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MODELING NONLINEAR RESONANCE: A MODIFICATION TO THE STOKES' PERTURBATION EXPANSION

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The Stokes' series is a small amplitude perturbation expansion for nonlinear, steadily translating waves of the form $u(x-ct)$. We have developed a modification to the Stokes' perturbation expansion to cope with the type of resonance that occurs when two different wavenumbers have identical phase speeds. Although the nonlinear wave is smooth and bounded at the resonance, the traditional Stokes' expansion fails because of the often-encountered "small denominator" problem. The situation is rectified by adding the resonant harmonic into the expansion at lowest order. The coefficient of the resonant wave is determined at higher order. Near resonance is treated by expanding the dispersion parameter in terms of the amplitude. As an example, we have chosen the Korteweg de Vries equation with an additional fifth degree dispersion term. However, the method is applicable to the amplitude expansions of much more complicated problems, such as the double cnoidal waves of the Korteweg de Vries equation, the problem that motivated this study.

1. Introduction

A sine wave is an exact solution to most wave equations in the limit of infinitesimal amplitude. When the amplitude is finite, however, nonlinearity modifies the wave in two ways. First, the phase speed is a function of amplitude, as shown by Sir G.G. Stokes and discussed in detail by Whitham [1]. Second, the steadily translating wave becomes the sum of the fundamental sine wave plus its harmonics.

Some nonlinear wave equations can be solved exactly via the inverse scattering transform, but unfortunately, this method applies to only a handful of special equations. For general nonlinear equations, however, one can calculate an appropriate analytic solution in the form of an asymptotic series, the "Stokes' expansion" [1-3]. Both the phase speed and the dependent variable are expanded in powers of the amplitude parameter.

Whenever the parameters of the problem are such that two waves of different wavenumber have the same nonlinear phase speed, they become harmonically resonant. In a traditional perturbation expansion, the denominator of a higher order term vanishes, causing the expansion to break down. Nayfeh [4] shows that the remedy is to add in the resonant term at lower order. Here, we apply a similar strategy to the Stokes' expansion.

Wave resonance is a real phenomenon which appears in many physical systems. We first became interested in the problem of resonance while studying a two-dimensional wave equation. Where we expected a rapid monotonic decrease in the amplitudes of the succeeding waves, we instead found that one of the higher harmonics was anomalously large since it was resonant with one of the lower order wave components. Thus, it is useful to determine a method to study such resonances.

The remainder of this paper describes the details of the Stokes' expansion and its modification for resonant cases. In Section 2, we introduce the Super Korteweg de Vries equation and apply the Stokes'

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perturbation expansion to it in Section 3. We show what happens when the parameters are such that a resonance occurs. The Stokes' expansion is modified in Section 4 to treat resonance. In Section 5, we develop the Galerkin numerical solution to the problem. The generalization to "near" resonance (small but nonzero denominators) is dealt with in Section 6. Section 7 is used to discuss some specific observations, such as higher wavenumber resonance, limits of validity, and relation to double cnoidal waves. Conclusions are summarized in the final section.

2. The Super Korteweg de Vries equation (SKDV)

As a test problem, we have chosen the Korteweg de Vries equation with an additional fifth degree dispersion term,

$$u_t + uu_x + \mu u_{xxx} - \nu u_{xxxxx} = 0. \tag{2.1}$$

We will refer to (2.1) as the Super Korteweg de Vries (SKDV) equation.

The SKDV equation is a model for several physical phenomena including shallow water waves near a critical value of surface tension, magneto-acoustic waves propagating at an angle to an external magnetic field, and waves in a nonlinear LC circuit with mutual inductance between neighboring inductors [5].

Since (2.1) has two independent variables, one dependent variable, and two free parameters, two of its degrees of freedom may be fixed without loss of generality. If u is a solution to (2.1), a one-parameter family of new solutions may be generated via the similarity transform

$$u(x, t) \rightarrow \lambda^3 u(\lambda x, \lambda^4 t), \tag{2.2}$$

with

$$\mu \rightarrow \lambda\mu, \qquad \nu \rightarrow \lambda^{-1}\nu,$$

where λ is a constant. We may therefore choose $\mu = 1$ without losing generality. A second useful similarity transform is

$$u(x, t) \rightarrow \alpha^2 u(\alpha x, \alpha^3 t), \tag{2.3}$$

with

$$\mu \rightarrow \mu, \qquad \nu \rightarrow \alpha^{-2}\nu,$$

where α is a constant. This implies that we may also choose an arbitrary period (2π) without loss of generality.

We are interested in traveling wave solutions of the form $u(x - ct)$ where c represents the phase speed so (2.1) becomes

$$(u - c)u_X + u_{XXX} - \nu u_{XXXXX} = 0, \tag{2.4}$$

where $X = x - ct$ is the phase variable. The linearized dispersion relation is

$$c = -k^2 - \nu k^4. \tag{2.5}$$

The SKDV has only three polynomial-type conserved quantities, in contrast to the KDV equation ($\nu = 0$) which has an infinite number of conservation laws. Therefore, the SKDV is nonintegrable whereas the KDV can be solved via the inverse scattering transform. Although no *general* solution is known, exact solutions of the SKDV have been found for special cases of both solitary waves and periodic cnoidal waves by Kano and Nakayama [6] and Yamamoto and Takizawa [7].

By varying the parameters μ and ν, the SKDV becomes a rich source of nonlinear phenomena. Our study is only the most recent in a line of investigations that have used it to better understand nonintegrable equations. Kawahara [8] found that the fifth degree term gives the tails of the solitary wave a structure of damped oscillations. Gorshkov, Ostrovskii, and Papko [9] studied the interactions of solitary waves of the SKDV both theoretically and experimentally, showing that a bound state could be formed. Further interaction studies were done by Nagashima and Kawahara [10]. Yoshimura and Watanabe [5] used the SKDV to study recurrence and the onset of turbulence. More recently, Boyd [2] has used a form of eq. (2.4) to test a variety of analytical and numerical techniques.

For the remainder of this paper, we will work with the SKDV in the form of eq. (2.4) with $\mu = 1$, a period of 2π, and ν and the wave amplitude a as the only free parameters.

For purposes of comparison, an "exact" solution was computed via a Fourier/Newton-Kantorovich algorithm similar to Boyd [2], but using Galerkin's method instead of the pseudospectral.

3. The nonresonant Stokes' expansion

Boyd [2] showed that for small amplitude, the periodic solutions of the SKDV may be accurately represented by expanding both the independent variable, u, and the phase speed, c, in terms of a, the coefficient of $\cos(X)$. Henceforth, we will refer to a simply as "the amplitude" although it is possible to define other less convenient measures of the size of the wave. Specifically, we let

$$u = \sum_{i=1}^{N} a^i u_i(X), \qquad c = \sum_{i=0}^{N-1} a^i c_i. \tag{3.1a, b}$$

The lowest order solution is taken to be

$$u_1 = \cos(X), \tag{3.2a}$$

from which we find

$$c_0 = -1 - \nu. \tag{3.2b}$$

Continuing the expansion by matching powers of a at each order, we find the ith order equation is

$$a^i: \quad -\nu u_{i,XXXXX} + u_{i,XXX} - c_0 u_{i,X} = F_i(X), \tag{3.3}$$

where

$$F_i(X) = -\sum_{j=1}^{i-1} (u_j - c_j) u_{i-j,X} .$$

At each order in a, we expand the solution, u_i, in a finite Fourier cosine series and the right-hand side, F_i, in a sine series,

$$u_i(X) = \sum_{k=1}^{N} u_{ik} \cos(kX), \qquad F_i(X) = \sum_{k=1}^{N} f_{ik} \sin(kX).$$

Substituting into eq. (3.3), we find

$$u_{ik} = \frac{f_{ik}}{\nu k^5 + k^3 + c_0 k}. \tag{3.4}$$

Since for $k=1$ the denominator of u_{i1} is zero, the solvability condition is to cancel any nonlinearly forced wavenumber 1 components on the right-hand side, F_i. This is easily accomplished by choosing c_{i-1} so that $f_{i1}=0$. Then all f_{ik} are completely determined by the lower order solutions and (3.4) gives the solution at ith order.

Completing the solutions to fifth order in a with the help of the REDUCE algebraic manipulation language, we find:

$$u_2 = \frac{1}{12(5\nu+1)} \cos(2X), \tag{3.5a}$$

$$u_3 = \frac{1}{192(10\nu+1)(5\nu+1)} \cos(3X), \tag{3.5b}$$

$$u_4 = \frac{(35\nu^2+12\nu+1)\cos(4X) - (425\nu^2+42\nu+1)\cos(2X)}{3456(17\nu+1)(1250\nu^4+875\nu^3+225\nu^2+25\nu+1)}, \tag{3.5c}$$

$$u_5 = \frac{5(650\nu^3+245\nu^2+28\nu+1)\cos(5X) - 9(9870\nu^3+1437\nu^2+68\nu+1)\cos(3X)}{331\,776(26\nu+1)(212\,500\nu^6+182\,500\nu^5+63\,125\nu^4+11\,200\nu^3+1070\nu^2+52\nu+1)}. \tag{3.5d}$$

The corrections to the phase speeds are:

$$c_1 = c_3 = c_5 = 0, \qquad c_2 = \frac{1}{24(5\nu+1)}, \tag{3.6a, b}$$

$$c_4 = \frac{-35\nu+1}{13\,824(1250\nu^4+875\nu^3+225\nu^2+25\nu+1)}. \tag{3.6c}$$

Note, however, that when the free parameter ν is chosen so that the denominator of (3.4) becomes zero for wavenumbers other than one, the corresponding term in the Stokes' expansion (3.5) is infinite. This "small denominator" problem is the sign of a harmonic resonance. In our case, the resonance condition of wavenumber k is

$$\nu = \nu_{\text{res}}(k) = -\frac{1}{k^2+1}. \tag{3.7}$$

In such resonance, the phase speed of the kth wave is precisely the same as that of the lowest order solution for $k=1$ as shown by the linearized dispersion relation, eq. (2.5). Although (3.7) is the condition for an exact resonance, it should be noted that there is actually an amplitude-dependent neighborhood about ν_{res} for which the denominator is so small that the Stokes' expansion is invalidated. This neighborhood shrinks to a point for infinitesimal a, but widens like a fan in the $\nu - a$ plane as a is increased. An improved perturbation theory that is accurate for $\nu \simeq \nu_{\text{res}}$ will be discussed in Section 6.

Numerical calculations (Section 5) show that the Fourier coefficients are finite and well-behaved for all parameter values. Only the perturbation theory is singular. However, in the numerical solution, the resonant wave varies as a different power of a than expected. Therefore, in the next section, we will specifically add it in at lower order in the expansion.

For instance, if $\nu = -\frac{1}{10}$, we find a resonance of wavenumber $k = 3$. The denominator of (3.4) vanishes and the linearized phase speeds of wavenumbers 1 and 3 are both $c_0 = -0.9$. Table 1 shows the numerical solutions for this value of ν. Comparison of the mantissas for the two amplitudes of 0.001 and 0.01 shows that the computation is highly accurate at these small amplitudes. Since the amplitude changed by one order of magnitude, the difference of the exponents between the two cases indicates the appropriate order in amplitude. As expected, wavenumber 2 is second order. One would expect wavenumber 3 to be third order. However, it is first order, i.e., $O(a)$. The higher coefficients are all bumped up one or more powers of a due to the nonlinear interaction of wavenumber 3 with the other wave components.

The resonances need not occur at lowest order in a. Table 2 lists the numerical Fourier coefficients for a wavenumber 5 resonance. Each coefficient through wavenumber 4 appears at its anticipated order. But

Table 1

Fourier coefficients of SKDV solution for $k = 3$ resonance ($\nu = -\frac{1}{10}$) as computed by the Newton-Kantorovich Galerkin algorithm

Wavenumber	$a = 0.001$	$a = 0.01$
1	$1.0000 \cdot 10^{-3}$	$1.0000 \cdot 10^{-2}$
2	$-1.4961 \cdot 10^{-8}$	$-1.4961 \cdot 10^{-6}$
3	$-5.4488 \cdot 10^{-4}$	$-5.4488 \cdot 10^{-3}$
4	$2.5947 \cdot 10^{-8}$	$2.5947 \cdot 10^{-6}$
5	$-4.3377 \cdot 10^{-13}$	$-4.3377 \cdot 10^{-10}$
6	$-7.8544 \cdot 10^{-10}$	$-7.8544 \cdot 10^{-8}$
7	$3.8863 \cdot 10^{-14}$	$3.8863 \cdot 10^{-11}$
8	$-8.9930 \cdot 10^{-19}$	$-8.9930 \cdot 10^{-15}$
9	$-3.7151 \cdot 10^{-16}$	$-3.7150 \cdot 10^{-13}$
10	$2.3270 \cdot 10^{-20}$	$2.3270 \cdot 10^{-16}$
11	$-6.9486 \cdot 10^{-25}$	$-6.9486 \cdot 10^{-20}$
12	$-1.3232 \cdot 10^{-22}$	$-1.3232 \cdot 10^{-18}$

Table 2

Numerically computed coefficients for $k = 5$ resonance ($\nu = -\frac{1}{26}$) for $a \ll 1$

Wavenumber	a_n †
1	$1.0000 \cdot a^1$
2	$0.10317 \cdot a^2$
3	$0.010479 \cdot a^3$
4	$0.0076881 \cdot a^4$
5	$0.064036 \cdot a^3$
6	$-0.0021623 \cdot a^4$
7	$-0.000050157 \cdot a^5$
8	$-0.0000021046 \cdot a^6$
9	$-0.0000013313 \cdot a^7$
10	$-0.0000035898 \cdot a^6$
11	$0.00000016030 \cdot a^7$
12	$0.00000000082829 \cdot a^8$

† These values were computed by comparing numerical coefficients for $a = 0.001$ with those for $a = 0.01$, as was done in Table 1.

the coefficient of $k=5$ is third order in a instead of the nonresonant fifth order. Coefficients for wavenumbers 6–12 are once again of lower order than expected due to their nonlinear interaction with the resonant wave.

The Stokes' expansion in its present form cannot be expected to handle this type of resonance since the solution is computed stepwise and it is presupposed that wavenumber k does not appear until kth order in the amplitude series. This supposition is not true when one is sufficiently close to resonance.

4. Resonant Stokes' expansion

We show here how adding in the resonant wave at the appropriate order gives us an accurate solution. For simplicity, we will restrict ourselves to $\nu = \nu_{\text{res}}(k)$ in this section; we will discuss variable ν in Section 6.

In Table 1, we saw that the resonant wavenumber 3 varied linearly with a for $\nu = -\frac{1}{10}$. To modify the Stokes' expansion, we simply add in the wavenumber 3 solution at lowest order with a variable amplitude, b, which will be determined at higher order. In particular, we let

$$u_1 = \cos(X) + b\cos(3X). \tag{4.1}$$

At higher order, we must cancel not only terms proportional to $\cos(X)$, but also terms in F_i proportional to $\cos(3X)$ to avoid a zero denominator in eq. (3.4). However, we now have two quantities, c_{i-1} and b, which may be chosen to allow these cancellations. At second order, the right-hand side, F_2 is

$$F_2 = -c_1\sin(X) + (\tfrac{1}{2}+b)\sin(2X) - 3bc_1\sin(3X) + 2b\sin(4X) + \tfrac{3}{2}b^2\sin(6X). \tag{4.2}$$

Thus, $c_1 = 0$ to cancel terms in both $\sin(X)$ and $\sin(3X)$, and

$$u_2 = (\tfrac{1}{6}+\tfrac{1}{3}b)\cos(2X) - \tfrac{1}{21}b\cos(4X) - \tfrac{1}{378}b^2\cos(6X). \tag{4.3}$$

At second order, b still appears as a variable. We must continue to third order to determine it, where the right-hand side becomes

$$F_3 = [9(12b^2+21b-84c_2+7)\sin(X) + 3(-b^3-756bc_2+108b+63)\sin(3X)$$
$$+ 25b(25b+9)\sin(5X) - 133b^2\sin(7X) - 9b^3\sin(9X)]/756. \tag{4.4}$$

We must eliminate terms proportional to $\sin(X)$ and $\sin(3X)$, giving two simultaneous equations to solve for c_2 and b,

$$12b^2+21b-84c_2+7=0, \qquad b^3+756bc_2-108b-63=0. \tag{4.5a, b}$$

There are three roots, which are

$$b^{(1)} = -0.5449, \qquad b^{(2)} = -1.784, \qquad b^{(3)} = 0.5947, \tag{4.6a}$$

producing

$$c_2^{(1)} = -0.01048, \qquad c_2^{(2)} = 0.09200, \qquad c_2^{(3)} = 0.2825. \tag{4.6b}$$

The first of these roots is the one appearing in Table 1. The other two roots were also verified by the Newton–Kantorovich Galerkin program. All roots exist for values of the amplitude a well over 1. It is suspected that only one of these roots is stable, but investigation of their stability is beyond the scope of this present study. Here, we simply point out that the modification to the Stokes' expansion did find the correct resonant amplitude and correctly predicted the existence of multiple modes. The relationship of these modes to the double cnoidal wave is discussed in Section 7.

It should be noted that this resonant amplitude, b, is correct only to lowest order. At order three, we happily obtained two equations in two unknowns when we used the two solvability conditions. If we view b as a constant, we have the problem at succeeding orders that we must again cancel two terms and will thus obtain two equations, but we will have only a single unknown quantity, c_{i-1}. Therefore, b cannot be constant; it must also be expanded in the amplitude parameter as:

$$b = b(a) = b_0 + ab_1 + a^2 b_2 + \cdots . \tag{4.7}$$

The three roots found at third order, (4.6a) are thus solutions for b_0. Continuing the expansion to fourth order, we find $b_1 = c_3 = 0$. At fifth order, we obtain a linear relation for b_2 and c_4 as functions of b_0 and c_2. We find values for b_2 and c_4 corresponding to each of the three roots of b_0.

$$b_2^{(1)} = 5.152 \cdot 10^{-4}, \qquad b_2^{(2)} = 2.106 \cdot 10^{-3}, \qquad b_2^{(3)} = -9.031 \cdot 10^{-3}; \tag{4.8a}$$

$$c_4^{(1)} = 9.094 \cdot 10^{-6}, \qquad c_4^{(2)} = -6.046 \cdot 10^{-3}, \qquad c_4^{(3)} = -6.274 \cdot 10^{-2}. \tag{4.8b}$$

For small amplitudes, $b \simeq b_0$. The values listed for b in subsequent calculations refer to the lowest order solution, b_0.

What about higher order resonances, such as the $k = 5$ resonance ($\nu = -\frac{1}{26}$) which we saw in Table 2 appeared at third order? Normally, we would not expect to find the fifth wavenumber appearing until fifth order. Carrying through the above procedure, we simply add it in at third order. We carry the expansion through third order as a nonresonant expansion. After the unmodified Stokes' series solution for third order is found, we add the additional term. Thus, u_1 and u_2 are given by (3.2a) and (3.5a) respectively, but instead of (3.5b), the complete third order solution is

$$u_3 = \frac{1}{192(50\nu^2 + 15\nu + 1)} \cos(3X) + b\cos(5X). \tag{4.9}$$

The coefficient of $\cos(5X)$ does not appear at order four. At fifth order, however, the nonlinear forcing produces a term proportional to $\cos(5X)$ which we must set to zero simultaneously with the coefficient of $\cos(X)$. We have

$$-256\,048\,128 c_4 + 134\,017 = 0, \qquad -145\,152 b + 9295 = 0. \tag{4.10a, b}$$

We find

$$b = 0.06404, \qquad c_4 = 5.234 \cdot 10^{-4}. \tag{4.11a, b}$$

These values are substantiated by the numerical solution in Table 2. One would have to progress to order seven to compute an improvement to b.

Table 3 compares the zeroth order computed coefficients, b_0, with the corresponding numerical coefficients. Note that the numerical coefficients were divided by a raised to the power of the order of resonance to correspond to our b (e.g. for $k = 5$, the Galerkin coefficient is divided by a^3). Thus b is really a resonance factor—the ratio of the amplitude of the resonant wave to that of wavenumber 1. We see that the modification to the Stokes' expansion works well in expressing the amplitude of the resonant wave.

The remaining issue is how to determine the order of resonance without the benefit of a numerical solution such as was used in Tables 1 and 2. The solution is to go ahead and apply the modified Stokes' expansion with b expanded as in (4.7) so that b_0 is at lowest order. When this method is used for the $k = 5$ (third order) resonance, we find that b_0 and b_1 are identically 0 and $b_2 = 0.06404$, in agreement with (4.11a). Although this method requires carrying a couple extra variables through the calculation, it is effective in determining the order of resonance.

Table 3

Comparison of amplitude factors, b computed by the resonant Stokes' expansion with the "exact" Galerkin solution

Wavenumber	Order of resonance	Resonant Stokes' expansion	Galerkin ($a=0.001$)
2	first	0.7071	0.7071
		−0.7071	−0.7071
3	first	−0.5449	−0.5449
		−1.784	−1.784
		0.5947	0.5947
4	second	4.425	4.425
5	third	0.06404	0.06404

5. Newton–Kantorovich Galerkin solution

We digress a bit at this point to discuss our numerical techniques. Boyd [2] describes the Newton–Kantorovich method for solving nonlinear differential equations. The crux of this iterative procedure is to express both u and c as a guess (u^n and c^n) plus a correction (Δ and δ_c).

$$u^{n+1} = u^n + \Delta, \qquad c^{n+1} = c^n + \delta_c. \tag{5.1a, b}$$

Substituting into eq. (2.4), we obtain a linear equation for the corrections by neglecting terms of $O(\Delta^2)$ and $O(\delta_c \Delta)$,

$$L\Delta - u_X^n \delta_c = f, \tag{5.2}$$

where

$$L = -\nu \frac{d^5}{dX^5} + \frac{d^3}{dX^3} + (u^n - c^n)\frac{d}{dX} + u_X^n, \qquad f = \nu u_{XXXXX}^n - u_{XXX}^n - (u^n - c^n) u_X^n .$$

The right-hand side is completely determined by the first guess.

Next, the corrections to u^n, Δ, are expanded in a Fourier cosine series.

$$\Delta = \sum_{k=2}^{N} \delta_k \cos(kX). \tag{5.3}$$

Finally, the Galerkin technique is applied by multiplying (5.2) by $\sin(iX)$ and integrating from $-\pi$ to π to produce the matrix equation:

$$\boldsymbol{H}\underset{\sim}{\delta} + \boldsymbol{h}\delta_c = \boldsymbol{g}, \tag{5.4}$$

where

$$H_{ik} = (\sin(iX), L\cos(kX)), \qquad h_i = (\sin(iX), -u_X^n), \qquad g_i = (\sin(iX), f(X)).$$

The outer parentheses denote inner products—multiplication and integration from $-\pi$ to π.

There is a modest complication in applying (5.4) numerically—both the operator, L, and the right-hand side, f, include nonlinear terms. However, since both u^n and Δ are expressed as Fourier cosine series, the nonlinearity is easily dealt with by applying trigonometric identities which express the product of trigonometric functions as sums and differences of linear trigonometric functions.

We represent this by breaking H_{ik} into its linear and nonlinear components.

$$H_{ik} = D_{kk} + K_{ik}, \tag{5.5}$$

where

$$D_{kk} = \pi(\nu k^5 + k^3 + c^n k) \tag{5.6}$$

is the linear component and K_{ik} is the nonlinear portion defined as

$$K_{ik} = (\sin(iX), (-u^n \sin(kX) + u_X^n \cos(kX))) \tag{5.7}$$

u^n is a cosine series and u_X^n is a sine series so that the trigonometric identities may be applied term-by-term.

We expect a full matrix for a nonlinear problem. In our case, we can think of this as being caused by the nonlinear interaction of the waves, or equivalently, as the products of the trigonometric functions being transformed to sums and differences producing additional wavenumbers in each row. Plus, of course, we need an additional full column to solve for δ_c. Actually, this column replaces the first column to solve for δ_1, the coefficient of $\cos(X)$, since we choose δ_1 (equivalent to a) as the fixed parameter. The resulting matrix equation looks like

$$\begin{pmatrix} h_1 & K_{12} & K_{13} & \cdots & K_{1N} \\ h_2 & D_{22}+K_{22} & K_{23} & \cdots & K_{2N} \\ h_3 & K_{32} & D_{33}+K_{33} & & \vdots \\ \vdots & \vdots & \vdots & \ddots & \\ h_N & K_{N2} & K_{N3} & \cdots & D_{NN}+K_{NN} \end{pmatrix} \begin{pmatrix} \delta_c \\ \delta_2 \\ \delta_3 \\ \vdots \\ \delta_N \end{pmatrix} = \begin{pmatrix} g_1 \\ g_2 \\ g_3 \\ \vdots \\ g_N \end{pmatrix}. \tag{5.8}$$

Given an adequate first guess, the matrix equation is solved for the δ's. After applying the corrections to the coefficients of (5.3), the process is iterated until the corrections are sufficiently small.

This Galerkin technique produces well-behaved solutions, even when $\nu = \nu_{res}(k)$. It does not break down for a harmonic resonance as does the Stokes' expansion. Each step of the Stokes' expansion is equivalent to solving a Galerkin matrix equation (5.8) with $K_{ik} = 0$ and including h_1 but not the higher order phase speed components, i.e., the matrix is diagonal. A resonant condition is manifested by one of the diagonal elements, D_{kk} as defined in (5.6) vanishing, which is exactly the same condition as a zero denominator of (3.4). For a diagonal matrix, a zero diagonal element produces a zero matrix determinant, precluding solution of the matrix equation. In contrast, the Galerkin matrix is not diagonal; it is a full matrix including the nonlinear elements, K_{ik}. Therefore, a vanishing diagonal element, indicating resonance, does not make the matrix singular, and the system is solvable even for cases of resonance.

However, the Galerkin method will break down for vanishing values of amplitude. Both the K_{ik} and h_i elements are proportional to powers of a. Therefore, a zero value of a will produce a singular matrix equation. (In the language of bifurcation theory, $a = 0$ is a "fold point" or "limit point".) We had no problem, however, getting the Galerkin method to converge for values of a as small as 10^{-5}.

6. Expansion near a resonance

In Section 3 we showed that the Stokes' expansion worked well as long as we are not at a resonance which produces a zero denominator of (3.4). The Stokes' expansion was modified in Section 4 to handle resonance. What about cases where the denominator is not quite zero, but small enough so that the nonresonant Stokes' expansion is still inaccurate? A resonance is not a discontinuous phenomenon that we can expect to disappear when the parameters are very slightly off the exact resonant values. There is a neighborhood about the exact resonance parameter for which a resonance is evident. We can further modify the resonant Stokes' expansion to express this blending from resonance to nonresonance by expanding the resonance parameter, ν, in powers of the amplitude: $\nu = \nu_0 + a\nu_1 + a^2\nu_2 + \cdots$. Since this is the expansion of a free parameter, we may then express a near resonant condition in terms of these ν's.

When u, c, and ν are all expanded in powers of a and like powers of a matched, the ith equation is

$$a^i: \quad -\nu_0 u_{i,XXXXX} + u_{i,XXX} - c_0 u_{i,X} = F_i'(X), \tag{6.1}$$

where

$$F_i'(X) = -\sum_{j=1}^{i-1} [-\nu_j u_{i-j,XXXXX} + (u_j - c_j) u_{i-j,X}].$$

Expanding u_i and F_i' in Fourier cosine and sine series respectively, the kth coefficient of u_i is

$$u_{i,k} = \frac{f_{i,k}}{\nu_0 k^5 + k^3 + c_0 k}. \tag{6.2}$$

For a resonance of wavenumber 3, we let $u_1 = \cos(X) + b\cos(3X)$, finding that $\nu_0 = -\frac{1}{10}$ and $c_0 = -1 - \nu_0 = -\frac{11}{10}$. At second order we find it convenient to choose $\nu_1 = 0$ so that the solution is unaltered. With $c_1 = 0$, the second order solution is

$$u_2 = (\tfrac{1}{3}b + \tfrac{1}{6})\cos(2X) - (\tfrac{1}{21}b\cos(4X) - \tfrac{1}{378}b^2\cos(6X). \tag{6.3}$$

So far, this solution is identical to that of Section 4. At third order, applying the solvability conditions in the nonlinear forcing produces two equations in the two unknowns (b and c_2), but now includes the free parameter ν_2,

$$12b^2 + 21b - 84c_2 - 84\nu_2 + 7 = 0, \tag{6.4}$$

$$-b^3 - 756bc_2 - 61\,236b\nu_2 + 108b + 63 = 0. \tag{6.5}$$

Eliminating c_2, we obtain a cubic for $b(\nu_2)$,

$$k = 3: \quad b^3 + 1.734b^2 + (-0.4128 + 554.9\nu_2)b - 0.5780 = 0. \tag{6.6}$$

The values of b computed by (6.6) are actually the lowest order in an expansion of b in powers of a. One could go to higher order to produce corrections to b. Table 4 compares the values of b as a function of ν_2 computed from equation (6.6) with the "exact" Galerkin solution for one of the roots. The b's computed from the nonresonant Stokes' expansion are also included. Note how the results of (6.6) track the exact solution closely for small ν_2 but the unmodified Stokes' expansion is in error. As ν_2 increases, we move away from the resonance and the unmodified Stokes' expansion becomes more accurate but the blended Stokes' expansion loses accuracy. Similar results were evident for the other two roots. The behavior of the solution for various ranges of ν_2 is further explored in Section 7.

The above procedure was repeated for resonances of wavenumbers 2-5. For the $k = 2$ resonance, ν_1 was nonzero at second order,

$$k = 2: \quad b = \tfrac{1}{2}(30\nu_1 \pm \sqrt{900\nu_1^2 + 2}). \tag{6.7}$$

Figure 1 compares the value of b computed from the positive root of (6.7) with the Galerkin solution and the unmodified Stokes' expansion. The higher wavenumber resonances all showed vanishing values

Table 4

Comparison of resonance factors b for the first root of the $k=3$ resonance as a function of ν_2 for the resonant Stokes' expansion, the "exact" Galerkin solution, and the nonresonant Stokes' expansion

ν_2	Modified Stokes' expansion	Exact Galerkin solution	Nonresonant Stokes' expansion
0	$-5.449 \cdot 10^{-1}$	$-5.449 \cdot 10^{-1}$	∞
10^{-4}	$-5.671 \cdot 10^{-1}$	$-5.671 \cdot 10^{-1}$	$7.654 \cdot 10^{0}$
10^{-3}	$-9.625 \cdot 10^{-1}$	$-9.625 \cdot 10^{-1}$	$9.940 \cdot 10^{-1}$
10^{-2}	$1.083 \cdot 10^{-1}$	$1.083 \cdot 10^{-1}$	$1.043 \cdot 10^{-1}$
10^{-1}	$1.049 \cdot 10^{-2}$	$1.049 \cdot 10^{-2}$	$1.042 \cdot 10^{-2}$
10^{0}	$1.042 \cdot 10^{-3}$	$1.041 \cdot 10^{-3}$	$1.041 \cdot 10^{-3}$
10^{1}	$1.166 \cdot 10^{-4}$	$1.031 \cdot 10^{-4}$	$1.031 \cdot 10^{-4}$
10^{2}	$-7.027 \cdot 10^{-5}$	$9.470 \cdot 10^{-6}$	$9.470 \cdot 10^{-6}$

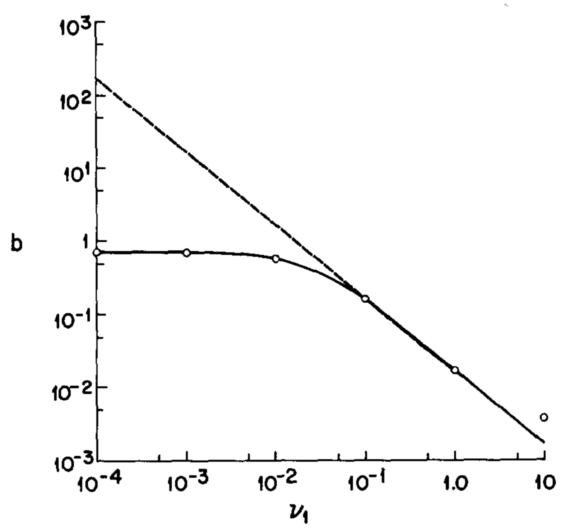

Fig. 1. Amplitude b of the $k=2$ resonance as a function of ν_1. The solid line is the modified Stokes' solution, the dashed line is the nonresonant Stokes' expansion, and the circles denote the "exact" Galerkin solution. The modified Stokes' solution is quite close to the exact solution over the range shown. The nonresonant Stokes' expansion is only accurate sufficiently far from exact resonance while the modified Stokes' expansion produces good results over the entire range depicted.

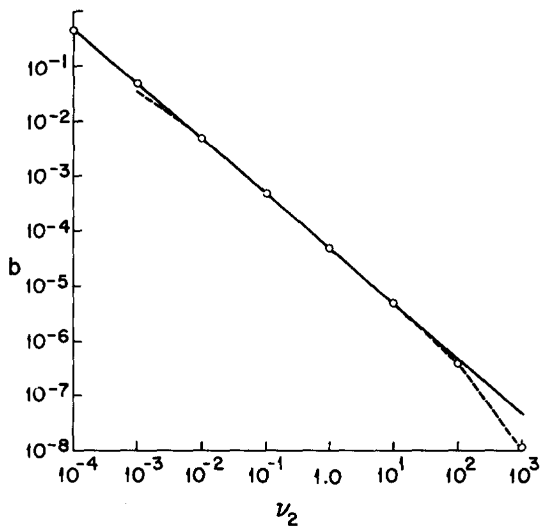

Fig. 2. Amplitude b of the $k=4$ resonance as a function of ν_2. The nonresonant Stokes expansion (dashed line) is not accurate near resonance but tracks closer to the "exact" Galerkin solution (circles) far from resonance. The modified Stokes' expansion (solid line) does not depart from the exact solution until far from the resonance.

of ν_1 at second order. Values of b as a function of ν_2 found at higher order are

$$k=4: \quad b=\frac{425}{96(90\,720\nu_2+1)}, \tag{6.8}$$

$$k=5: \quad b=\frac{120835}{145\,152(399\,168\nu_2+13)}. \tag{6.9}$$

Figure 2 is similar to Fig. 1 but for $k=4$. Note how closely the expansions track the Galerkin solution. Plots of b vs. ν_2 for the $k=3$ and $k=5$ resonances produce similar results.

7. Other issues

7.1. Range of validity

All singular perturbation methods assume the expansion parameter to be small. It is useful to determine how large the perturbation parameter (in our case, the amplitude a) can be for the expansion to be valid. We may check the lowest order approximation by examining the amplitude dependence of the exact solution to see how far b is independent of a.

We find that for $a \in [0, 1]$, b remained constant to at least two decimal places except for the $k = 4$ resonance.

Figure 3 compares the Newton–Kantorovich Galerkin solution (circles) for the first root of $k = 3$ with the modified Stokes' expansion solution to lowest order (solid line) and to second order (dashed line) for the first of the three roots. Note the very narrow scale of the ordinate axis. The lowest order modified Stokes' solution is, if course, independent of amplitude. The second order solution tracks the exact solution very closely, even for values of a close to 1 where we expect the expansion to begin to break down.

In Fig. 4, we see the $k = 5$ resonance plotted to higher amplitude. The exact solution for b is nearly independent of a until approximately $a = 1$, and then changes rapidly. Similar plots of the other resonances showed the same conclusion: little change in the exact value of b until $a \simeq 1$, then a significant change. This is to say that if b were expanded in powers of a, ($b_1 = 0$) we would find $|b_2| < 1$.

We thus conclude that the modified Stokes' expansion yields solutions accurate to a couple of decimal places until a becomes close to 1. We can change our definition of a, of course, but the cutoff must change as the definition changes. Better accuracy may be achieved by computing b to higher order. However, as for any asymptotic expansion, the series must fail for large values of the perturbation parameter, no matter what order we take.

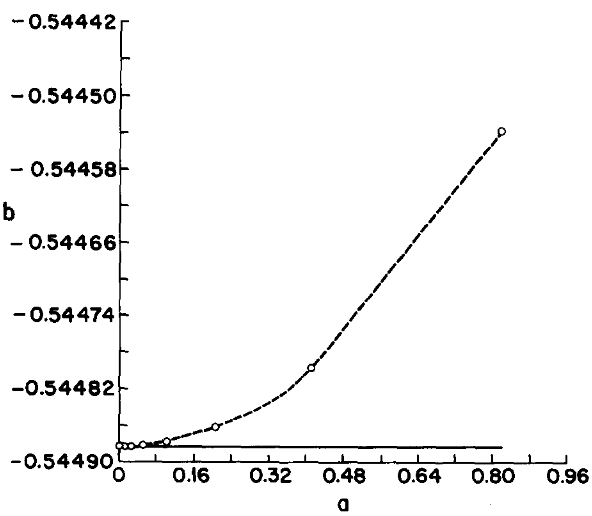

Fig. 3. Amplitude b of the first root of the $k = 3$ resonance as a function of the amplitude a. The "exact" Galerkin solution (circles) is compared with the results of the modified Stokes' expansion—the solid line is the zeroth order solution and the dashed line is the solution to second order. Even the lowest order solution was accurate to three significant figures over the range shown here. The second order solution, $b_0 + a^2 b_2$, tracks the Galerkin solution closely.

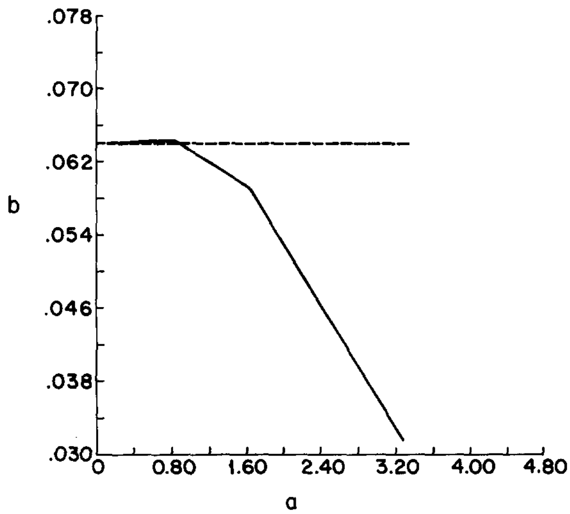

Fig. 4. Amplitude b of the $k = 5$ resonance as a function of the base amplitude a. The "exact" Galerkin solution (solid line) is compared to the zeroth order Stokes' expansion (dashed line). They agree quite well until $a \simeq 1$, then the exact solution changes rapidly. Higher order solutions are therefore necessary for larger values of the amplitude. The unmodified Stokes' expansion is not included since it produces infinite b for $\nu = \nu_{res}$.

7.2. Higher wavenumber resonances

So far we have looked for resonances of higher wavenumbers with wavenumber one only. However, a resonance may occur between any two waves. The criteria is that their phase speeds be equal. For the SKDV equation, we prescribed an exact resonance with wavenumber one by setting the variable parameter, ν equal to $\nu_{res}(k)$ as determined by (3.7). To compute the value of ν for which any two wavenumbers k_1 and k_2 are resonant, we set their phase speeds (described by eq. (2.5)) equal to find

$$\nu = \nu_{res}(k_1, k_2) = \frac{-1}{k_1^2 + k_2^2}. \tag{7.1}$$

Certain higher order resonances, however, are merely similarity transforms of resonances with wavenumber one. For instance, it is irrelevant to search for the resonance of wavenumber 4 with 2 because setting $\alpha = 2$ in (2.3) shows that it is the same as a resonance of wavenumbers 2 and 1 with ν multiplied by one-fourth and u rescaled by a factor of four.

The resonance we will investigate here is $k_2 = 3$ with $k_1 = 2$ which implies $\nu_{res}(2, 3) = -\frac{1}{13}$. The resonance is first order, so the lowest order solution becomes:

$$u_1 = \cos(2X) + b\cos(3X), \qquad c_0 = -2.769. \tag{7.2a, b}$$

Second order produces:

$$u_2 = -[6552b\cos(X) + 936\cos(4X) + 468b\cos(5X) + 91b^2\cos(6X)]/24\,192, \tag{7.3a}$$

$$c_1 = 0. \tag{7.3b}$$

At third order, we determine the lowest order solution for b. We find two roots,

$$b \simeq b_0 = \pm 0.9370, \qquad c_2 = -0.1467. \tag{7.4a, b}$$

For $a_2 = 0.001$ (where a_2 is the amplitude of the $k_1 = 2$ wave), (7.4) agreed with the pseudospectral Newton–Kantorovich solution to three significant figures. To maintain accuracy for higher amplitudes, one may easily proceed to higher order to obtain corrections to b. As in Section 6, it is easy to extend the method from $\nu = \nu_{res}$ to a small neighborhood about ν_{res}.

In a similar manner, the resonance of any two other wavenumbers could be investigated.

7.3. Resonance as limiting case of double cnoidal wave

A normal Stokes' expansion produces periodic cnoidal wave solutions to a nonlinear wave equation. For many nonlinear equations, double cnoidal wave solutions also exist. These double cnoidal waves are merely two distinct waves of differing amplitude and phase speed on each period interval. In the limit of large amplitude, these two waves interact like solitary waves with peaks of two different heights within each period.

For a double cnoidal wave, the lowest order Stokes' expansion is

$$u_1 = a[\cos(X) + b\cos(Y)], \tag{7.5}$$

where X and Y are phase variables of the two waves,

$$X = x - c_1 t, \qquad Y = k(x - c_2 t). \tag{7.6a, b}$$

In general, the phase speeds of the two waves, c_1 and c_2 are independent. In the special case where $c_1 = c_2$, we have a single parameter family and the solution collapses to a single cnoidal wave.

In comparison, we also have two distinct dominant components in the modified Stokes' expansion model for resonance waves. The resonance factor b functions the same as for the double cnoidal wave expansion in (7.5). Indeed, the modified Stokes' series for the single cnoidal wave is a special case of that for the double cnoidal wave—special in that $c_1 = c_2$.

Thus we may consider wave resonance for the ordinary cnoidal wave to be the limiting case of double cnoidal waves with precisely equal phase speeds.

7.4. Multiple roots and double cnoidal waves

We have seen that for some resonances, multiple roots of the resonance factor b were found in the modified Stokes' expansion and verified numerically. Specifically, for an exact resonance of wavenumber 3 with wavenumber 1, three real roots were identified. More can be learned about these multiple roots by examining them in the neighborhood of the resonance. To do this, we use the results of the modified Stokes' expansion in Section 6 with ν expanded in powers of a, which produce eq. (6.6). We are interested in results away from the exact resonance, that is, $|\nu_2| \gg 0$. We examine (6.6) for two separate cases.

Case 1. For $|b| \ll 1$, an order of magnitude analysis reduces eq. (6.6) to

$$554.9 b\nu_2 - 0.5780 \sim 0, \tag{7.7}$$

producing

$$b \sim \frac{1.042 \cdot 10^{-3}}{\nu_2}. \tag{7.8}$$

This is the root which appears in Table 4.

Case 2. For $|b| \gg 1$, eq. (6.6) instead reduces to

$$b^3 + 554.9 b\nu_2 \sim 0, \tag{7.9}$$

producing the other two roots for b:

$$b \sim \pm\sqrt{-554.9\nu_2}. \tag{7.10}$$

There are two possible subcases for eq. (7.10). The first is $\nu_2 \gg 0$, for which the roots b are both imaginary and no physical solution is possible. The second subcase is $\nu_2 \ll 0$, for which two real roots exist.

The above cases and subcases are shown pictorially in Fig. 5, which is a graph of the three roots for the $k=3$ resonance for (a) $\nu_2 > 0$, and (b) $\nu_2 < 0$. In both plots, the root of Case 1 is seen as the apparently straight line at about $b=0$ for sufficiently large $|\nu_2|$. The other two roots are real in both graphs for small $|\nu_2|$, but the two plots demonstrate the differences in the subcases for Case 2. For $\nu_2 > 0$ (Fig. 5(a)), the roots reach a bifurcation point at $\nu_2 = 1.054 \cdot 10^{-3}$, beyond which they no longer represent a physical solution. The situation is different, however, when $\nu_2 < 0$ (Fig. 5(b)) and the values of the two additional real roots are approximated by (7.10).

The remaining issue is to attach a meaning to the two additional roots of the $k=3$ resonance for $\nu_2 < 0$. In the previous subsection, we saw that a double cnoidal wave collapses to a resonance of a single cnoidal wave when $c_1 = c_2$. We may thus interpret the additional two roots in terms of a double cnoidal wave of the SKDV consisting of wavenumbers 1 and 3. When the SKDV, eq. (2.1), with $\mu = 1$ is written in terms

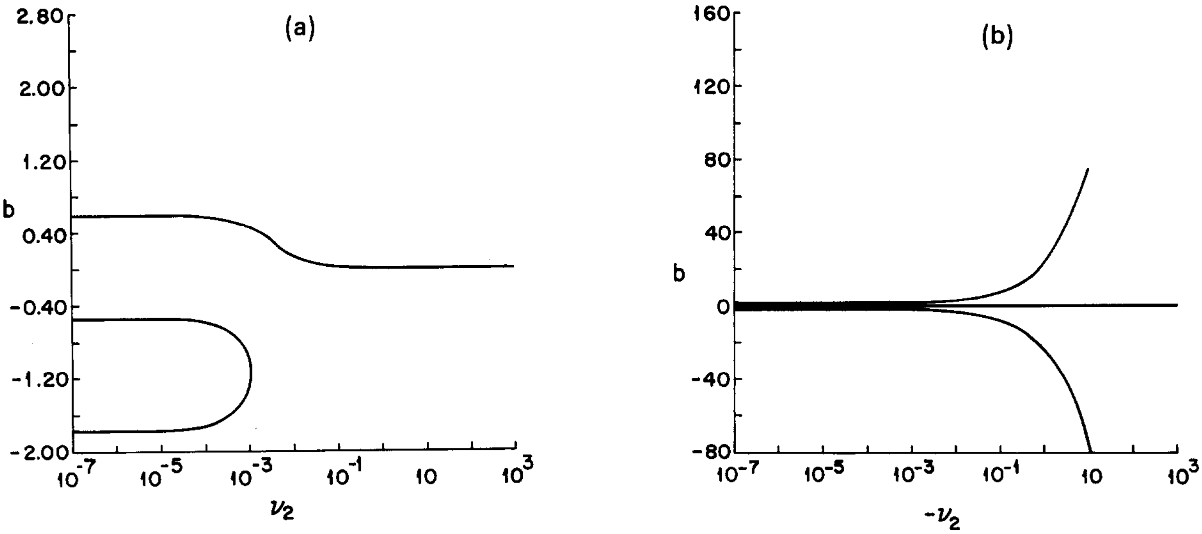

Fig. 5. Amplitude b of all three roots of the $k=3$ resonance as a function of ν_2 as computed by the Galerkin algorithm: (a) For $\nu_2>0$; (b) For $\nu_2<0$.

of the two phase variables (7.6a and b), it becomes

$$-\nu(u_{XXXXXX}+5ku_{XXXXXY}+10k^2u_{XXXXYY}+10k^3u_{XXXYYY}+5k^4u_{XXYYYY}+k^5u_{YYYYYY})$$
$$+u_{XXX}+3ku_{XXY}+3k^2u_{XYY}+k^3u_{YYY}-c_1u_X-kc_2u_Y+u(u_X+ku_Y)=0. \tag{7.11}$$

Expanding u, c_1, c_2, and ν all in powers of a; taking the lowest order solution as (7.5); and using $k=3$ produces phase speeds to second order of

$$c_1=-1-(\nu_0+a^2\nu_2)+a^2\frac{(620a^2\nu_2-12b^2-7)}{12(550a^2\nu_2-7)}, \tag{7.12}$$

$$c_2=-9-81(\nu_0+a^2\nu_2)+a^2\frac{(-620a^2b^2\nu_2+17\,280a^2\nu_2+7b^2-75b)}{756(710a^2\nu_2-7)}. \tag{7.13}$$

By setting $c_1=c_2$ for $\nu_0=-\frac{1}{10}$ (the value for resonance of wavenumbers 1 and 3) and assuming $|a|\ll 1$, we obtain a relation for b:

$$b\sim\pm\sqrt{-554.9\nu_2+0.413}. \tag{7.14}$$

Eq. (7.14) is the same as (7.10) for large negative ν_2 when the wavenumber 3 component is large in comparison to wavenumber 1. Therefore, the two additional modes of the $k=3$ resonance represent the collapse of the double cnoidal wave to a single cnoidal wave.

The third root, the only one which exists for large positive ν_2, may be the limit of a double cnoidal wave too, but we have not examined this question.

8. Conclusions

Here, we have made a rather simple modification to the Stokes' expansion to study small amplitude waves even when resonance occurs.

The resonant wave is simply added in at the order of the observed resonance. For a lowest order resonance, this is equivalent to taking the lowest order approximation to be $u_1 = a[\cos(X) + b\cos(kX)]$ where k is the wavenumber of the resonant wave and b is the resonance factor. Since the resonant wave interacts nonlinearly with other components, applying solvability conditions at higher order determines b and the higher order corrections to the phase speed.

When there is resonance, we must expand the resonance factor, $b(a)$ as well as $u(X)$ and the phase speed in powers of the amplitude a. In the neighborhood of the resonance, where the phase speeds of the two waves are nearly equal, we may expand the resonance parameter (in our case, ν) in powers of the amplitude as well, expressing the blending from resonance to nonresonance. When all these variables are expanded in powers of a, the algebraic computations can become quite complicated. Luckily, algebraic manipulation languages such as REDUCE make the work less formidable. As shown here, low order solutions give reasonable accuracy for moderate amplitude.

This study has used a simple one-dimensional model as a testbed. Work in progress will extend the technique to compute double cnoidal waves of the Korteweg de Vries and Regularized Long Wave equations, which are solutions of two-dimensional eigenvalue problems.

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