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A SIMPLE METHOD FOR CALCULATING THE LINEAR EXTRAPOLATION DISTANCE AT THE SURFACE OF A BLACK BODY

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Abstract—We adapt a method introduced by Fuchs for calculating evaporation rates from small droplets to the evaluation of the linear extrapolation distance of the neutron density at the surface of a black body immersed in an infinite, absorbing medium. Explicit results are obtained for spheres and cylinders and, by comparison with some very accurate calculations carried out by others, these results are shown to be accurate to within about 9% for a range of parameters.

Modifications are introduced which enable the method to deal with absorbing and scattering bodies which are not black.

INTRODUCTION

A simple theory of evaporation of small droplets was developed many years ago by Fuchs (1959). The need for this simple model arose from the failure of classical diffusion theory to treat particles that were comparable in size to the mean free path (m.f.p.) of a gas molecule and the difficulty of employing transport theory. Fuchs therefore assumed the diffusion equation to be valid up to a certain distance from the drop, what he called the "vapour jump" length Δ, which was in fact roughly equal to a m.f.p. Within the layer $a \leqslant r \leqslant a+\Delta$, where a is the particle radius, it was assumed that elementary gas kinetic theory applied. Fuchs' model was very successful in predicting evaporation rates and has only recently been superseded by more accurate methods using the linear transport equation (Sahni, 1966; Loyalka, 1973; Williams, 1975).

In an interdisciplinary spirit, therefore, it seems reasonable to apply the concepts used by Fuchs to the problem of absorption of neutrons by small black bodies. We illustrate the technique by reference to spheres and cylinders in absorbing and scattering media. Also by means of an albedo condition we can extend our results to bodies which scatter as well as absorb neutrons.

THEORY

In order to illustrate the basic ideas of the method we first apply it to a very simple case. Thus, we envisage a black sphere of radius a in an infinite, nonabsorbing medium. Around the sphere, and concentric with it, we construct a hypothetical sphere of radius $a+\Delta$. In the region $r > a+\Delta$ the classical diffusion equation is expected to be valid and the solution appropriate to the situation can be written:

$$N(r) = N_\infty - (N_\infty - N')\frac{(a+\Delta)}{r}, \tag{1}$$

where N' is the density at $r = a+\Delta$.

Now the current through the surface $a+\Delta$ according to diffusion theory is:

$$\begin{aligned} J_{\text{in}} &= 4\pi(a+\Delta)^2 Dv \left.\frac{\mathrm{d}N}{\mathrm{d}r}\right|_{r=a+\Delta} \\ &= 4\pi(a+\Delta)Dv(N_\infty - N'), \end{aligned} \tag{2}$$

where D is the neutron diffusion coefficient and v the mean neutron velocity. In the region $a \leqslant r \leqslant a+\Delta$ we assume that simple kinetic theory applies and that the current into the sphere at $r = a$ is

$$4\pi a^2 \frac{N'v}{4}. \tag{3}$$

Now since there is no absorption in the region $a \leqslant r \leqslant a+\Delta$, these two expressions for the current must be equal, thus:

$$(a+\Delta)D(N_\infty - N') = \frac{a^2 N'}{4}. \tag{4}$$

Solving for N' and substituting into equation (1) we find:

$$N(r) = N_\infty \left\{1 - \frac{a^2}{a^2+4D(a+\Delta)}\frac{(a+\Delta)}{r}\right\}. \tag{5}$$

The linear extrapolation distance, λ, at the surface of the sphere is defined as:

$$\lambda = N \Big/ \left.\frac{\mathrm{d}N}{\mathrm{d}r}\right|_{r=a}, \tag{6}$$

whence:

$$\lambda = 4D - \frac{a\Delta}{a+\Delta}. \tag{7}$$

Recalling that $D = l_t/3$, where l_t is the transport m.f.p., we find:

$$\lambda = \frac{4}{3}l_t - \frac{a\Delta}{a+\Delta}. \tag{8}$$

For $a \to 0$, $\lambda \to 4\ l_t/3$ as expected from transport theory (Davison, 1951). On the other hand, for $a \to \infty$, we find:

Table 1. The black sphere

a	A	B	C	D	E	Error E (%)	Error B (%)
0.1		1.2635		1.2965	1.247	3.8	2.5
0.2		1.2164	1.2557	1.2543	1.182	5.7	3.1
0.4			1.1757		1.090		
0.5	1.134	1.1208		1.1549	1.056	8.6	3.0
0.6			1.1339		1.028		
0.7		1.0755		1.1037	1.004		2.6
0.8			1.0761		0.983		
1	1.024	1.0229	1.0535	1.0431	0.949	9.0	2.0
1.3		0.9826		0.9967	0.912	8.5	1.4
1.5		0.9605	0.9876	0.9718	0.893	8.1	1.1
2		0.9171		0.9248	0.858	7.2	0.8
2.5		0.8845		0.8920	0.835	6.4	0.8
5	0.81	0.8057		0.8128	0.779	4.2	0.9
∞	0.7083	0.7071	0.7113	0.7104	0.7104		0.5

A = Marshak's variational method (Marshak, 1947).
B = Kushneriuk and Mckay's variational method (Sahni, 1966).
C = Conkie's iterative method (Conkie, 1961).
D = Sahni's numerical integration (Sahni, 1966).
E = Present method.

$$\lambda = \frac{4}{3}l_t - \Delta. \qquad (9)$$

Now we know the exact result in this case, namely $\lambda = 0.7104\ l_t$ (Davison, 1951). Thus we may fix Δ as $0.623\ l_t$ and obtain:

$$\lambda = \frac{4}{3} - \frac{a}{1+1.605a}. \qquad (10)$$

Over the complete range of a where distances are now in units of l_t.

Table 1 shows the values of λ obtained from equation (10) compared with some other calculations. Sahni's results are claimed by him to be accurate to the number of significant figures shown and the error in our results is obtained by comparison with this work. The maximum error of 9% occurs at $a = 1$. Clearly, the method is not as accurate as the variational technique but it is very simple and can easily be applied to other geometries for which a solution of the diffusion equation is readily available.

A better comparison with the exact result which arises from the neutron transport equation can be had by examining the limiting case obtained by Davison (1951). For a large black sphere, Davison shows that:

$$\lambda_D = 0.7104 + \frac{0.5047}{a} + \frac{0.2336}{a^2} - \frac{1}{4}\frac{\log a}{a^3} - \frac{0.1704}{a^3} + O\left(\frac{\log^2 a}{a^4}\right). \qquad (11)$$

An expansion of equation (10) to the same order leads to:

$$\lambda = 0.7104 + \frac{0.388}{a} - \frac{0.242}{a^2} + \frac{0.151}{a^3} + O\left(\frac{1}{a^4}\right). \qquad (12)$$

The first terms of equations (11) and (12) are exact by definition and the second term has the correct analytical form but the amplitude is in error. Thereafter, the comparison rapidly deteriorates since the approximate method does not predict the logarithmic dependence.

For a small black sphere, Davison shows that

$$\lambda_D = \frac{4}{3} - \frac{5}{9}a - 0.97827a^2\log a - 1.4002a^2 + O(a^2\log^k a). \qquad (13)$$

The approximate formula leads to:

$$\lambda = \frac{4}{3} - a + 1.605a^2 + O(a^3). \qquad (14)$$

Thus, as before, the leading term is exact but subsequent terms are in error either in amplitude or form. Nevertheless, for values of a over the complete range the error is never greater than 9%.

A similar calculation in infinite, cylindrical geometry leads to:

$$\lambda = \frac{4}{3} - a\log\left(1 + \frac{0.623}{a}\right) \qquad (15)$$

and the comparison with Davison is as follows:

for a cylinder with a large radius,

$$\lambda_D = 0.7104 + \frac{0.2524}{a} + \frac{0.0949}{a^2} - \frac{5}{64a^3}\log a - \frac{0.1634}{a^3} + O\left(\frac{\log^2 a}{a^4}\right), \qquad (16)$$

$$\lambda = 0.7104 + \frac{0.1941}{a} - \frac{0.0806}{a^2} + O\left(\frac{1}{a^3}\right); \qquad (17)$$

for a cylinder with a small radius,

$$\lambda_D = \frac{4}{3} + 0.4596a\log a - 0.2164a + O(a^2\log^2 a), \qquad (18)$$

$$\lambda = \frac{4}{3} + a\log a - 0.473a - 1.605a^2 - 1.605a^2 + O(a^3). \qquad (19)$$

The best available results of λ for the black cylinder have been reported by Pellaud (1968) and later by Kavenoky

Table 2. The black cylinder

a	A	Error A (%)	B	Error B (%)	C	D
0.1	1.1355	5.8	1.1805	2.0	1.205	1.206
0.2	1.0504	7.9	1.1118	1.6	1.136	1.136
0.3	0.9962	8.1	1.0638	1.9	1.084	1.087
0.4	0.9577					
0.5	0.9288	8.1	0.9971	1.4	1.011	1.015
0.6	0.9061					
0.7	0.8877	7.8	0.9515	1.2	0.963	0.966
0.8	0.8726					
0.9	0.8599					
1	0.8491	6.9	0.9043	0.8	0.912	0.916
1.3	0.8244					
1.5	0.8123	5.7	0.8550	0.7	0.861	
2	0.7910					
2.5	0.8123	3.8	0.8040	0.5	0.808	0.810
3	0.7673					
3.5	0.7600					
4	0.7543					
5	0.7462	2.0	0.7575	0.5	0.761	0.762
∞	0.7104	0	0.7071	0.5	0.7104	

A = Present method.
B = Variational method (Mckay, 1960).
C = From Pellaud (1968).
D = Kavenoky (1978).

(1978). Some variational results using the method of Kushneriuk and Mckay have also been given by Mckay (1960). Table 2 compares these results with those obtained by the methods described in this paper. We see that our present method is accurate to about 8%. The variational method is significantly more accurate but at the expense of much greater labour.

MODIFICATIONS TO INCLUDE ABSORPTION AND SCATTERING

Scattering in the absorber

We have assumed the absorbing body to be black, i.e. to absorb all neutrons incident upon it. However, in practice, the body will both absorb and scatter neutrons. These processes can be incorporated into our formalism by the introduction of the blackness, β. The blackness is the fraction of neutrons incident on the body which are actully absorbed by it and is given by the expression (Bell and Glasstone, 1971):

$$\beta = \frac{(1-c_R)\bar{l}_R\Sigma_t^R(1-P_c)}{1-c_R P_c}. \tag{20}$$

c_R is the ratio of scattering to total cross section in the body, $\bar{l}_R$ is the mean chord length of the body, Σ_t^R is the total macroscopic cross section of the body and P_c is the first flight-collision probability. This expression for β is approximate but accurate (Bell and Glasstone, 1971). In the case of a black body $P_c \to 1$ and $\bar{l}_R\Sigma_t^R(1-P_c) \to 1$ leading to $\beta = 1$ which is the case we have considered above. To include these modifications it is necessary only to multiply the number of neutrons incident on the body, viz: $SN'v/4$ (where S is surface area), by β. In this case it is readily shown that equation (10) becomes:

$$\lambda = \frac{4}{3\beta} - \frac{a}{1+a/z(\beta)}, \tag{21}$$

where

and equation (15) becomes:

$$\lambda = \frac{4}{3\beta} - a\log\left(1+\frac{z(\beta)}{a}\right). \tag{22}$$

We have tested the accuracy of equation (22) by comparing it with a variational treatment given by Kushneriuk and Mckay (1954). Table 3 shows the extrapolation distance for an absorbing (but not black) cylinder embedded in a nonabsorbing moderator. The maximum error of 18% arises at a rod radius of 0.7 m.f.p. This error is disappointingly large but the variational results of Kushneriuk and Mckay are themselves approximate and so no definitive conclusions can be drawn. Moreover, we shall see below, that when the method is compared with some essentially exact results in the absorbing moderator case the error is much smaller.

Table 3. Absorbing cylinder

a	$a\Sigma_a^R$	λ_{vm}	λ_{app}	Error (%)
0.07	0.1	7.424	7.212	2.9
0.1	0.2	4.125	3.845	6.8
0.15	0.3	2.926	2.677	8.6
0.2	0.5	2.033	1.806	11
0.3	0.7	1.633	1.406	14
0.5	0.8	1.464	1.207	18
0.7	1	1.276	1.047	18
1	1.3	1.110	0.9270	16
1.5	1.6	0.9907	0.8509	14
2.5	2	0.8857	0.7931	7.2
5	2.5	0.8088	0.7516	5.0
8	3	0.7740	0.7358	3.3
12	4	0.7519	0.7269	

λ_{vm} = the variational result of Kushneriuk and Mckay (1954).
λ_{app} = the present method.

ABSORBTION IN THE MODERATOR

When the moderator absorbs as well as scatters, the character of the solution there changes. Both Pellaud (1968) and Sahni and Sjöstrand (1983) have calculated the effect of moderator absorption on the extrapolation distance at the surface of cylinders and we shall refer to their results below.

In order to modify our method to include moderator absorption it is necessary to solve the following diffusion equation in the moderator:

$$D\nabla^2 N - \Sigma_{am} N + \frac{S}{v} = 0, \tag{23}$$

where S is a constant source term.

We illustrate the result for the sphere because the algebra is simple. Thus the general solution of equation (23) for the neutron density surrounding the sphere is:

$$N(r) = \frac{S}{v\Sigma_{am}} + A\frac{e^{-\nu r}}{r}, \tag{24}$$

where $1/\nu$ is the diffusion length and is found as a root of:

$$\frac{c}{2\nu}\log\left(\frac{1+\nu}{1-\nu}\right) = 1. \tag{25}$$

Distances are measured in units of the total moderator m.f.p. and $c = \Sigma_{sm}/\Sigma_{tm}$. To be consistent we should also define the diffusion coefficient D as:

$$D = \frac{1-c}{\nu^2}, \tag{26}$$

since it will then apply for moderately strong absorption.

As before, we construct a fictitious sphere of radius $a+\Delta$ around the actual sphere and require the density there to be N'. Thus we can write the neutron density as:

$$N(r) = \frac{S}{v\Sigma_{am}} + \left(N' - \frac{S}{v\Sigma_{am}}\right)\frac{(a+\Delta)}{r}e^{-\nu(r-a-\Delta)}. \tag{27}$$

The boundary condition in the Fuchs' shell is:

$$4\pi(a+\Delta)^2 Dv\left.\frac{dN}{dr}\right|_{a+\Delta} = 4\pi a^2\frac{N'v}{4}\beta, \tag{28}$$

where we have included the blackness condition but have assumed no absorbtion in the thin Fuchs' shell. This is justified on the basis that $(1-c)$ is small.

Solving for N' and using the definition of λ leads to

$$\lambda = \frac{\beta a^2 + 4D(a+\Delta)[1+\nu(a+\Delta)] - \beta a(a+\Delta)e^{\nu\Delta}}{\beta(a+\Delta)(1+\nu a)e^{\nu\Delta}}. \tag{29}$$

To obtain Δ, we allow $a \to \infty$ when:

$$\lambda(a=\infty) = \frac{1}{\nu}\left[\left(1+\frac{4D\nu}{\beta}\right)e^{-\nu\Delta} - 1\right]. \tag{30}$$

But $\lambda(a=\infty)$ is the extrapolation distance for two adjacent half-spaces—one of which contains a source. This problem is readily solved by the Wiener–Hopf technique or approximately by the variational method. We call the value λ_M and rewrite Δ as:

$$\Delta = \frac{1}{\nu}\log\left(\frac{1+4D\nu/\beta}{1+\nu\lambda_M}\right). \tag{31}$$

Thus we have a complete expression for λ as a function of radius and absorbing and scattering properties.

An interesting limiting case of equation (29) is when $a \to 0$. Then we find:

$$\lambda(a=0) = \frac{4D}{\beta}(1+\nu\Delta)e^{-\nu\Delta}. \tag{32}$$

We can also show that as the absorption in the moderator tends to zero, the value of λ becomes the same as that already given by equation (21).

In the case of a black absorber, the value of λ_M can be obtained very easily since then the problem reduces to one of a half-space with a source and no incident flux. This result is well known (Auerbach, 1961) and leads to:

$$1+\nu\lambda_M = c\left\{\frac{c-1+\nu^2}{2(1-c)(1-\nu^2)}\right\}^{1/2} e^{\nu z_0}, \tag{33}$$

where z_0 is the conventional Milne problem extrapolation distance (Case *et al.*, 1953).

We can repeat these calculations for an infinite cylinder of radius a, and find:

$$\lambda = \frac{\beta a K_0[\nu(a+\Delta)] + 4D\nu(a+\Delta) \times K_1[\nu(a+\Delta)] - \beta a K_0(\nu a)}{\beta a \nu K_1(\nu a)}, \tag{34}$$

where K_0 and K_1 are modified Bessel functions. The value of Δ is the same as that for the sphere.

Table 4. Absorbing moderator

a \ c	0.9		0.93		0.95		0.97		0.99	
0.2	1.191	11.0	1.146	10.0	1.120	9.0	1.095	8.0	1.072	7.0
0.5	1.115	8.5	1.064	7.6	1.033	7.0	1.002	6.4	0.9698	5.9
1.0	1.065	5.1	1.010	4.2	0.9577	3.8	0.9415	3.4	0.9030	3.3
1.5	1.041	3.1	0.9839	2.5	0.9477	2.0	0.9121	1.6	0.8712	1.7
2.0	1.026	2.1	0.9677	1.4	0.9307	1.0	0.8940	0.6	0.8520	0.6
3.0	1.008	0.7	0.9484	0.2	0.9103	−0.1	0.8725	−0.6	0.8292	−0.7
5.0	0.9906	0.0	0.9297	−0.5	0.8904	−0.8	0.8514	−1.0	0.8069	−1.4
∞	0.9566	0.0	0.8920	0.0	0.8496	0.0	0.8066	0.0	0.7568	0.0

Results from the present method. The second column under the c-value denotes the percentage error compared with the results of Pellaud (1968) and Sahni and Sjöstrand (1983).

In the limit as $a \to 0$ we find that:

$$\lambda(a = 0) = \frac{4D}{\beta} v\Delta K_1(v\Delta) \tag{35}$$

and we also note that, as $c \to 1$, equation (22) is recovered.

Now numerical results for this problem have been presented by Pellaud (1968) and by Sahni and Sjöstrand (1983), thus we may check the accuracy of the method for cylinders. Table 4 shows the results for a black cylinder in a moderator of varying absorbing power. We note that the error is largest for small radius rods but rapidly reduces to less than 9% and at rods of 1 m.f.p. radius the error is better than 5%. As expected, the error increases as c decreases. In fact our method fails for $c \leqslant 0.7$ because Δ becomes negative. However, such a value of c rarely arises in practice.

It should be noted that Kavenoky (1978) has presented values of the extrapolation distance for the source free problem where the flux is allowed to become unbounded as $r \to \infty$. Although our method is applicable to that situation, we have not carried the calculations through. Thus only in the case $c = 1$ can the work of Kavenoky be used for comparison purposes since then both source and source-free problems become identical.

SUMMARY AND CONCLUSIONS

Whilst the problem discussed here is a very old one, dating back to the late 1940s, it nevertheless remains a fundamental transport theory calculation. We have also noted that there exist accurate solutions of this problem. Nevertheless, it is of considerable interest to examine simple models such as the one discussed here particularly because of its interdisciplinary nature. Moreover, there is a practical advantage. If it can be shown that the results obtained by the simple method are acceptable for these simple geometries then it may also be applied to more complicated geometries where the diffusion equation is much more easily solved than the transport equation. Extensions to include energy dependence and absorption are also possible at the expense of greater numerical effort but nevertheless still in the context of diffusion theory. The method is not dissimilar in philosophy therefore to the well known ABH or SPECTROX ideas of Leslie (1963).

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