), m$ cancel.

This cancellation shows that for any n-labeling m which is not column bijective we have

$$\sum_D \operatorname{sgn}(D)\, \alpha^{w(D, m)} = 0,$$

where the sum is over all r–c digraphs D consistent with m. Hence

$$g_\lambda(x_1, ..., x_n; \alpha) = \sum_{(D, m) \in B_\lambda^{(n)}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} x_m.$$

But if m is a column bijective labeling of V_λ of type μ then λ must dominate μ, which completes the result.

This triangularity property is one of the defining conditions for the Jack symmetric functions.

PROPERTY 7. *Let $l = l(\lambda)$ and $\hat{\lambda}$ be the partition obtained from λ by subtracting* 1 *from every part. Then*

$$g_\lambda(x_1, ..., x_l; \alpha) = \left(\prod_{i=1}^{l} ((l-i+1) + (\lambda_i - 1)\,\alpha)\right) (x_1 \cdots x_l)\, g_{\hat{\lambda}}(x_1, ..., x_l; \alpha).$$

Proof. By Theorem 3.6 and condition (C6) we have

$$\begin{aligned} g_\lambda(x_1, ..., x_l; \alpha) &= \sum_{(f, (\hat{D}, \hat{m}))} (\alpha^{\deg f}) \operatorname{sgn}(\hat{D})\, \alpha^{w(\hat{D}, \hat{m})}(x_1, ..., x_l)\, x_{\hat{m}}. \\ &= \left(\sum_f \alpha^{\deg f}\right) (x_1 \cdots x_l) \left(\sum_{(\hat{D}, \hat{m})} \operatorname{sgn}(\hat{D})\, \alpha^{w(\hat{D}, \hat{m})} x_{\hat{m}}\right) \\ &= \left(\sum_f \alpha^{\deg f}\right) (x_1 \cdots x_l)\, g_{\hat{\lambda}}(x_1, ..., x_l; \alpha). \end{aligned}$$

It is easy to see that

$$\left(\sum_f \alpha^{\deg f}\right) = \left(\prod_{i=1}^{l} ((l-i+1) + \lambda_i - 1)\,\alpha)\right)$$

which completes the result.

Property 7 is one of the many things shown to be true about the Jack symmetric functions by R. P. Stanley [7]. A related result, also obtained by Stanley for the Jack symmetric functions is the following.

PROPERTY 8. *The coefficient of $m_\lambda(x_1, ..., x_n)$ in the expansion of $g_\lambda(x_1, ..., x_n; \alpha)$ in terms of monomials is*

$$c_{\lambda\lambda}(\alpha) = \prod_{j=1}^{\lambda_1} \prod_{i=1}^{\lambda_j} ((\lambda_j - i + 1) + \alpha(\lambda_i - j)).$$

Proof. We begin with

$$g_\lambda(x_1, ..., x_n; \alpha) = \sum_{(D, m) \in B_\lambda^{(n)}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} x_m \tag{5.2}$$

We will compute the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} ... x_l^{\lambda_l}$ on the right hand side of (5.2). This will be the coefficient $c_{\lambda\lambda}(\alpha)$ by Property 1.

Suppose (D, m) contributes to the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} ... x_l^{\lambda_l}$ on the right-hand side of (5.2). Then m is column bijective so the labels assigned by m to the squares in column j of V_λ are $1, 2, ..., \lambda_j$. We define the notion of *totally hereditary* for pairs $(D, m) \in B_\lambda^{(l)}$ with the property that the labels assigned to the squares in column j are $1, 2, ..., \lambda_j$. We say (D, m) is *totally hereditary* if $(D, m) = \varphi(f, (\hat{D}, \hat{m}))$, where $(\hat{D}, \hat{m})$ is totally hereditary. It is easy to see that totally hereditary pairs (D, m) are in one to one correspondence with sequences $(f_1, f_2, ..., f_{\lambda_1})$, where f_j is a hook-choice function for column j of V_λ. The one to one correspondence is

$$(D, m) \leftrightarrow \varphi(f_1, \varphi(..., \varphi(f_{\lambda_1 - 1}, (\varphi(f_{\lambda_1}), \varphi))...)).$$

Given this correspondence, the sign of D is $+1$ and the weight of (D, m) is $\sum_j \deg(f_j)$. Let T_λ be the set of totally hereditary pairs (D, m) in $B_\lambda^{(n)}$.

CLAIM. $\sum_{(D, m)} \operatorname{sgn}(D)\, \alpha^{w(D, m)} = \sum_{(D, m) \in T_\lambda} \alpha^{w(D, m)}$, *where the sum on the left is over all pairs $(D, m) \in B_\lambda^{(n)}$ with $x_m = x_1^{\lambda_1} ... x_l^{\lambda_l}$.*

Proof of Claim. We prove this by induction on $|\lambda|$. By Theorem 3.6 we have

$$\sum_{(D, m)} \operatorname{sgn}(D)\, \alpha^{w(D, m)} = \sum_{(f_1, (\hat{D}, \hat{m}))} \operatorname{sgn}(\hat{D})\, \alpha^{\deg(f_1)} \alpha^{w(\hat{D}, \hat{m})}$$

$$= \sum_{f_1} \alpha^{\deg(f_1)} \left\{ \sum_{(\hat{D}, \hat{m})} \operatorname{sgn}(\hat{D})\, \alpha^{w(\hat{D}, \hat{m})} \right\}. \tag{5.3}$$

Note that the pairs $(\hat{D}, \hat{m})$ which can occur in the summation on the right-hand side of (5.3) are pairs in $B_{\hat{\lambda}}^{(n)}$ with the property that $x_{\hat{m}} = x_1^{\hat{\lambda}_1} x_2^{\hat{\lambda}_2} ...$. By our induction hypothesis it follows that

$$\sum_{(D, m)} \operatorname{sgn}(D)\, \alpha^{w(D, m)} = \sum_{f_1} \alpha^{\deg(f_1)} \left\{ \sum_{(\hat{D}, \hat{m}) \in T_{\hat{\lambda}}} \alpha^{w(\hat{D}, \hat{m})} \right\}$$

$$= \sum_{(D, m) \in T_\lambda} \alpha^{w(D, m)},$$

the last equality obtained using the definition of T_λ and using (C6) to obtain $w(D, m) = \deg(f_1) + w(\hat{D}, \hat{m})$.

Given this claim we have

$$c_{\lambda\lambda}(\alpha) = \sum_{(D, m) \in T_\lambda} \alpha^{w(D, m)} = \sum_{(f_1, \ldots, f_{\lambda_1})} \alpha^{\Sigma \deg(f_i)}.$$

It is easy to see that

$$\sum_{(f_1, \ldots, f_{\lambda_1})} \alpha^{\Sigma \deg(f_j)} = \prod_{j=1}^{\lambda_1} \prod_{i=1}^{\lambda_j'} ((\lambda_j - i + 1) + \alpha(\lambda_i - j))$$

and this completes the result.

PROPERTY 9. *If we set all n variables equal to* 1 *in* $g_\lambda(x_1, \ldots, x_n; \alpha)$, *the value is*

$$g_\lambda(1, 1, \ldots, 1; \alpha) = \prod_{(i, j) \in \lambda} (n + (j-1)\,\alpha - (i+1)).$$

Proof. We will prove this by induction on $|\lambda|$. For $|\lambda| = 1$ the result is easy. For a general shape λ let $k = (u, \lambda_u)$ be a corner square of λ. Note that k is a southeast corner in every r–c digraph D of shape λ.

Let λ_- be the partition λ with the corner k removed. Let $\lambda_0, \lambda_1, \ldots, \lambda_{u-1}$; $\sigma_0, \sigma_1, \ldots, \sigma_{\lambda_u - 1}$; R_{λ_-}; C_{λ_-} be as in Section 3, part A. We begin by writing $g_\lambda(1, 1, \ldots, 1; \alpha)$ as

$$g_\lambda(1, 1, \ldots, 1; \alpha) \sum_{(D, m) \in B_\lambda^{(n)}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} \tag{5.4}$$

We split the right-hand side of (5.4) into 4 sums

$$\sum_{(D, m) \in B_\lambda^{(n)}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} = S_1 + S_2 + S_3 + S_4,$$

where

$$S_1 = \sum_{\substack{\gamma_- \in C_{\lambda_-} \\ \sigma_- \in R_{\lambda_-}}} \sum_m \operatorname{sgn}(\gamma_i)\, \alpha^{w(D(\gamma_0\gamma_-,\, \sigma_0\sigma_-),\, m)}$$

$$S_2 = \sum_{j=1}^{\lambda_u - 1} \sum_{\substack{\gamma_- \in C_{\lambda_-} \\ \sigma_- \in R_{\lambda_-}}} \sum_m \operatorname{sgn}(\gamma_-)\, \alpha^{w(D(\gamma_0\gamma_-,\, \sigma_j\sigma_-),\, m)}$$

$$S_3 = \sum_{i=1}^{u-1} \sum_{\substack{\gamma_- \in C_{\lambda_-} \\ \sigma_- \in R_{\lambda_-}}} \sum_m -\operatorname{sgn}(\gamma_-)\, \alpha^{w(D(\gamma_i\gamma_-,\, \sigma_0\sigma_-),\, m)}$$

and

$$S_4 = \sum_{i=1}^{u} \sum_{j=1}^{\lambda_u - 1} \sum_{\substack{\gamma_- \in C_{\lambda_-} \\ \sigma_- \in R_{\lambda_-}}} \sum_{m} \operatorname{sgn}(D(\gamma_i \gamma_-, \sigma_j \sigma_-))\, \alpha^{w(D(\gamma_i \gamma_-, \sigma_j \sigma_-), m)}.$$

Combining Lemma 3.3 with condition (C4) part 4 we have $S_4 = 0$. In S_1 the digraphs $D(\gamma_0 \gamma, \sigma_0 \sigma_-)$ have a loop at the corner k. So for each labeling m_- of V_{λ_-} consistent with $D(\gamma_-, \sigma_-)$, there are exactly n labelings of V_λ consistent with $D(\gamma_0 \gamma_-, \sigma_0 \sigma_-)$ which restrict to m_-. Condition (C4) part 1 says that for all these labelings m we have

$$w(D(\gamma_0 \gamma_-, \sigma_0 \sigma_-), m) = w(D(\gamma_-, \sigma_-), m_-).$$

So S_1 can be rewritten as

$$\begin{aligned} S_1 &= n \left(\sum_{(D, m) \in A^{(n)}_{\lambda_-}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} \right) \\ &= n \left(\prod_{(i, j) \in \lambda_-} (n + (j-1)\,\alpha - (i-1)) \right). \end{aligned}$$

The latter equality following by our induction hypothesis.

In the digraphs $D(\gamma_0 \gamma_-, \sigma_j \sigma_-)$ summed in S_2, the corner k is a cycle which includes points of V_{λ_-}. So for each labeling m_- consistent with $D(\gamma_-, \sigma_-)$ there is a unique labeling m consistent with $D(\gamma_0 \gamma_-, \sigma_j \sigma_-)$ which restricts to m_-. Using (C4) part 3 we have

$$\begin{aligned} S_2 &= \sum_{j=1}^{\lambda_\mu - 1} \left(\sum_{(D, m) \in A^{(n)}_{\lambda_-}} \operatorname{sgn}(D)\, \alpha^{w(D, m) + 1} \right) \\ &= (\lambda_u - 1)\,\alpha \left(\prod_{(i, j) \in \lambda_-} (n + (j-1)\,\alpha - (i-1)) \right). \end{aligned}$$

Lastly, by a similar argument we obtain

$$\begin{aligned} S_3 &= \sum_{i=1}^{u-1} \left(\sum_{(D, m) \in A^{(n)}_{\lambda_-}} \operatorname{sgn}(D)\, \alpha^{w(D, m)} \right) \\ &= -(u-1) \left(\prod_{(i, j) \in \lambda_-} (n + (j-1)\,\alpha + (i-1)) \right). \end{aligned}$$

Summing S_1, S_2, S_3 we obtain

$$\begin{aligned} g_\lambda(1, 1, ..., 1; \alpha) &= (n + (\lambda_u - 1)\,\alpha - (u-1)) \\ &\quad \times \left(\sum_{(i, j) \in \lambda_-} (n + (j-1)\,\alpha - (i-1)) \right) \\ &= \prod_{(i, j) \in \lambda} (n + (j-1)\,\alpha - (i-1)). \end{aligned}$$

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