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Two Banach Space Methods and Dual Operator Algebras

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In this paper we present several new slightly nonlinear variants of the bipolar and the open mapping theorems in Banach spaces, which we abstracted from the recent developments in the theory of dual operator algebras.

A new application of our techniques to the theory of operator algebras is also given.

1. INTRODUCTION

The basic problem that we discuss in this paper is the surjectivity and openness of bilinear maps. We consider bilinear maps

$$\tau\colon \mathscr{D}(\tau) \to \mathscr{X},$$

where $\mathscr{D}(\tau) \subset \mathscr{H} \times \mathscr{K}$, and $\mathscr{H}$, $\mathscr{K}$, $\mathscr{X}$ are Banach spaces. By analogy with Banach's open mapping theorem (for linear operators) we will be concerned with the richness of the sets

$$B = \{\tau(h, k)\colon (h, k) \in \mathscr{D}(\tau),\ \|h\| \leqslant 1,\ \|k\| \leqslant 1,\ \|\tau(h, k)\| \leqslant M\}, \qquad (1.1)$$

where $M > 0$. Thus we will consider two problems.

1.2. *Problem.* Find conditions under which the closure of B necessarily contains an open ball centered at the origin in $\mathscr{X}$.

1.3. *Problem.* Find conditions under which B necessarily contains a ball centered at the origin in $\mathscr{X}$.

We will present two approaches to these problems. In Part I we give some answers to Problem 1.2 in an abstract framework in which we give

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ourselves the set B rather than the bilinear map τ. In Part II we give answers to both problems. Even though there are similarities between the approaches in Parts I and II, the results do not overlap completely. In fact we will see in our application in Part III that the two approaches can be combined to yield new results.

We want to emphasize some new ideas introduced in this paper. In Part I we consider a new type of convexity and dominancy in Banach spaces. The basic observation is as follows. Let B be a bounded balanced set in a Banach space $\mathscr{X}$. Even if B is not absolutely convex, there may be many points $x \in \mathscr{X}$ such that all absolutely convex combinations $\alpha x + \beta y$, $y \in B$, $|\alpha| + |\beta| \leqslant 1$, are in the closure of B. The set of all such points x forms a closed absolutely convex set $D(0)$ contained in $\bar{B}$. Under certain circumstances this allows us to conclude that $\bar{B}$ contains a given convex set C (see Section 4 for the precise statements). It is seen that the set B defined in (1.1) does sometimes satisfy the conditions in Part I, and this is based on the observation that $\alpha\tau(h, k) + \beta\tau(h', k')$ is very close to $\tau(\alpha^{1/2}h + \beta^{1/2}h', \alpha^{1/2}k + \beta^{1/2}k')$ if $\tau(h, k')$ and $\tau(h', k)$ are very close to zero. See Section 10 for an application of this observation.

In Part II we consider Problem 1.2 and we give sufficient conditions for τ to be open (at every point). It is interesting that our conditions imply with little additional work the solvability of arbitrary systems of the form

$$\tau(h_i, k_j) = x_{ij}, \qquad 0 \leqslant i, j < \infty, \tag{1.4}$$

where $\{x_{ij}: i, j \geqslant 0\}$ is a given array in $\mathscr{X}$.

In Part III we give an application of the methods developed in Parts I and II to the structure theory of contractions on Hilbert space. Our results are formulated using the concept of an H^p-functional calculus. More precisely, let T be a contraction on a Hilbert space $\mathscr{H}$; assume that the unitary part of T is absolutely continuous, and let $\varphi \in H^p$ with $p \geqslant 2$. One can then define an operator $\varphi(T)$ acting continuously on a Banach space denoted H^q_T, $1/p + 1/q = \frac{1}{2}$. The space H^q_T is a dense linear manifold in $\mathscr{H}$, and $H^2_T = \mathscr{H}$. In Section 8 we define this functional calculus under the additional assumption that T is of class C_{00}. The general case will be treated elsewhere [9].

The basic result in Part III is that, if T is of class C_{00} and there is $p \in [2, +\infty)$ such that

$$\|\varphi(T)\| \geqslant \gamma \, \|\varphi\|_p, \qquad \varphi \in H^p,$$

for some $\gamma > 0$ then T belongs to the class $A_{\aleph_0}$ defined in [5].

We conclude this introduction with a few remarks on the history of the problems treated here. The question of openness for bilinear maps was considered by Cohen, who showed in [15] that a surjective bilinear map is not necessarily open at the origin. Later Horowitz [17] provided an example in this direction with $\tau\colon \mathbb{C}^3 \times \mathbb{C}^3 \to \mathbb{C}^4$. The abstract results in this paper were inspired by certain techniques that proved to be very powerful in the theory of operators on Hilbert space. (As a matter of fact, in operator theory it is convenient to work with maps $\tau(h, k)$ that are linear in h and conjugate linear in k. Our results apply in this situation because any such map τ can be traded for a bilinear one upon replacing $\mathscr{K}$ by its conjugate version.) The first open mapping theorem of the kind proved in Part II was proved by S. Brown, who used it in [13] to show that all subnormal operators have invariant subspaces. The idea of solving systems of the form (1.4) first emerged in [12] and was further used in [5, 11, 3, 7], etc.

We believe that the abstract results in this paper will prove fruitful in operator theory and in other areas of functional analysis. In particular, the concept of super-dominancy (cf. Section 2) seems to point to new interesting properties in the geometry of Banach spaces. The H^p-functional calculus introduced in Part III represents a new application of function theory to the analysis of absolutely continuous contractions, and it is expected to yield further insights about the structure of such contractions.

We thank Professor D. van Dulst for his pertinent remarks on the first version of this paper. These remarks are included here with due reference.

PART I: SUPER-DOMINANCY

2. Statement of the Problem

Let $\mathscr{X}$ be a separable complex Banach space. For two bounded subsets $A, B \subset \mathscr{X}$ we set

$$\begin{aligned}\operatorname{Dist}(A, B) &= \sup\{\operatorname{dist}(a, B)\colon a \in A\}\\ &= \sup_{a \in A} \inf_{b \in B} \|a - b\|,\end{aligned}$$

where $\|\cdot\|$ denotes the norm on $\mathscr{X}$.

Let B denote a balanced (i.e., $\lambda B \subset B$ for $\lambda \in \mathbb{C}$, $|\lambda| \leqslant 1$) bounded subset of $\mathscr{X}$, and let C be a closed absolutely convex set. The sets B and C will remain fixed throughout Sections 2 and 3. We will say that B *dominates* C if

$$\sup\{|f(x)|\colon x \in B\} \geqslant \sup\{|f(x)|\colon x \in C\}$$

for every $f \in \mathscr{X}^*$. By the bipolar theorem, this is obviously equivalent to the inclusion

$$(\text{aco}(B)^- \supset C,$$

where $\text{aco}(B)$ denotes the absolutely convex hull of B. The following stronger condition of dominancy occurred naturally in the study of dual operator algebras (cf. [7]).

2.1. DEFINITION. The set B is said to *super-dominate* C if for every $f \in \mathscr{X}^*$, every finite subset F of B, and every $\varepsilon > 0$ there exists $x \in B$ such that

(i) $f(x) + \varepsilon > \sup\{|f(y)| : y \in C\}$, and
(ii) $\text{Dist}(\text{aco}\{x, b\}, B) < \varepsilon$ for every $b \in F$.

A useful property of super-dominancy is the obvious fact that it is preserved by continuous linear maps. That is, if $T: \mathscr{X} \to \mathscr{Y}$ is a continuous linear map and B super-dominates C, then TB super-dominates TC. (This property also holds for usual dominancy.)

The solution of several major problems in the theory of dual operator algebras can be reduced to the following question for certain Banach spaces $\mathscr{X}$ (cf. Part III below).

2.2. *Problem.* Assume that B super-dominates C. Is then B strongly dense in C, i.e., is $C \subset \bar{B}$?

Unfortunately, the answer to this problem is NO, as shown by the following simple example due to D. van Dulst:

2.3. EXAMPLE. Let $\mathscr{X}$ be the space L^1 of (classes of) Lebesgue integrable functions on $\mathbb{T} = \{e^{it}: t \in [0, 2\pi)\}$ endowed with the usual norm

$$\|x\|_1 = \frac{1}{2\pi} \int_0^{2\pi} |x(e^{it})| \, dt, \qquad x \in L^1.$$

Set $C = \{x \in \mathscr{X}: \|x\| \leqslant 1\}$ and $B = \{x \in C: x = 0$ on a subset θ_x of $\mathbb{T}$ of measure $(1/2\pi) \int_{\theta_x} dt > \frac{1}{2}\}$. Then it is obvious that B is not strongly dense in C. But B super-dominates C since for every essentially bounded measurable function u on $\mathbb{T}$ there exist $x_j \in C$, $j \geqslant 1$, such that

$$\langle u, x_j \rangle = \frac{1}{2\pi} \int_0^{2\pi} u(e^{it}) \, x_j(e^{it}) \, dt$$
$$\to \text{ess max}\{|u(e^{it})| : t \in [0, 2\pi)\}$$

for $j \to \infty$, and

$$\text{meas}\{e^{it}: x_j(e^{it}) = 0\} \to 0.$$

We will show, however, that there are many useful cases in which Problem 2.2 has an affirmative answer.

There is a version of super-dominancy for the weak topology.

2.4. Defintion. The set B is said to *super-dominate C weakly* if for every $f \in \mathscr{X}^*$, every finite subset $F \subset B$, every finite subset $\{g_1, g_2, ..., g_n\} \subset \mathscr{X}^*$, and every $\varepsilon > 0$, there exists $x \in B$ such that

(i) $f(x) + \varepsilon > \sup\{|f(y)|: y \in C\}$, and

(ii) $\text{Dist}(\text{aco}\{Tx, Tb\}, TC) < \varepsilon$, $b \in F$, where $T: \mathscr{X} \to \mathbb{C}^n$ is given by $Ty = (g_1(y), g_2(y), ..., g_n(y))$, $y \in \mathscr{X}$.

The remarks above show that super-dominancy implies weak super-dominancy. Of course the two notions coincide if $\mathscr{X}$ is finite-dimensional. We will show in Section 4 (cf. Theorem 4.2) that the analogue of Problem 2.2 always has a positive answer for weak super-dominancy.

3. Asymptotic Convex Structures

In this section we introduce certain techniques that are relevant in the study of Problem 2.2. These techniques will allow us to prove in Section 4 that Problem 2.2 has an affirmative answer for a rich family of Banach spaces.

It is convenient to write $\varepsilon_F = 1/(\text{card}(F) + 1)$ for every finite set F. Now, if F is a finite subset of B, we define

$$B_F = \{x \in \mathscr{X}: \text{Dist}(\text{aco}\{x, b\}, B) < \varepsilon_F \text{ for every } b \in F\}. \tag{3.1}$$

It is clear that the sets B_F are balanced since $\text{aco}\{\lambda x, y\} \subset \text{aco}\{x, y\}$ if $x, y \in \mathscr{X}$ and $|\lambda| \leqslant 1$. It is also obvious that $B_\varnothing = \mathscr{X}$, $\{x: \|x\| < \varepsilon_F\} \subset B_F \subset \{x: \text{dist}(x, B) < \varepsilon_F\}$ for $F \neq \varnothing$, and that $B_{F \cup F'} \subset B_F \cap B_{F'}$ for all finite sets $F, F' \subset B$.

A basic property of the sets B_F is given in the following result.

3.2. Lemma. *Given a finite set $F \subset B$ and $x \in B_F$, there exists a finite set F', $F \subset F' \subset B$, such that $\lambda x + \mu y \in B_F$ for all $y \in B_{F'}$ and $\lambda, \mu \in \mathbb{C}$ satisfying the inequality $|\lambda| + |\mu| \leqslant 1$.*

Proof. The lemma is obvious for $F=\varnothing$, so there is no loss of generality in assuming that $F\neq\varnothing$. We must find F' such that

$$\operatorname{dist}(\alpha b+\beta(\lambda x+\mu y), B)<\varepsilon_F \tag{3.3}$$

for all $b\in F$, $y\in B_{F'}$ and all pairs (α,β), $(\lambda,\mu)\in\Gamma$, where $\Gamma=\{(\xi,\eta)\in\mathbb{C}^2\colon |\xi|+|\eta|\leqslant 1\}$. Let us set $M=\sup\{\|z\|\colon z\in B\}$, and

$$\varepsilon=\min\{\varepsilon_F-\operatorname{Dist}(\operatorname{aco}\{x,b\}, B)\colon b\in F\}.$$

Obviously $M<\infty$ and $\varepsilon>0$. Fix $\delta=\varepsilon/5(5M+2)$, and let $\Gamma_0\subset\Gamma$ be a finite δ-net in Γ (i.e., every point of Γ is at distance at most δ from some point in Γ_0). Of course we can, and shall, assume that $\lambda\mu\neq 0$ for every $(\lambda,\mu)\in\Gamma_0$.

For every (α_0,β_0) and (λ_0,μ_0) in Γ_0 an $b\in F$, we have

$$\operatorname{dist}\left(\frac{a_0 b+\beta_0\lambda_0 x}{|\alpha_0|+|\beta_0\lambda_0|}, B\right)\leqslant\varepsilon_F-\varepsilon,$$

and hence there is $x_0\in B$ such that

$$\left\|\frac{\alpha_0 b+\beta_0\lambda_0 x}{|\alpha_0|+|\beta_0\lambda_0|}-x_0\right\|<\varepsilon_F-\frac{\varepsilon}{2}. \tag{3.4}$$

Choose now a finite set $F'\supset F$ containing all the points x_0 constructed above, and satisfying the inequality $\varepsilon_{F'}<\delta$.

For $(\alpha,\beta),(\lambda,\mu)\in\Gamma$ we choose $(\alpha_0,\beta_0),(\lambda_0,\mu_0)\in\Gamma_0$ such that $|\alpha-\alpha_0|<\delta$, $|\beta-\beta_0|<\delta$, $|\lambda-\lambda_0|<\delta$, and $|\mu-\mu_0|<\delta$. Then, for every $b\in F$ and $y\in B_{F'}$ we have

$$\begin{aligned}
\operatorname{dist}(\alpha b+\beta(\lambda x+\mu y), B)&\leqslant\operatorname{dist}(\alpha_0 b+\beta_0(\lambda_0 x+\mu_0 y), B)\\
&\quad+\|\alpha b+\beta(\lambda x+\mu y)-\alpha_0 b-\beta_0(\lambda_0 x+\mu_0 y)\|\\
&\leqslant\operatorname{dist}(\alpha_0 b+\beta_0(\lambda_0 x+\mu_0 y), B)+\delta M+2\delta\,\|x\|+2\delta\,\|y\|\\
&\leqslant\operatorname{dist}(\alpha_0 b+\beta_0(\lambda_0 x+\mu_0 y), B)+\delta M+2\delta(M+\varepsilon_F)+2\delta(M+\varepsilon_{F'})\\
&<\operatorname{dist}(\alpha_0 b+\beta_0(\lambda_0 x+\mu_0 y), B)+\varepsilon/4,
\end{aligned}$$

and hence in order to prove (3.3) it will suffice to show that

$$\operatorname{dist}(\alpha_0 b+\beta_0(\lambda_0 x+\mu_0 y), B)\leqslant\varepsilon_F-\varepsilon/4.$$

With the vector x_0 as it occurs in (3.4) we have

$$\begin{aligned}
&\operatorname{dist}(\alpha_0 b + \beta_0 \lambda_0 x + \beta_0 \mu_0 y, B)\\
&\quad \leqslant \operatorname{dist}((|\alpha_0| + |\beta_0 \lambda_0|)\, x_0 + \beta_0 \mu_0 y, B)\\
&\qquad + (|\alpha_0| + |\beta_0 \lambda_0|) \left\| \frac{\alpha_0 b + \beta_0 \lambda_0 x}{|\alpha_0| + |\beta_0 \lambda_0|} - x_0 \right\|\\
&\quad \leqslant \varepsilon_{F'} + \varepsilon_F - \frac{\varepsilon}{2} < \varepsilon_F - \frac{\varepsilon}{4}
\end{aligned}$$

by the choice of F' and δ. This concludes the proof.

We note for further use that the calculations in the above proof show that

$$\operatorname{dist}(\alpha b + \beta(\lambda x + \mu y), B) \leqslant \varepsilon_F - \eta, \qquad \eta = \varepsilon/4 - \varepsilon_{F'},$$

for all $y \in B_{F'}$ and $(\alpha, \beta), (\lambda, \mu) \in \Gamma$. This inequality can be rewritten as

$$\operatorname{Dist}(\operatorname{aco}\{b, x, y\}, B) \leqslant \varepsilon_F - \eta, \qquad y \in B_{F'}, \quad b \in F.$$

Let us define now

$$d_F(x) = \operatorname{dist}(x, B_F), \qquad x \in \mathscr{X},$$

for every finite set $F \subset B$. It is easy to see that

$$d_F(x) \leqslant d_{F'}(x) \leqslant \|x\|, \qquad x \in \mathscr{X},$$

if $F \subset F'$ are finite subsets of B. Furthermore, we have

$$d_F(x) \geqslant \operatorname{dist}(x, B) - \varepsilon_F, \qquad x \in \mathscr{X}, \quad F \neq \varnothing,$$

because $B_F \subset \{y : \operatorname{dist}(y, B) < \varepsilon_F\}$,

$$d_F(\lambda x) \leqslant |\lambda|\, d_F(x), \qquad x \in \mathscr{X}, \quad |\lambda| \leqslant 1,$$

because B_F is balanced, and

$$|d_F(x) - d_F(y)| \leqslant \|x - y\|, \qquad x, y \in \mathscr{X}.$$

We see now that the limit

$$d(x) = \lim_F d_F(x) = \sup\{d_F(x) : F \subset B,\ F \text{ finite}\}$$

exists for every $x \in \mathscr{X}$ and enjoys the properties

$$\begin{aligned} |d(x) - d(y)| &\leqslant \|x - y\|, && x, y \in \mathscr{X}, \\ d(\lambda x) &\leqslant |\lambda|\, d(x), && x \in \mathscr{X}, \quad |\lambda| \leqslant 1, \\ d(x) &\geqslant \operatorname{dist}(x, B), && x \in \mathscr{X}. \end{aligned} \tag{3.5}$$

3.6. COROLLARY. (i) *For every finite set $F \subset B$ and every $x \in B_F$ there exists a finite set F' such that*

$$d_F(\lambda x + \mu y) \leqslant |\mu|\, d_{F'}(y) \leqslant |\mu|\, d(y)$$

for every $y \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| + |\mu| \leqslant 1$.

(ii) *For every finite set $F \subset B$, every $x \in \mathscr{X}$, and every $\varepsilon > 0$, there exists a finite set F' such that*

$$d_F(\lambda x + \mu y) \leqslant \varepsilon + |\lambda|\, d_F(x) + |\mu|\, d_{F'}(y)$$

for every $y \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| + |\mu| \leqslant 1$.

Proof. (i) Let F' be given by Lemma 3.2, let $y \in \mathscr{X}$, and let η be an arbitrary positive number. Choose $y' \in B_{F'}$ such that $\|y - y'\| < d_{F'}(y) + \eta$. Then, by virtue of Lemma 3.2,

$$\begin{aligned} d_F(\lambda x + \mu y) &\leqslant d_F(\lambda x + \mu y') + |\mu|\, \|y - y'\| \\ &= |\mu|\, \|y - y'\| \\ &\leqslant |\mu|\, d_{F'}(y) + \eta, \qquad \lambda, \mu \in \mathbb{C}, \quad |\lambda| + |\mu| \leqslant 1. \end{aligned}$$

Part (i) of the corollary follows since η is arbitrary.

(ii) Choose $x_0 \in B_F$ such that $\|x - x_0\| < d_F(x) + \varepsilon$, and choose F' as in the proof of (i), with x replaced by x_0. Then we have

$$\begin{aligned} d_F(\lambda x + \mu y) &\leqslant \|\lambda x - \lambda x_0\| + d_F(\lambda x_0 + \mu y) \\ &\leqslant \varepsilon + |\lambda|\, d_F(x) + |\mu|\, d_{F'}(y) \end{aligned}$$

for $y \in \mathscr{X}$ and $|\lambda| + |\mu| \leqslant 1$, as desired. Thus the proof is complete.

3.7. COROLLARY. *Let $F \subset B$ be a finite set. Then*

$$d_F(\lambda x + \mu y) \leqslant |\lambda|\, d_F(x) + |\mu|\, d(y)$$

for all $x, y \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| + |\mu| \leqslant 1$.

Proof. Fix $x, y \in \mathscr{X}$, $\lambda, \mu \in \mathbb{C}$, $|\lambda| + |\mu| \leqslant 1$. It follows from Corollary 3.6(ii) that

$$d_F(\lambda x + \mu y) \leqslant \varepsilon + |\lambda|\, d_F(x) + |\mu|\, d(y)$$

for every $\varepsilon > 0$. The conclusion follows immediately.

3.8. PROPOSITION. *The function d is absolutely convex, i.e.,*

$$d(\lambda x + \mu y) \leqslant |\lambda|\, d(x) + |\mu|\, d(y)$$

for all $x, y \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$ with $|\lambda| + |\mu| \leqslant 1$.

Proof. Fix $x, y \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| + |\mu| \leqslant 1$. By the preceding corollary

$$\begin{aligned} d(\lambda x + \mu y) &= \lim_F d_F(\lambda x + \mu y) \leqslant \lim_F \left(|\lambda|\, d_F(x) + |\mu|\, d(y)\right) \\ &= |\lambda|\, d(x) + |\mu|\, d(y). \end{aligned}$$

The proposition is proved.

For every number $\theta \geqslant 0$ we define now the set

$$D(\theta) = \{x \in \mathscr{X} : d(x) \leqslant \theta\}.$$

The following properties of the sets $D(\theta)$ readily follow from Proposition 3.8 and (3.5).

3.9. COROLLARY. *For every $\theta \geqslant 0$, $D(\theta)$ is closed and absolutely convex,*

$$\{x : \|x\| \leqslant \theta\} \subset D(\theta) \subset \{x : \operatorname{dist}(x, B) \leqslant \theta\},$$

and

$$D(\theta) = \bigcap_F \{x : \operatorname{dist}(x, B_F) \leqslant \theta\},$$

where the intersection is taken over all finite subsets F of B. In particular, $D(0) \subset \bar{B}$.

4. SUPER-DOMINANCY AND APPROXIMATION

We come now to a first positive answer to Problem 2.2 in an easy case.

4.1. LEMMA. *Assume that B super-dominates C. If B is relatively (strongly) compact then $D(0)$ contains C.*

Proof. Fix $f \in \mathscr{X}^*$ and observe from Definition 2.1 that for each finite set $F \subset B$ we can choose a vector $x_F \in B_F \cap B$ such that

$$\liminf_F f(x_F) \geqslant \sup\{|f(y)| : y \in C\}.$$

Since B is relatively compact there exists $x \in \bar{B}$ such that

$$x \in \bigcap_F \{x_G : G \supset F, G \text{ finite}\}^- \subset \bigcap_F \bar{B}_F = D(0);$$

clearly $f(x) \geqslant \sup\{|f(y)| : y \in C\}$. Since $D(0)$ is closed and absolutely convex, the conclusion follows from the bipolar theorem.

4.2. THEOREM. *Assume that B super-dominates C weakly. Then C is contained in the weak closure of B.*

Proof. Fix $x_0 \in C$ and an arbitrary weak neighborhood of x_0 given by

$$V = \{x \in \mathscr{X} : |f_j(x - x_0)| < 1, j = 1, 2, ..., n\},$$

where $f_1, f_2, ..., f_n \in \mathscr{X}^*$. The map $T: \mathscr{X} \to \mathbb{C}^n$ defined by

$$Tx = (f_1(x), f_2(x), ..., f_n(x)), \qquad x \in \mathscr{X},$$

is linear and continuous and clearly TB super-dominates TC weakly. Since $\mathbb{C}^n$ is finite-dimensional, TB super-dominates TC. By Lemma 4.1, $\overline{TB} \supset TC$ and, in particular, there exists $b \in B \cap V$. This completes the proof.

4.3. COROLLARY. *Assume that B super-dominates C. Then C is contained in the weak closure of $B_F \cap B$ for every finite set $F \subset B$.*

Proof. The corollary follows directly from Theorem 4.2 and the remark that $B_F \cap B$ super-dominates C for every finite set $F \subset B$. To prove this, fix $f \in \mathscr{X}^*$, $\varepsilon > 0$, and a finite set $F_0 \subset B_F \cap B$. We must show that there exists $y \in B_F \cap B$ such that

$$\text{Dist}(\text{aco}\{y, x\}, B_F \cap B) < \varepsilon, \qquad x \in F_0,$$

and

$$f(y) + \varepsilon > \sup\{|f(z)| : z \in C\}.$$

By Definition 2.1, for every finite set $F' \subset B$ we can find $y \in B_{F'}$ such that

$$f(y) + \varepsilon > \sup\{|f(z)| : z \in C\},$$

so it will suffice to produce a set F' such that

$$\operatorname{Dist}(\operatorname{aco}\{y, x\}, B_F \cap B) < \varepsilon, \qquad x \in F_0, \quad y \in B_{F'}.$$

A repeated application of the remark following the proof of Lemma 3.2 shows the existence of $\eta > 0$ and F' such that

$$\operatorname{Dist}(\operatorname{aco}\{b, x, y\}, B) \leqslant \varepsilon_F - \eta, \qquad b \in F, \quad x \in F_0, \quad y \in B_{F'}.$$

We may of course assume that $F' \supset F_0$ and $\varepsilon_{F'} < \min\{\varepsilon, \eta\}$. Now, if $\lambda, \mu \in \mathbb{C}$, $|\lambda| + |\mu| \leqslant 1$, $x \in F_0$, and $y \in B_{F'}$, we can find $z \in B$ such that

$$\|\lambda x + \mu y - z\| < \varepsilon_{F'}.$$

Hence for $b \in F$ we have

$$\begin{aligned}\operatorname{Dist}(\operatorname{aco}\{b, z\}, B) &\leqslant \operatorname{Dist}(\operatorname{aco}\{b, \lambda x + \mu y\}, B) + \varepsilon_{F'}\\ &\leqslant \operatorname{Dist}(\operatorname{aco}\{b, x, y\}, B) + \varepsilon_{F'}\\ &\leqslant \varepsilon_F - \eta + \varepsilon_{F'} < \varepsilon_F,\end{aligned}$$

and thus $z \in B_F \cap B$. We deduce that

$$\operatorname{Dist}(\operatorname{aco}\{y, x\}, B_F \cap B) \leqslant \varepsilon_{F'} < \varepsilon, \qquad x \in F_0, \quad y \in B_{F'},$$

as desired. This concludes the proof.

The following results show that Problem 2.2 has a positive answer in a large number of cases. We start with a statement which is a substantial improvement, due to van Dulst, of our original result concerning uniformly convex Banach spaces.

4.4. PROPOSITION. *Assume that $B \subset C$, and B super-dominates C. Then every strongly exposed point of C belongs to $D(0)$. If, in particular, $\mathscr{X}$ has the Radon–Nikodym property, then B is norm-dense in C.*

Proof. If $\mathscr{X}$ has the Radon–Nikodym property then C is the closed convex hull of its strongly exposed points (Phelps's theorem, cf. Proposition 5.14 of [16]). Thus, the second part of the proposition follows from the first part and the fact that $D(0)$ is convex and closed.

To prove the first part fix a strongly exposed point $x \in C$, and a strongly exposing functional $f \in \mathscr{H}^*$. In other words,

$$f(x) = \sup\{\operatorname{Re} f(y) \colon y \in C\},$$

and

$$\lim_{\varepsilon \downarrow 0} \operatorname{diam} S_\varepsilon(C, x, f) = 0,$$

where

$$S_\varepsilon(C, x, f) = \{ y \in C \colon \operatorname{Re} f(y) > f(x) - \varepsilon \}.$$

By the remark made in the proof of Corollary 4.3, we can find $x_F \in B \cap B_F \subset C$ satisfying the relation

$$\operatorname{Re} f(x_F) > f(x) - \varepsilon_F$$

or, equivalently, $x_F \in S_{\varepsilon_F}(C, x, f)$. We clearly have then $x = \lim_F x_F$, and this concludes the proof of our proposition.

As a consequence of Proposition 4.4 we see that in all separable dual Banach spaces, Problem 2.2 has an affirmative answer provided that $B \subset C$. Indeed, every separable dual Banach space has the Radon–Nikodym property (cf. Corollary 6.3 in [16]), and hence Problem 2.2 has a positive answer for such spaces.

The following is the main result given by the methods developed in Section 3.

4.5. THEOREM. *Assume that B super-dominates C. If the weak topology on $B \cup C$ is metrizable, then $D(0)$ contains C.*

Proof. Fix $x \in C$ and a basis $V_0 \supset V_1 \supset V_2 \supset \cdots$ of weak neighborhoods of x_0 in $B \cup C$. By Corollary 3.9 it suffices to prove that $d_F(x) = 0$ for every finite subset $F \subset B$. Fix such a set F. We construct inductively sequences $\{F_j : j \geqslant 0\}$ and $\{x_j : j \geqslant 0\} \subset B$ with the following properties: $F_0 = F$, $x_j \in B_{F_j} \cap B \cap V_j$, and

$$d_{F_j}(\lambda x_j + \mu y) \leqslant |\mu| \, d_{F_{j+1}}(y),$$

for all $y \in \mathscr{X}$ and $\lambda, \mu \in \mathbb{C}$ such that $|\lambda| + |\mu| \leqslant 1$. This construction is clearly possible by virtue of Corollaries 3.6 and 4.3. Since the sequence $\{x_j : j \geqslant 0\}$ converges weakly to x, a classical theorem of Mazur implies that for every $\eta > 0$ there exists a finite sequence $\{\lambda_k : 0 \leqslant k \leqslant n\} \subset \mathbb{C}$ such that $\lambda_n \neq 0$, $\sum_{k=0}^n |\lambda_k| \leqslant 1$, and $\|x - \sum_{k=0}^n \lambda_k x_k\| < \eta$. We have then

$$\begin{aligned} d_F\left(\sum_{k=0}^{n} \lambda_k x_k\right) &= d_{F_0}\left(\sum_{k=0}^{n} \lambda_k x_k\right) \\ &\leqslant \left(\sum_{k=1}^{n} |\lambda_k|\right) d_{F_1}\left(\frac{\sum_{k=1}^{n} \lambda_k x_k}{\sum_{k=1}^{n} |\lambda_k|}\right) \\ &\leqslant \left(\sum_{k=2}^{n} |\lambda_k|\right) d_{F_2}\left(\frac{\sum_{k=2}^{n} \lambda_k x_k}{\sum_{k=2}^{n} |\lambda_k|}\right) \\ &\leqslant |\lambda_n|\, d_{F_n}(x_n) = 0, \end{aligned}$$

and consequently $d_F(x) < \eta$. The theorem follows by letting η tend to zero.

The following is an immediate consequence of Theorem 4.5.

4.6. COROLLARY. *Assume that B super-dominates C. If $\mathscr{X}^*$ is separable then $D(0) \supset C$.*

We conclude this section with a remarkable property of the sets $D(\theta)$.

4.7. THEOREM. *Let $\gamma > \theta \geqslant 0$ be fixed. Then $D(\theta) \supset \{x\colon \|x\| \leqslant \gamma\}$ if and only if $D(0) \supset \{x\colon \|x\| \leqslant \gamma - \theta\}$.*

Proof. Assume first that $D(0) \supset \{x\colon \|x\| \leqslant \gamma - \theta\}$ and let $z \in \mathscr{X}$ with $\|z\| \leqslant \gamma$. We have then

$$d(z) = d\left(\frac{\theta}{\gamma} z + \left(1 - \frac{\theta}{\gamma}\right) z\right) \leqslant \frac{\theta}{\gamma} \|z\| + d\left(\left(1 - \frac{\theta}{\gamma}\right) z\right) = \frac{\theta}{\gamma} \|z\| \leqslant \theta$$

and therefore $z \in D(\theta)$, thus $D(\theta) \supset \{x\colon \|x\| \leqslant \gamma\}$.

Before proving the converse we note an additional property of the function d_F, where $F \subset B$ is finite. Let $\beta > 0$, $x \in \mathscr{X}$, and $y \in B_F$. Then we have

$$\begin{aligned} d_F\left(\frac{x}{1+\beta}\right) &= d_F\left(\frac{1}{1+\beta} y + \frac{\beta}{1+\beta} \frac{x-y}{\beta}\right) \\ &\leqslant \frac{1}{1+\beta} d_F(y) + \frac{\beta}{1+\beta} d\left(\frac{x-y}{\beta}\right) = \frac{\beta}{1+\beta} d\left(\frac{x-y}{\beta}\right). \end{aligned} \tag{4.8}$$

Assume now that $D(\theta) \supset \{x\colon \|x\| \leqslant \gamma\}$ and note that this is equivalent to

$$d(x) \leqslant \beta \|x\|, \qquad x \in \mathscr{X}, \quad \|x\| \leqslant \gamma, \tag{4.9}$$

where $\beta = \theta/\gamma$. We will use (4.8) to show that (4.9) implies

$$d(z) \leqslant \beta^2 \|z\|, \qquad z \in \mathscr{X}, \quad \|z\| \leqslant \frac{\gamma}{1+\beta}. \tag{4.10}$$

Indeed, fix $z \in \mathscr{X}$ such that $0 < \|z\| \leqslant \gamma/(1+\beta)$, a finite set $F \subset B$, and $\beta' \in (\beta, 1)$. Write $x = (1+\beta) z$. From (4.9) we have $d(x) < \beta' \|x\|$ so that $\|x-y\| < \beta' \|x\|$ for some $y \in B_F$. Again by (4.9) applied to $(x-y)/\beta'$, we have

$$\begin{aligned} d\left(\frac{x-y}{\beta}\right) &\leqslant d\left(\frac{x-y}{\beta'}\right) + \frac{\|x-y\| \, (\beta'-\beta)}{\beta\beta'} \\ &\leqslant \beta \frac{\|x-y\|}{\beta'} + \frac{\gamma(\beta'-\beta)}{\beta} \\ &< \beta \, \|x\| + \frac{\gamma(\beta'-\beta)}{\beta}. \end{aligned}$$

An application of (4.8) yields now

$$d_F(z) \leqslant \frac{\beta}{1+\beta}\left(\beta \, \|x\| + \frac{\gamma(\beta'-\beta)}{\beta}\right) \leqslant \beta^2 \, \|z\| + \frac{\gamma\beta(\beta'-\beta)}{(1+\beta)\,\beta},$$

and hence $d_F(z) \leqslant \beta^2 \|z\|$ since $\beta' \in (\beta, 1)$ is arbitrary. Relation (4.10) follows by taking the supremum over F.

An inductive application of the previous argument shows that we have

$$d(z) \leqslant \beta_n \, \|z\|, \qquad z \in \mathscr{X}, \quad \|z\| \leqslant \gamma_n, \quad n = 0, 1, 2 \ldots,$$

where $\beta_0 = \beta$, $\gamma_0 = \gamma$, and $\beta_{n+1} = \beta_n^2$, $\gamma_{n+1} = \gamma_n/(1+\beta_n)$. It is clear that $\beta_n = \beta^{2^n}$, and

$$\gamma_n = \gamma \prod_{k=0}^{n-1} \frac{1}{(1+\beta^{2^k})} = \frac{\gamma(1-\beta)}{1-\beta^{2^n}} \geqslant \gamma(1-\beta) = \gamma - \theta.$$

Thus, for $z \in \mathscr{X}$ with $\|z\| \leqslant \gamma - \theta$, we have

$$d(z) \leqslant \lim_{n \to \infty} \beta_n \, \|z\| = 0.$$

This last relation means exactly that $D(0) \supset \{x\colon \|x\| \leqslant \gamma - \theta\}$, so the theorem is proved.

PART II: AN OPEN MAPPING THEOREM FOR BILINEAR MAPS

5. Notation and Preliminary Results

Let $\mathscr{H}$, $\mathscr{K}$, and $\mathscr{X}$ be normed spaces, and let

$$\tau\colon \mathscr{D}(\tau) \subset \mathscr{H} \times \mathscr{K} \to \mathscr{X}$$

be a partially defined bilinear map. The fact that τ is bilinear means, in particular, that whenever (h, k), (h', k), and (h, k') are $\mathscr{D}(\tau)$, the pairs $(h+h', k)$ and $(h, k+k')$ also belong to $\mathscr{D}(\tau)$ and

$$\tau(h+h', k) = \tau(h, k) + \tau(h', k), \qquad \tau(h, k+k') = \tau(h, k) + \tau(h, k').$$

Note that $\mathscr{D}(\tau)$ is not generally a linear manifold. We will make, however, the following assumption.

5.1. *Assumption.* There exist linear manifolds $\mathscr{H}_0 \subset \mathscr{H}$ and $\mathscr{K}_0 \subset \mathscr{K}$ (not necessarily closed) such that $\mathscr{H}_0 \times \mathscr{K}_0 \subset \mathscr{D}(\tau)$.

Our results will be based on the richness of the sets $\mathscr{X}_\theta = \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$ defined as follows for $\theta \geqslant 0$.

5.2. DEFINITION. Assume that 5.1 holds and $\theta \geqslant 0$. The set $\mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$ consists of all vectors $x \in \mathscr{X}$ with the following property: Given an integer p, vectors $\xi_1, \xi_2, ..., \xi_p \in \mathscr{H}_0$, vectors $\eta_1, \eta_2, ..., \eta_p \in \mathscr{K}_0$, and a number $\varepsilon > 0$, there exist vectors $h \in \mathscr{H}_0$, $k \in \mathscr{K}$ such that

(i) $\|h\| \leqslant 1$, $\|k\| \leqslant 1$,

(ii) $\|x - \tau(h, k)\| < \theta + \varepsilon$,

(iii) $\|\tau(\xi_i, k)\| < \varepsilon$, $\|\tau(h, \eta_i)\| < \varepsilon$ for $1 \leqslant i \leqslant p$.

5.3. LEMMA. *The sets* $\mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$ are closed.

Proof. Let $x \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)^-$, $\xi_1, \xi_2, ..., \xi_p \in \mathscr{H}_0$, $\eta_1, \eta_2, ..., \eta_p \in \mathscr{K}_0$, and $\varepsilon > 0$. There exists then $x' \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$ such that $\|x - x'\| < \varepsilon/2$, and by Definition 5.2 there are $h \in \mathscr{H}_0$ and $k \in \mathscr{H}_0$ such that $\|h\| \leqslant 1$, $\|k\| \leqslant 1$, $\|x' - \tau(h, k)\| < \theta + \varepsilon/2$, and $\|\tau(\xi_i, k)\| < \varepsilon$, $\|\tau(h, \eta_i)\| < \varepsilon$ for $1 \leqslant i \leqslant p$. The lemma follows now easily because

$$\begin{aligned}\|x - \tau(h, k)\| &\leqslant \|x - x'\| + \|x' - \tau(h, k)\| \\ &< \varepsilon/2 + \theta + \varepsilon/2 = \theta + \varepsilon.\end{aligned}$$

We can now define the property of τ which will be relevant to our results.

5.4. DEFINITION. Let τ be a bilinear map satisfying Assumption 5.1, and let $\gamma > 0$. The map τ is said to have *property* $(A_{\theta,\gamma})$ *relative to* $\mathscr{H}_0$ *and* $\mathscr{K}_0$ if

$$\mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0) \supset \{x \in \mathscr{X} : \|x\| \leqslant \gamma\}.$$

Let us note that $\mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$ always contains $\{x \in \mathscr{X} : \|x\| \leqslant \theta\}$, and hence τ always has property $(\Delta_{\theta,\gamma})$ if $\gamma \leqslant \theta$. Thus, when working with the properties $(\Delta_{\theta,\gamma})$, we will normally assume that $\gamma > \theta \geqslant 0$.

5.5. *Remark.* Let $x \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$, $\xi_1, \xi_2, ..., \xi_p \in \mathscr{H}_0$, $\eta_1, \eta_2, ..., \eta_p \in \mathscr{K}_0$, and $\varepsilon > 0$. Then there are vectors $h \in \mathscr{H}_0$, $k \in \mathscr{K}_0$ such that

(i) $\|h\| < 1$, $\|k\| < 1$,

(ii) $\|x - \tau(h, k)\| < \theta + \varepsilon$,

(iii) $\|\tau(\xi_i, k)\| < \varepsilon$, $\|\tau(h, \eta_i)\| < \varepsilon$ for $1 \leqslant i \leqslant p$.

Indeed, by Definition 5.2 there are h, k satisfying (ii) and (iii), and $\|h\| \leqslant 1$, $\|k\| \leqslant 1$. We can replace h and k by αh and αk, where $\alpha \in (0, 1)$ is chosen sufficiently close to one such that $\|x - \alpha^2 \tau(h, k)\| < \theta + \varepsilon$.

From this point on it will always be assumed that Assmption 5.1 is satisfied and τ is closed, i.e., the graph $\{(h, k, x)\colon (h, k) \in \mathscr{D}(\tau),\ x = \tau(h, k)\}$ is a closed set in $\mathscr{H} \times \mathscr{K} \times \mathscr{X}$. Denote by $\mathscr{H}_0^-$ and $\mathscr{K}_0^-$ the closures of $\mathscr{H}_0$ and $\mathscr{K}_0$. Observe that in this case the formulas

$$N_{\mathscr{H}} = \{h \in \mathscr{H}_0^- : \{h\} \times \mathscr{K}_0 \subset \mathscr{D}(\tau) \text{ and } \tau(h, k) = 0 \text{ for } k \in \mathscr{K}_0\}$$

$$N_{\mathscr{K}} = \{k \in \mathscr{K}_0^- : \mathscr{H}_0 \times \{k\} \subset \mathscr{D}(\tau) \text{ and } \tau(h, k) = 0 \text{ for } h \in \mathscr{H}_0\}$$

define closed subspaces of $\mathscr{H}$ and $\mathscr{K}$, respectively. In fact it also follows that

$$(N_{\mathscr{H}} \times \mathscr{K}_0^-) \cup (\mathscr{H}_0^- \times N_{\mathscr{K}}) \subset \mathscr{D}(\tau),$$

and

$$\tau(h, k) = 0 \qquad \text{if} \quad (h, k) \in (N_{\mathscr{H}} \times \mathscr{K}_0^-) \cup (\mathscr{H}_0^- \times N_{\mathscr{K}}).$$

It is clear that the sets $\mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$ (cf. Definition 5.2) do not change if we replace $\mathscr{H}_0$ and $\mathscr{K}_0$ by $\mathscr{H}_0 + N_{\mathscr{H}}$ and $\mathscr{K}_0 + N_{\mathscr{K}}$, respectively. Therefore the following technical assumption will not be restrictive.

5.6. *Assumption.* $N_{\mathscr{H}} \subset \mathscr{H}_0$ and $N_{\mathscr{K}} \subset \mathscr{K}_0$.

5.7. *Remark.* Let $\tau\colon \mathscr{D}(\tau) \subset \mathscr{H} \times \mathscr{K} \to \mathscr{X}$ be a closed bilinear map satisfying Assumptions 5.1 and 5.6. Assume in addition that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces. Then we have $\mathscr{H}_0 \ominus N_{\mathscr{H}} \subset \mathscr{H}_0$, $\mathscr{K}_0 \ominus N_{\mathscr{K}} \subset \mathscr{K}_0$, and

$$\mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0) = \mathscr{X}_\theta(\tau, \mathscr{H}_0 \ominus N_{\mathscr{H}}, \mathscr{K}_0 \ominus N_{\mathscr{K}}), \qquad \theta \geqslant 0. \tag{5.8}$$

Indeed, if h and k satisfy conditions (i), (ii), and (iii) of Remark 5.5, then $h' = P_{\mathscr{H}_0 \ominus N_{\mathscr{H}}} h$ and $k' = P_{\mathscr{K}_0 \ominus N_{\mathscr{K}}} k$ also satisfy these conditions.

We conclude this section by showing that properties $(\Delta_{\theta,\gamma})$ are substantially easier to verify in the case in which $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces.

5.9. LEMMA. *Let* $\tau: \mathscr{D}(\tau) \subset \mathscr{H} \times \mathscr{K} \to \mathscr{X}$ *be a closed bilinear map satisfying Assumptions* 5.1 *and* 5.6. *Assume in addition that* $\mathscr{H}$ *and* $\mathscr{K}$ *are Hilbert spaces. If* $\theta \geqslant 0$, $x \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$, $\varepsilon > 0$, p *is an integer, and* $\xi_1, \xi_2, ..., \xi_p \in \mathscr{H}_0$, $\eta_1, \eta_2, ..., \eta_p \in \mathscr{K}_0$, *then there are* $h \in \mathscr{H}_0$ *and* $k \in \mathscr{K}_0$ *such that*

(i) $\|h\| \leqslant 1$, $\|k\| \leqslant 1$,

(ii) $\|x - \tau(h, k)\| < \theta + \varepsilon$,

(iii) $\|\tau(\xi_i, k)\| < \varepsilon$, $\|\tau(h, \eta_i)\| < \varepsilon$, $1 \leqslant i \leqslant p$,

(iv) $|(h \mid \xi_i)| < \varepsilon$, $|(k \mid \eta_i)| < \varepsilon$, $1 \leqslant i \leqslant p$, *where* $(\cdot \mid \cdot)$ *denotes the scalar product.*

Proof. Let $F \subset \mathscr{H}_0$ and $G \subset \mathscr{K}_0$ be finite sets. By (5.8) and Definition 5.2, we can find $h_{F,G} \in \mathscr{H}_0 \ominus N_{\mathscr{H}}$, $k_{F,G} \in \mathscr{K}_0 \ominus N_{\mathscr{K}}$ such that

$$\|h_{F,G}\| \leqslant 1, \qquad \|k_{F,G}\| \leqslant 1, \qquad \|x - \tau(h_{F,G}, k_{F,G})\| < \theta + \varepsilon_{F,G},$$

and

$$\|\tau(\xi, k_{F,G})\| \leqslant \varepsilon_{F,G}, \qquad \|\tau(h_{F,G}, \eta)\| \leqslant \varepsilon_{F,G}, \qquad (\xi, \eta) \in F \times G, \tag{5.10}$$

where $\varepsilon_{F,G} = 1/(\text{card}(F) + \text{card}(G))$. The idea is to show that the weak limit of the net $\{(h_{F,G}, k_{F,G})\}_{F,G} \subset \mathscr{H} \times \mathscr{K}$ is (0, 0). Indeed, if this is shown, it will suffice to take F, G sufficiently large so that $h = h_{F,G}$ and $k = k_{F,G}$ satisfy conditions (i)–(iv). By symmetry it suffices to show that the weak limit of the net $\{h_{F,G}\}_{F,G}$ is zero. Let ξ denote an aribtrary weak accumulation point of this net, and fix $\varepsilon > 0$ and $\eta \in \mathscr{K}_0$. Choose F and G such that $\varepsilon_{F,G} < \varepsilon$ and $\eta \in G$ and note that by Mazur's theorem ξ belongs to the norm-closed absolutely convex hull of the set $\{h_{F',G'}: F' \supset F,\ G' \supset G\}$. Thus there are $F_1, F_2, ..., F_n$, $G_1, G_2, ..., G_n$ with $F_i \supset F$, $G_i \supset G$, $1 \leqslant i \leqslant n$, and there are constants $\alpha_1, \alpha_2, ..., \alpha_n \in \mathbb{C}$ such that $\sum_{j=1}^n |\alpha_j| \leqslant 1$ and $\|\xi - \sum_{j=1}^n \alpha_j h_{F_j,G_j}\| < \varepsilon$. Note further that (5.10) implies the inequality

$$\left\| \tau\left(\sum_{j=1}^n \alpha_j h_{F_j,G_j}, \eta \right) \right\| \leqslant \varepsilon_{F,G} < \varepsilon$$

since $\varepsilon_{F_j,G_j} \leqslant \varepsilon_{F,G}$, $1 \leqslant j \leqslant n$. But $\varepsilon > 0$ is arbitrary; so we conclude that the tiple $(\xi, \eta, 0)$ belongs to the closure of the graph of τ, and hence $(\xi, \eta) \in \mathscr{D}(\tau)$ and $\tau(\xi, \eta) = 0$. Now, $\eta \in \mathscr{K}_0$ is arbitrary, whence $\xi \in N_{\mathscr{H}}$. But we also have $\xi \in \mathscr{H} \ominus N_{\mathscr{H}}$ because $h_{F,G} \in \mathscr{H}_0 \ominus N_{\mathscr{H}}$. We conclude that necessarily $\xi = 0$, and this completes our proof.

5.11. PROPOSITION. *Let* $\tau: \mathscr{D}(\tau)$ $(\subset \mathscr{H} \times \mathscr{K}) \to \mathscr{X}$ *be a closed bilinear map satisfying Assumption* 5.1. *If* $\mathscr{H}$ *and* $\mathscr{K}$ *are Hilbert spaces, then the sets* $\mathscr{X}_\theta(\tau, \mathscr{H}, \mathscr{K})$ *are cllosed and absolutely convex for* $\theta \geqslant 0$.

Proof. As we noted above, we may assume that 5.6 holds. We already know from Lemma 5.3 that the sets $\mathscr{X}_\theta(\tau, \mathscr{H}, \mathscr{K})$ are closed. Fix $\theta \geqslant 0$, $x_1, x_2 \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$, and $\alpha_1, \alpha_2 \in \mathbb{C}$ such that $|\alpha_1| + |\alpha_2| \leqslant 1$; we want to show that $\alpha_1 x_1 + \alpha_2 x_2 \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$. Choose then $\varepsilon > 0$ and $\xi_1, \xi_2, ..., \xi_p \in \mathscr{H}_0$, $\eta_1, \eta_2, ..., \eta_p \in \mathscr{K}_0$, and use Remark 5.5 and Lemma 5.9 to find $h_1, h_2 \in \mathscr{H}_0$, $k_1, k_2 \in \mathscr{K}_0$ with the following properties:

$$\|h_1\| < 1, \qquad \|h_2\| \leqslant 1, \qquad \|k_1\| < 1, \qquad \|k_2\| \leqslant 1,$$

$$\|x_1 - \tau(h_1, k_1)\| < \theta + \varepsilon/2, \qquad \|x_2 - \tau(h_2, k_2)\| < \theta + \varepsilon/2,$$

$$\|\tau(h_1, \eta_i)\| < \varepsilon/2, \qquad \|\tau(h_2, \eta_i)\| < \varepsilon/2,$$

$$\|\tau(\xi_i, k_1)\| < \varepsilon/2, \qquad \|\tau(\xi_i, k_2)\| < \varepsilon/2, \qquad 1 \leqslant i \leqslant p,$$

$$\|\tau(h_1, k_2)\| < \varepsilon/4, \qquad \|\tau(h_2, k_1)\| < \varepsilon/4,$$

$$|(h_1 \mid h_2)| \leqslant \tfrac{1}{2}(1 - |\alpha_2| - |\alpha_1| \, \|h_1\|^2),$$

$$|(k_1 \mid k_2)| \leqslant \tfrac{1}{2}(1 - |\alpha_2| - |\alpha_1| \, \|k_1\|^2).$$

Of course, in order to satisfy the last inequalities, the pair (h_2, k_2) is chosen after (h_1, k_1). If $\alpha_1 = 0$ we can choose $h_1 = 0$ and $k_1 = 0$. Fix some square roots $\alpha_1^{1/2}$ and $\alpha_2^{1/2}$, and define

$$h = \alpha_1^{1/2} h_1 + \alpha_2^{1/2} h_2, \qquad k = \alpha_1^{1/2} k_1 + \alpha_2^{1/2} k_2.$$

We have

$$\begin{aligned} \|h\|^2 &= |\alpha_1| \, \|h_1\|^2 + |\alpha_2| \, \|h_2\|^2 + 2 \operatorname{Re}(\alpha_1^{1/2} h_1 \mid \alpha_2^{1/2} h_2) \\ &\leqslant |\alpha_1| \, \|h_1\|^2 + |\alpha_2| + 2 \, |(h_1 \mid h_2)| \leqslant 1 \end{aligned}$$

and, analogously, $\|k\| \leqslant 1$.

Next,

$$\begin{aligned} &\|\alpha_1 x_1 + \alpha_2 x_2 - \tau(h, k)\| \\ &\quad\leqslant \|\alpha_1 x_1 + \alpha_2 x_2 - \alpha_1 \tau(h_1, k_1) - \alpha_2 \tau(h_2, k_2)\| \\ &\qquad + \|\alpha_1^{1/2} \alpha_2^{1/2} \tau(h_1, k_2)\| + \|\alpha_1^{1/2} \alpha_2^{1/2} \tau(h_2, k_1)\| \\ &\quad\leqslant |\alpha_1| \, \|x_1 - \tau(h_1, k_1)\| + |\alpha_2| \, \|x_2 - \tau(h_2, k_2)\| \\ &\qquad + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < (|\alpha_1| + |\alpha_2|) \left(\theta + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2} \leqslant \theta + \varepsilon, \end{aligned}$$

and

$$\|\tau(h, \eta_i)\| \leqslant \|\tau(h_1, \eta_i)\| + \|\tau(h_2, \eta_i)\| < \varepsilon, \qquad 1 \leqslant i \leqslant p,$$

$$\|\tau(\xi_i, k)\| \leqslant \|\tau(\xi_i, k_1)\| + \|\tau(\xi_i, k_2)\| < \varepsilon, \qquad 1 \leqslant i \leqslant p.$$

We conclude that $x \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$, as desired. The proposition is proved.

An easy consequence of Proposition 5.11 and the bipolar theorem is the following.

5.12. COROLLARY. *Let $\tau\colon \mathscr{D}(\tau)\ (\subset \mathscr{H} \times \mathscr{K}) \to \mathscr{X}$ be a closed bilinear map satisfying Assumption* 5.1. *Assume, in addition, that $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces. Then τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ if and only if*

$$\sup\{|f(x)|: x \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)\} \geqslant \gamma\ \|f\|$$

for every $f \in \mathscr{X}^*$.

6. THE OPEN MAPPING THEOREM

Let $\mathscr{H}$ and $\mathscr{K}$ be Hilbert spaces, let $\mathscr{X}$ be a normed space, and let

$$\tau\colon \mathscr{D}(\tau)(\subset \mathscr{H} \times \mathscr{K}) \to \mathscr{X}$$

be a closed bilinear map satisfying Assumptions 5.1 and 5.6. These objects will remain fixed throughout this section.

6.1. LEMMA. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$. Let $x \in \mathscr{X}$, $h, \xi_1, \xi_2, ..., \xi_p \in \mathscr{H}_0$, $k, \eta_1, \eta_2, ..., \eta_p \in \mathscr{K}_0$, and $\varepsilon > 0$ be given. Then there exist $h' \in \mathscr{H}_0$ and $k' \in \mathscr{K}_0$ with the following properties*:

$$\text{(i)}\quad \|h'\| \leqslant \left(\|h\|^2 + \frac{1}{\gamma}\ \|x - \tau(h, k)\| + \varepsilon\right)^{1/2},$$

$$\|k'\| \leqslant \left(\|k\|^2 + \frac{1}{\gamma}\ \|x - \tau(h, k)\| + \varepsilon\right)^{1/2},$$

$$\text{(ii)}\quad \|h' - h\| \leqslant \left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2}, \qquad \|k' - k\| \leqslant \left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2},$$

$$\text{(iii)}\quad \|x - \tau(h', k')\| \leqslant \frac{\theta + \varepsilon}{\gamma}\ \|x - \tau(h, k)\|,$$

$$\text{(iv)}\quad \|\tau(h' - h, \eta_i)\| < \varepsilon, \qquad \|\tau(\xi_i, k' - k)\| < \varepsilon, \qquad 1 \leqslant i \leqslant p,$$

and

$$\text{(v)}\quad |(h' - h \mid \xi_i)| < \varepsilon, \qquad |(k' - k \mid \eta_i)| < \varepsilon, \qquad 1 \leqslant i \leqslant p.$$

Proof. If $x = \tau(h, k)$ we can choose $h' = h$ and $k' = k$. If $x \neq \tau(h, k)$ then the vector $x_1 = (\gamma/\|x - \tau(h, k)\|)(x - \tau(h, k))$ has norm γ and hence

$x_1 \in \mathscr{X}_\theta(\tau, \mathscr{H}_0, \mathscr{K}_0)$. Fix $\delta > 0$ and choose $(h_1, k_1) \in \mathscr{H}_0 \times \mathscr{K}_0$ with the following properties:

$$\|h_1\| \leqslant 1, \qquad \|k_1\| \leqslant 1,$$
$$\|x_1 - \tau(h_1, k_1)\| < \theta + \delta,$$
$$\|\tau(h_1, \eta_1)\| < \delta, \qquad \|\tau(\xi_1, k_1)\| < \delta, \qquad 1 \leqslant i \leqslant p,$$
$$\|\tau(h_1, k)\| < \delta, \qquad \|\tau(h, k_1)\| < \delta,$$
$$|(h_1 \mid \xi_i)| < \delta, \qquad |(k_1 \mid \eta_i)| < \delta, \qquad 1 \leqslant i \leqslant p,$$

and

$$|(h_1 \mid h)| < \delta, \qquad |(k_1 \mid k)| < \delta.$$

We define now

$$h' = h + \left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2} h_1, \qquad k' = k + \left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2} k_1,$$

and we will see that δ can be chosen so small that (i)–(v) are satisfied. Condition (ii) is clearly satisfied. We have

$$\|h'\|^2 \leqslant \|h\|^2 + \frac{\|x - \tau(h, k)\|}{\gamma} \|h_1\|^2 + 2\left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2} |(h \mid h_1)|$$
$$\leqslant \|h\|^2 + \frac{\|x - \tau(h, k)\|}{\gamma} + 2\delta\left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2}.$$

An analogous calculation for k' shows that (i) is satisfied if

$$2\delta\left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2} < \varepsilon. \tag{6.2}$$

Moreover, if (6.2) holds we also have

$$\|\tau(h' - h, \eta_i)\| = \left(\frac{\|x - \tau(h, k)\|}{\gamma}\right)^{1/2} \|\tau(h_1, \eta_i)\| < \frac{\varepsilon}{2},$$

so that (iv) and (v) are also satisfied. Finally,

$$\begin{aligned}
\|x-\tau(h',k')\| &\leqslant \left\| x-\tau(h,k)-\frac{\|x-\tau(h,k)\|}{\gamma}\tau(h_1,k_1)\right\| \\
&\quad+\left(\frac{\|x-\tau(h,k)\|}{\gamma}\right)^{1/2}(\|\tau(h,k_1)\|+\|\tau(h_1,k)\|) \\
&\leqslant \frac{\|x-\tau(h,k)\|}{\gamma}\|x_1-\tau(h_1,k_1)\|+2\delta\left(\frac{\|x-\tau(h,k)\|}{\gamma}\right)^{1/2} \\
&< \frac{\theta+\delta}{\gamma}\|x-\tau(h,k)\|+2\delta\left(\frac{\|x-\tau(h,k)\|}{\gamma}\right)^{1/2},
\end{aligned}$$

and we see that (iii) is satisfied if

$$\begin{aligned}
&\frac{\theta+\delta}{\gamma}\|x-\tau(h,k)\|+2\delta\left(\frac{\|x-\tau(h,k)\|}{\gamma}\right)^{1/2} \\
&\qquad<\frac{\theta+\varepsilon}{\gamma}\|x-\tau(h,k)\|. \qquad (6.3)
\end{aligned}$$

It is clear that δ can be chosen to satisfy both (6.2) and (6.3), and this concludes the proof.

6.4. COROLLARY. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$. Then τ also has property $(\Delta_{\theta_1,\gamma_1})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$, where $\theta_1=\theta^2/(\gamma+\theta)$ and $\gamma_1=\gamma^2/(\gamma+\theta)$.*

Proof. Let $x\in\mathscr{X}$ be such that $\|x\|<\gamma_1$, and let $\xi_1,\ldots,\xi_p\in\mathscr{H}_0$, $\eta_1,\ldots,\eta_p\in\mathscr{K}_0$, $\varepsilon>0$, be given. Choose δ so small that the following inequalities are satisfied:

$$\frac{\theta+\gamma+\delta}{\gamma^2}\|x\|+\delta\leqslant 1$$

and

$$\left(\frac{\theta+\delta}{\gamma}\right)^2\|x\|<\frac{\theta^2}{\theta+\gamma}+\varepsilon=\theta_1+\varepsilon.$$

An application of Lemma 6.1 (with $h=0$, $k=0$, and ε replaced by δ) yields vectors $h'\in\mathscr{H}_0$ and $k'\in\mathscr{K}_0$ such that

$$\|h'\| \leqslant \left(\frac{\|x\|}{\gamma}\right)^{1/2}, \qquad \|k'\| \leqslant \left(\frac{\|x\|}{\gamma}\right)^{1/2},$$

$$\|x - \tau(h', k')\| \leqslant \frac{\theta + \delta}{\gamma} \|x\|,$$

$$\|\tau(h', \eta_i)\| < \varepsilon/2, \qquad \|\tau(\xi_i, k')\| < \varepsilon/2, \qquad 1 \leqslant i \leqslant p.$$

A second application of Lemma 6.1, with h and k replaced by h' and k', respectively, yields $h'' \in \mathscr{H}_0$, $k'' \in \mathscr{K}_0$ such that

$$\|h''\| \leqslant \left(\|h'\|^2 + \frac{1}{\gamma} \|x - \tau(h', k')\| + \delta \right)^{1/2},$$

$$\|k''\| \leqslant \left(\|k'\|^2 + \frac{1}{\gamma} \|x - \tau(h', k')\| + \delta \right)^{1/2},$$

$$\|x - \tau(h'', k'')\| \leqslant \frac{\theta + \delta}{\gamma} \|x - \tau(h', k')\|,$$

and

$$\|\tau(h'' - h', \eta_i)\| < \varepsilon/2, \qquad \|\tau(\xi_i, k'' - k')\| < \varepsilon/2, \qquad 1 \leqslant i \leqslant p.$$

Note that

$$\|h''\| \leqslant \frac{\|x\|}{\gamma} + \frac{1}{\gamma}\left(\frac{\theta + \delta}{\gamma}\right) \|x\| + \delta = \frac{\theta + \gamma + \delta}{\gamma^2} \|x\| + \delta \leqslant 1,$$

$$\|x - \tau(h'', k'')\| \leqslant \left(\frac{\theta + \delta}{\gamma}\right)^2 \|x\| < \theta_1 + \varepsilon,$$

and

$$\|\tau(h'', \eta_1)\| \leqslant \|\tau(h', \eta_i)\| + \|\tau(h'' - h', \eta_i) < \varepsilon,$$

$$\|\tau(\xi_i, k'')\| \leqslant \|\tau(\xi_i, k')\| + \|\tau(\xi_i, k'' - k')\| < \varepsilon,$$

$$1 \leqslant i \leqslant p.$$

This clearly implies that $x \in \mathscr{X}_{\theta_1}(\tau, \mathscr{H}_0, \mathscr{K}_0)$, and we conclude from Lemma 5.3 that τ has property $(\Delta_{\theta_1, \gamma_1})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$. The corollary is proved.

6.5. Proposition. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$. Then τ also has property $(\Delta_{0,\gamma-\theta})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$.*

Proof. It follows from Corollary 3.4 that τ has property $(\Delta_{\theta_n,\gamma_n})$, *where* $\theta_0=\theta$, $\gamma_0=\gamma$, *and* $\theta_{n+1}=\theta_n^2/(\gamma_n+\theta_n)$, $\gamma_{n+1}=\gamma_n^2/(\gamma_n+\theta_n)$. Note that $\theta_{n+1}/\gamma_{n+1}=(\theta_n/\gamma_n)^2$, and by induction $\theta_n/\gamma_n=(\theta/\gamma)^{2^n}$. Thus

$$\begin{aligned}\theta_{n+1}&=\theta_n\cdot\frac{\theta_n/\gamma_n}{1+\theta_n/\gamma_n}=\cdots=\theta\left(\prod_{i=1}^{n}\frac{\theta_i/\gamma_i}{1+\theta_i/\gamma_i}\right)=\theta\left(\prod_{i=1}^{n}\frac{(\theta/\gamma)^{2^i}}{1+(\theta/\gamma)^{2^i}}\right)\\&=\theta\left(\frac{\theta}{\gamma}\right)^{2^{n+1}-1}\left(\frac{1-\theta/\gamma}{1-(\theta/\gamma)^{2^{n+1}}}\right)\end{aligned}$$

and, analogously,

$$\gamma_{n+1}=\frac{\gamma_n}{1+\theta_n/\gamma_n}=\gamma\left(\frac{1-\theta/\gamma}{1-(\theta/\gamma)^{2^{n+1}}}\right).$$

Clearly we have $\lim_{n\to\infty}\theta_n=0$, $\lim_{n\to\infty}\gamma_n=\gamma-\theta$, and $\gamma_n\geqslant\gamma-\theta$. Suppose now that $x\in\mathscr{X}$ with $\|x\|\leqslant\gamma-\theta$, $\varepsilon>0$, and $\xi_1,\xi_2,\dots,\xi_p\in\mathscr{H}_0$, $\eta_1,\eta_2,\dots,\eta_p\in\mathscr{K}_0$. Choose n so large that $\theta_n<\varepsilon/2$, and choose, by virtue of property $(\Delta_{\theta_n,\gamma_n})$, vectors $h\in\mathscr{H}_0$, $k\in\mathscr{K}_0$, with the properties

$$\|h\|\leqslant 1,\qquad \|k\|\leqslant 1,$$
$$\|x-\tau(h,k)\|<\theta_n+\varepsilon/2,$$
$$\|\tau(h,\eta_i)\|<\varepsilon,\qquad \|\tau(\xi_i,k)\|<\varepsilon,\qquad 1\leqslant i\leqslant p.$$

Since $\theta_n+\varepsilon/2<\varepsilon$ and $\xi_1,\dots,\xi_p$, $\eta_1,\dots,\eta_p$ were arbitrary, we deduce that τ has property $(\Delta_{0,\gamma-\theta})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$. The proposition is proved.

6.6. THEOREM. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$. Given $x\in\mathscr{X}$, $h\in\mathscr{H}_0$, $k\in\mathscr{K}_0$, and $\varepsilon>0$, we can find $(h',k')\in\mathscr{D}(\tau)$ such that*

(i) $x=\tau(h',k')$,

(ii) $$\|h'\|\leqslant\left(\|h\|^2+\frac{1}{\gamma-\theta}\|x-\tau(h,k)\|+\varepsilon\right)^{1/2},$$

$$\|k'\|\leqslant\left(\|k\|^2+\frac{1}{\gamma-\theta}\|x-\tau(h,k)\|+\varepsilon\right)^{1/2},$$

and

(iii) $$\|h'-h\|\leqslant\left(\frac{1}{\gamma-\theta}\|x-\tau(h,k)\|\right)^{1/2}+\varepsilon,$$

$$\|k'-k\|\leqslant\left(\frac{1}{\gamma-\theta}\|-\tau(h,k)\|\right)^{1/2}+\varepsilon.$$

Proof. Choose $\delta > 0$ such that

$$\left(\|h\|^2 + \frac{1}{\gamma + \theta}\|x - \tau(h, k)\| + \delta\right)^{1/2} + \frac{\delta^{1/2}}{(\gamma - \theta)^{1/2}(1 - \delta^{1/2})}$$
$$< \left(\|h\|^2 + \frac{1}{\gamma - \theta}\|x - \tau(h, k)\| + \varepsilon\right)^{1/2}$$

and

$$\frac{\delta^{1/2}}{(\gamma - \theta)^{1/2}(1 - \delta^{1/2})} < \varepsilon.$$

Keeping in mind the fact that τ has property $(\Delta_{0,\gamma-\theta})$, a first application of Lemma 6.1 (with ε replaced by $\delta\gamma/\|x - \tau(h, k)\|$) provides vectors $h_1 \in \mathscr{H}_0$ and $k_1 \in \mathscr{K}_0$ such that

$$\|h_1\| \leqslant \left(\|h\|^2 + \frac{1}{\gamma - \theta}\|x - \tau(h, k)\| + \delta\right)^{1/2},$$
$$\|k_1\| \leqslant \left(\|k\|^2 + \frac{1}{\gamma - \theta}\|x - \tau(h, k)\| + \delta\right)^{1/2},$$
$$\|h_1 - h\| \leqslant \left(\frac{1}{\gamma - \theta}\|x - \tau(h, k)\|\right)^{1/2},$$
$$\|k_1 - k\| \leqslant \left(\frac{1}{\gamma - \theta}\|x - \tau(h, k)\|\right)^{1/2},$$

and

$$\|x - \tau(h_1, k_1)\| < \delta.$$

Successive applications of the same lemma will yield sequences $\{h_n : n \geqslant 2\} \subset \mathscr{H}_0$ and $\{k_n : n \geqslant 2\} \subset \mathscr{K}_0$ with the properties

$$\|x - \tau(h_n, k_n)\| < \delta^n, \qquad n \geqslant 2$$
$$\|h_{n+1} - h_n\| \leqslant \left(\frac{\delta^n}{\gamma - \theta}\right)^{1/2}, \qquad \|k_{n+1} - k_n\| \leqslant \left(\frac{\delta^n}{\gamma - \theta}\right)^{1/2}, \qquad n \geqslant 1.$$

It is clear that $h' = \lim_{n \to \infty} h_n$ and $k' = \lim_{n \to \infty} k_n$ exist and, since $\lim_{n \to \infty} \tau(h_n, k_n) = x$ and τ is closed, it follows that $(h', k') \in \mathscr{D}(\tau)$ and $\tau(h', k') = x$. It is easy to see that h' and k' satisfy the other conditions in the theorem. Indeed,

$$\|h'\| \leqslant \|h_1\| + \sum_{n=1}^{\infty} \|h_{n+1} - h_n\| \leqslant \|h_1\| + \sum_{n=1}^{\infty} \left(\frac{\delta^n}{\gamma - \theta}\right)^{1/2}$$
$$= \|h_1\| + \frac{\delta^{1/2}}{(\gamma-\theta)^{1/2}(1-\delta^{1/2})}$$

and

$$\|h' - h\| \leqslant \|h_1 - h\| + \sum_{n=1}^{\infty} \|h_{n+1} - h_n\| \leqslant \|h_1 - h\|$$
$$+ \frac{\delta^{1/2}}{(\gamma-\theta)^{1/2}(1-\delta^{1/2})},$$

and the required inequalities follow from the way in which δ was chosen. This concludes our proof.

6.7. *Remark.* Under the conditions of Theorem 6.6, suppose that $\mathscr{X}$ is complete and we are given $\xi_1, ..., \xi_p \in \mathscr{H}_0$ and $\eta_1, ..., \eta_p \in \mathscr{K}_0$. Then h' and k' can be chosen such that $(h', \eta_i) \in \mathscr{D}(\tau)$, $(\xi_i, k) \in \mathscr{D}(\tau)$, and $\|\tau(h' - h, \eta_i)\| < \varepsilon$, $\|\tau(\xi_i, k' - k)\| < \varepsilon$ for $1 \leqslant i \leqslant p$. Indeed, the sequences $\{h_n\}$ and $\{k_n\}$ could be chosen such that (upon denoting $h_0 = h$ and $k_0 = k$) we have

$$\|\tau(h_{n+1} - h_n, \eta_i)\| < \frac{\varepsilon}{2^{n+1}}, \qquad \|\tau(\xi_i, k_{n+1} - k_n)\| < \frac{\varepsilon}{2^{n+1}},$$
$$1 \leqslant i \leqslant p, \quad n \geqslant 0.$$

It follows that the sequences $\{\tau(h_n, \eta_i): n \geqslant 0\}$, $\{\tau(\xi_i, k_n): n \geqslant 0\}$ converge for $1 \leqslant i \leqslant p$ and, since τ is closed, $(h', \eta_i) \in \mathscr{D}(\tau)$, $(\xi_i, k') \in \mathscr{D}(\tau)$. The estimate of $\tau(h' - h, \eta_i)$ and $\tau(\xi_i, k' - k)$ is now immediate. A similar argument shows that we can require the additional conditions $|(h' - h \mid \xi_i)| < \varepsilon$ and $|(k' - k \mid \eta_i)| < \varepsilon$ for $1 \leqslant i \leqslant p$.

A less precise form of Theorem 6.6 is the open mapping theorem referred to in the Introduction and in the title of this section.

6.8. THEOREM. *If τ is closed and has property $(\Delta_{\gamma,\theta})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$, then τ is surjective and open.*

7. SYSTEMS OF EQUATIONS

Let $\mathscr{H}$, $\mathscr{K}$, $\mathscr{X}$, and τ be as in the previous section. In order to treat the solutions of infinite systems of equations of the form

$$\tau(h_i, k_j) = x_{ij}, \qquad 0 \leqslant i, j < \infty,$$

we will consider another bilinear map

$$\tilde{\tau}: \mathscr{D}(\tilde{\tau}) \subset \tilde{\mathscr{H}} \times \tilde{\mathscr{K}} \to \tilde{\mathscr{X}}.$$

The Hilbert spaces $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{K}}$ are orthogonal direct sums of infinitely many copies of $\mathscr{H}$ and $\mathscr{K}$, respectively. Thus

$$\tilde{\mathscr{H}} = \left\{ \{h_j : j \geqslant 0\} : h_j \in \mathscr{H} \text{ and } \sum_{j=0}^{\infty} \|h_j\|^2 < \infty \right\}$$

and

$$\|\tilde{h}\|^2 = \sum_{j=0}^{\infty} \|h_j\|^2, \qquad \tilde{h} = \{h_j : j \geqslant 0\} \in \tilde{\mathscr{H}}.$$

The space $\tilde{\mathscr{X}}$ consists of all arrays $\{x_{ij} : i, j \geqslant 0\} \subset \mathscr{X}$ such that $\|\{x_{ij} : i, j \geqslant 0\}\| = \sum_{i,j=0}^{\infty} \|x_{ij}\| < \infty$. Finally, $\mathscr{D}(\tilde{\tau})$ consists of those pairs $(\{h_i : i \geqslant 0\}, \{k_j : j \geqslant 0\}) \in \tilde{\mathscr{H}} \times \tilde{\mathscr{K}}$ with the property that $(h_i, k_j) \in \mathscr{D}(\tau)$ for all $i, j \geqslant 0$, and $\sum_{i,j=0}^{\infty} \|\tau(h_i, k_j)\| < \infty$. If $(\{h_i : i \geqslant 0\}, \{k_j : j \geqslant 0\}) \in \mathscr{D}(\tilde{\tau})$, we set

$$\tilde{\tau}(\{h_i : i \geqslant 0\}, \{k_j : j \geqslant 0\}) = \{\tau(h_i, k_j) : i, j \geqslant 0\}.$$

It is easy to see that $\tilde{\tau}$ is a closed bilinear map (remember that τ was assumed to be closed in Section 6). However, $\tilde{\tau}$ is not generally continuous, even if τ is continuous. (In fact $\tilde{\tau}$ is continuous if and only if $\tau = 0$!)

Assume now that $\mathscr{H}_0 \subset \mathscr{H}$ and $\mathscr{K}_0 \subset \mathscr{K}$ are as in Assumption 5.1, i.e., $\mathscr{H}_0 \times \mathscr{K}_0 \subset \mathscr{D}(\tau)$. Then $\tilde{\mathscr{H}}_0 \times \tilde{\mathscr{K}}_0 \subset \mathscr{D}(\tilde{\tau})$, where $\tilde{\mathscr{H}}_0$ and $\tilde{\mathscr{K}}_0$ consist of all finitely nonzero sequences with elements in $\mathscr{H}_0$ and $\mathscr{K}_0$, respectively.

7.1. LEMMA. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$. Then $\tilde{\tau}$ has property $(\Delta_{\theta,\gamma})$ relative to $\tilde{\mathscr{H}}_0$ and $\tilde{\mathscr{K}}_0$.*

Proof. The ball of radius γ centered at the origin in $\tilde{\mathscr{X}}$ is the closed absolutely convex hull of all arrays $\{x_{ij} : i, j \geqslant 0\}$ such that $x_{ij} = 0$ for all pairs (i, j) except one (depending on the array), say (i_0, j_0), satisfying $\|x_{i_0 j_0}\| \leqslant \gamma$. It suffices then to show that such arrays belong to $\mathscr{X}_\theta(\tilde{\tau}, \tilde{\mathscr{H}}_0, \tilde{\mathscr{K}}_0)$. Assume therefore that $x = \{x_{ij} : i, j = 0\} \in \tilde{\mathscr{X}}$, $x_{ij} = 0$ for $(i, j) \neq (i_0, j_0)$, and $\|x_{i_0 j_0}\| \leqslant \gamma$. Let $\xi_1, \xi_2, ..., \xi_p \in \tilde{\mathscr{H}}_0$, $\eta_1, \eta_2, ..., \eta_p \in \tilde{\mathscr{K}}_0$, and $\varepsilon > 0$ be given. Write $\xi_n = \{\xi_i^{(n)} : i \geqslant 0\}$, $\eta_n = \{\eta_i^{(n)} : i \geqslant 0\}$, $1 \leqslant n \leqslant p$; then $\xi_i^{(n)} = 0$ and $\eta_i^{(n)} = 0$ for $i > N$ for some large enough N. We can now use property $(\Delta_{\theta,\gamma})$ for τ to find $h \in \mathscr{H}_0$ and $k \in \mathscr{K}_0$ such that

$$\|h\| \leqslant 1, \qquad \|k\| \leqslant 1,$$
$$\|x_{i_0 j_0} - \tau(h, k)\| < \theta + \varepsilon,$$
$$\|\tau(h, \eta_i^{(n)})\| < \varepsilon/N, \qquad \|\tau(\xi_i^{(n)}, k)\| < \varepsilon/N,$$
$$1 \leqslant n \leqslant p, \quad 0 \leqslant i \leqslant N.$$

We now define $\tilde{h} \in \tilde{\mathscr{H}}_0$, $\tilde{k} \in \tilde{\mathscr{K}}_0$ by $\tilde{h} = \{h_i : i \geqslant 0\}$, $\tilde{k} = \{k_i : \geqslant 0\}$, $h_i = \delta_{ii_0} h$, $k_j = \delta_{jj_0} k$. (Here, of course, $\delta_{ij} = 0$ or 1 according to whether $i \neq j$ or $i = j$.) It is not difficult to verify that

$$\|\tilde{h}\| = \|h\| \leqslant 1, \qquad \|\tilde{k}\| = \|k\| \leqslant 1,$$
$$\|x - \tilde{\tau}(\tilde{h}, \tilde{k})\| = \|x_{i_0 j_0} - \tau(h, k)\| < \theta + \varepsilon,$$

and

$$\|\tilde{\tau}(\tilde{h}, \eta_n)\| = \sum_{j=1}^{N} \|\tau(h, \eta_j^{(n)}\| < \varepsilon,$$
$$\|\tilde{\tau}(\xi_n, \tilde{k})\| = \sum_{i=1}^{N} \|\tau(\xi_i^{(n)}, k)\| < \varepsilon, \qquad 1 \leqslant n \leqslant p.$$

We conclude that $x \in \tilde{\mathscr{X}}_\theta(\tilde{\tau}, \tilde{\mathscr{H}}_0, \tilde{\mathscr{K}}_0)$, and the proof is complete.

We see now that all results proved in Section 6 can be applied to $\tilde{\tau}$. We will give only two applications which are very useful in operator theory:

7.2. THEOREM. *Assume that Assumption* 5.1 *is satisfied, and* τ *has property* $(A_{\theta,\gamma})$ *relative to* $\mathscr{H}_0$ *and* $\mathscr{K}_0$ *for some* $\gamma > \theta \geqslant 0$. *Then for every infinite array* $\{x_{ij} : i, j \geqslant 0\} \subset \mathscr{X}$ *there exist sequences* $\{h_i : i \geqslant 0\} \in \mathscr{H}$ *and* $\{k_j : j \geqslant 0\} \subset \mathscr{K}$ *such that* $(h_i, k_j) \in \mathscr{D}(\tau)$ and $\tau(h_i, k_j) = x_{ij}$ *for all* $i, j \geqslant 0$.

Proof. Determine first constants $\alpha_i > 0$ and $\beta_j > 0$ such that

$$\alpha_i \beta_j \|x_{ij}\| \leqslant 2^{-i-j}, \qquad i, j \geqslant 0.$$

We can then define an element $\tilde{x} = \{\alpha_i \beta_j x_{ij} : i, j \geqslant 0\} \in \tilde{\mathscr{X}}$. By virtue of Lemma 7.1 and Theorem 6.6 there exists $(\tilde{h}, \tilde{k}) \in \mathscr{D}(\tilde{\tau})$ such that $\tilde{\tau}(\tilde{h}, \tilde{k}) = \tilde{x}$. If $\tilde{h} = \{\xi_i : i \geqslant 0\}$ and $\tilde{k} = \{\eta_j : j \geqslant 0\}$, the vectors $h_i = \alpha_i^{-1} \xi_i$ and $k_j = \beta_j^{-1} \eta_j$ will satisfy the conditions of the theorem.

The following result about finite systems of equations requires the full strength of Theorem 6.6.

7.3. THEOREM. *Assume that Assumption* 5.1 *is satisfied, and* τ *has property* $(A_{\theta,\gamma})$ *relative to* $\mathscr{H}_0$ *and* $\mathscr{K}_0$ *for some* $\gamma > \theta \geqslant 0$. *Suppose* $n \in \mathbb{N}$,

$\alpha>0$, and let $\{h_0, h_1, ..., h_{n-1}\} \subset \mathscr{H}_0$, $\{k_0, k_1, ..., k_{n-1}\} \subset \mathscr{K}_0$, and $\{x_{ij}: 0 \leqslant i, j < n\} \subset \mathscr{X}$ be such that

$$\|x_{ij} - \tau(h_i, k_j)\| < \alpha, \qquad 0 \leqslant i, j < n.$$

Then there exist $\{h'_0, h'_1, ..., h'_{n-1}\} \in \mathscr{H}$, $\{k'_0, k'_1, ..., k'_{n-1}\} \in \mathscr{K}$ such that $(h'_i, k'_j) \in \mathscr{D}(\tau)$ for $0 \leqslant i, j < n$,

$$\tau(h'_i, k'_j) = x_{ij}, \qquad 0 \leqslant i, j < n,$$

and

$$\|h'_i - h_i\| \leqslant \frac{n\alpha^{1/2}}{(\gamma-\theta)^{1/2}}, \qquad \|k'_j - k_j\| \leqslant \frac{n\alpha^{1/2}}{(\gamma-\theta)^{1/2}}, \qquad 0 \leqslant i, j < n.$$

Proof. Define $\tilde{h} \in \tilde{\mathscr{H}}_0$, $\tilde{k} \in \tilde{\mathscr{K}}_0$, and $\tilde{x} \in \tilde{\mathscr{X}}$ such that $\tilde{h} = \{h_0, h_1, ..., h_{n-1}, 0, 0, ...\}$, $\tilde{k} = \{k_0, k_1, ..., k_{n-1}, 0, 0, ...\}$, and $\tilde{x} = \{u_{ij}: i, j \geqslant 0\}$ with $u_{ij} = x_{ij}$ if $i, j \leqslant n-1$ and $u_{ij} = 0$ if $\max(i, j) \geqslant n$. We have

$$\|\tilde{x} - \tilde{\tau}(\tilde{h}, \tilde{k})\| = \sum_{i,j=0}^{n-1} \|x_{ij} - \tau(h_i, k_j)\| < n^2\alpha.$$

Choose $\alpha' < \alpha$ and $\varepsilon > 0$ such that $\|\tilde{x} - \tilde{\tau}(\tilde{h}, \tilde{k})\| < n^2\alpha'$ and

$$\frac{n\alpha'^{1/2}}{(\gamma-\theta)^{1/2}} + \varepsilon \leqslant \frac{n\alpha^{1/2}}{(\gamma-\theta)^{1/2}}.$$

Theorem 6.9 implies the existence of $(\tilde{h}', \tilde{k}') \in \mathscr{D}(\tilde{\tau})$ such that $\tilde{\tau}(\tilde{h}', \tilde{k}') = \tilde{x}$ and

$$\|\tilde{h}' - \tilde{h}\| < \frac{n\alpha'^{1/2}}{(\gamma-\theta)^{1/2}} + \varepsilon, \qquad \|\tilde{k}' - \tilde{k}\| < \frac{n\alpha'^{1/2}}{(\gamma-\theta)^{1/2}} + \varepsilon.$$

If $\tilde{h}' = \{h'_i: i \geqslant 0\}$ and $\tilde{k}' = \{k'_j: j \geqslant 0\}$, it is easy to check that $\{h'_0, h'_1, ..., h'_{n-1}\}$ and $\{k'_0, k'_1, ..., k'_{n-1}\}$ satisfy the conditions of the theorem.

8. Remarks on the Case of Banach Spaces $\mathscr{H}$ and $\mathscr{K}$

Some of the results presented in Sections 6 and 7 remain valid in case $\mathscr{H}$ and $\mathscr{K}$ are Banach spaces. The estimates for the solutions h and k are somewhat different, and the technique for solving infinite systems of equations is much more complicated. We refer to [8] for the detailed proof of the results outlined below.

It will be assumed throughout this section that $\mathscr{H}$, $\mathscr{K}$, and $\mathscr{X}$ are Banach spaces, and $\tau\colon \mathscr{D}(\tau) \subset \mathscr{H} \times \mathscr{K} \to \mathscr{X}$ is a closed bilinear map satisfying Assumption 5.1.

8.1. Proposition. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$. Then τ also has property $(\Delta_{0,(\gamma^{1/2}-\theta^{1/2})^2})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$.*

8.2. Theorem. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$. Given $x \in \mathscr{X}$, $h \in \mathscr{H}_0$, $k \in \mathscr{K}_0$, and $\varepsilon > 0$, we can find $(h', k') \in \mathscr{D}(\tau)$ such that*

(i) $x = \tau(h', k')$,

(ii)
$$\|h' - h\| \leqslant \left(\frac{1}{\gamma^{1/2} - \theta^{1/2}}\right) \|x - \tau(h, k)\|^{1/2} + \varepsilon,$$
$$\|k' - k\| \leqslant \left(\frac{1}{\gamma^{1/2} - \theta^{1/2}}\right) \|x - \tau(h, k)\|^{1/2} + \varepsilon.$$

8.3. Theorem. *Assume that τ has property $(\Delta_{\theta,\gamma})$ relative to $\mathscr{H}_0$ and $\mathscr{K}_0$ for some $\gamma > \theta \geqslant 0$. Let $\{x_{ij}: i, j \geqslant 0\} \subset \mathscr{X}$ be an array such that $\sum_{j=0}^{\infty} \|x_{ij}\|^{1/2} < \infty$, $0 \leqslant i < \infty$, $\sum_{i=0}^{\infty} \|x_{ij}\|^{1/2} < \infty$, $0 \leqslant j < \infty$, and let $\{\varepsilon_i : i \geqslant 0\}$ be a sequence of positive numbers. Then there exist sequences $\{h_i : i \geqslant 0\} \in \mathscr{H}$, $\{k_j : j \geqslant 0\} \in \mathscr{K}$ such that*

(i) $(h_i, k_j) \in \mathscr{D}(\tau)$ *and* $\tau(h_i, k_j) = x_{ij}$ *for* $i, j \geqslant 0$,

(ii)
$$\|h_i\| \leqslant \varepsilon_i + \left(\frac{1}{\gamma^{1/2} - \theta^{1/2}}\right) \sum_{j=0}^{\infty} \|x_{ij}\|^{1/2}, \qquad 0 \leqslant i < \infty,$$
$$\|k_j\| \leqslant \varepsilon_j + \left(\frac{1}{\gamma^{1/2} - \theta^{1/2}}\right) \sum_{i=0}^{\infty} \|x_{ij}\|^{1/2}, \qquad 0 \leqslant j < \infty.$$

If $\mathscr{H}$ and $\mathscr{K}$ are Hilbert spaces then (ii) *can be replaced by*

(iii)
$$\|h_i\|^2 \leqslant \varepsilon_i + \left(\frac{1}{\gamma - \theta}\right) \sum_{j=0}^{\infty} \|x_{ij}\|, \qquad 0 \leqslant i < \infty,$$
$$\|k_j\|^2 \leqslant \varepsilon_j + \left(\frac{1}{\gamma - \theta}\right) \sum_{i=0}^{\infty} \|x_{ij}\|, \qquad 0 \leqslant j < \infty.$$

PART III: APPLICATION TO OPERATOR THEORY

9. A Basic Illustration from Operator Theory

Let $\mathscr{H}$ be a separable complex Hilbert space, and suppose T is a C_{00} contraction on $\mathscr{H}$, that is, $\|T\| \leqslant 1$ and $\lim_{n \to \infty} \|T^n h\| = \lim_{n \to \infty} \|T^{*n} h\| = 0$ for all $h \in \mathscr{H}$. It is well known (cf. [19, Chap. VI]) that T is unitarily equivalent to a functional model $S(\Theta)$, where $\Theta \in H^\infty(\mathscr{L}(\mathscr{E}))$, $\|\Theta\|_\infty \leqslant 1$, $\Theta(e^{it})$ is unitary for almost every $t \in [0, 2\pi)$, and $\mathscr{E}$ is a suitable Hilbert space. We recall the relevant definitions.

If $\mathscr{E}$ is a separable Hilbert space, and $1 \leqslant p < \infty$, we denote by $L^p(\mathscr{E})$ the space of measurable $\mathscr{E}$-valued functions f on $\mathbb{T} = \{e^{it} : t \in [0, 2\pi)\}$ such that

$$\|f\|_p = \left(\frac{1}{2\pi} \int_0^{2\pi} \|f(e^{it})\|^p \, dt\right)^{1/p} < \infty.$$

If $\mathscr{E} = \mathbb{C}$ we will write simply L^p for $L^p(\mathscr{E})$. The space $H^p(\mathscr{E})$ [resp., $H_0^p(\mathscr{E})$] consists of those functions $f \in L^p(\mathscr{E})$ such that $\int_0^{2\pi} e^{int} f(e^{it}) \, dt = 0$ for $n > 0$ [resp., $n \geqslant 0$]. We denote by $H^\infty(\mathscr{L}(\mathscr{E}))$ the set of all bounded, analytic, $\mathscr{L}(\mathscr{E})$-valued functions Φ on $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$. As before, we use the notation H^∞ in the numerical case. If $\Phi \in H^\infty(\mathscr{L}(\mathscr{E}))$ then the limit $\Phi(e^{it}) = \lim_{r \to 1} \Phi(re^{it})$ exists in the strong operator topology for almost every $t \in [0, 2\pi)$. We refer to [19] for a detailed account of these facts.

We can now define the operator $S(\Theta)$, where Θ is as above. Let S denote the unilateral shift in $H^2(\mathscr{E})$ (i.e., multiplication by e^{it}), and let $\mathscr{H}(\Theta) = H^2(\mathscr{E}) \ominus \Theta H^2(\mathscr{E})$. Then

$$S(\Theta) = PS \mid \mathscr{H}(\Theta),$$

where P denotes the orthogonal projection of $H^2(\mathscr{E})$ onto $\mathscr{H}(\Theta)$. Without loss of generality, in the sequel we take $\mathscr{H} = \mathscr{H}(\Theta)$ and $T = S(\Theta)$.

We note that for $f, g \in H^2(\mathscr{E})$ we can define a function $f \cdot g \in L^1$ by

$$(f \cdot g)(e^{it}) = (f(e^{it}) \mid g(e^{it})), \qquad t \in [0, 2\pi),$$

where the scalar product is taken in $\mathscr{E}$. It will be convenient to denote by $[\psi]$ the class in L^1/H_0^1 of a function $\psi \in L^1$.

The sesquilinear map $\tau_1 : \mathscr{H} \times \mathscr{H} \to L^1/H_0^1$ defined by $\tau_1(f, g) = [f \cdot g]$, $f, g \in H$, has proved a basic object in the study of T. It was shown that T displays a very rich structure (for instance, it is reflexive, and is a strong dilation of every strict contraction on a separable Hilbert space) whenever τ_1 is surjective (cf. [14, 1, 3, 5, 10, 7]). Indeed, if T is, as above, of class

C_{00}, and τ_1 is surjective, it was shown in [3] that T belongs to the class $\mathbb{A}_{\aleph_0}$, and hence the dilation theory of [5] can be applied to T. It is therefore important to characterize the operators T of class C_{00} for which τ_1 is surjective. It is important to realize that an operator T, for which τ_1 is surjective, necessarily satisfies the relation $\sigma(T) \supset \mathbb{T}$.

We recall now such a characterization which was first conjectured in [3]. For $u \in H^\infty$ define a bounded operator $u(T) \in \mathscr{L}(\mathscr{H})$ by

$$u(T)\,Pf = P(uf), \qquad f \in H^2(\mathscr{E}), \tag{9.1}$$

where P, as before, is the projection of $H^2(\mathscr{E})$ onto $\mathscr{H}$. The association $u \to u(T)$ is the H^∞-functional calculus for T (cf. [19]). Recall that $\|u(T)\| \leqslant \|u\|_\infty$, $u \in H^\infty$. It is easy to show that the H^∞-functional calculus is an isometry (i.e., $\|u(T)\| = \|u\|_\infty$, $u \in H^\infty$) whenever τ_1 is surjective. The conjecture states that the converse is also true, i.e., τ_1 is surjective if the H^∞-functional calculus is an isometry. In this paper we prove weaker forms of this statement, and the conjecture may be viewed as a limiting case of our results. Our difficulty in proving the conjecture lies in the fact that Problem 2.2 has a negative answer for $\mathscr{X} = L^1/H_0^1$ as shown by the following slight modification of van Dulst's example (see (2.3)).

9.2. Example. Let $X = L^1/H_0^1$, $C = \{[x] \in X \colon \|[x]\| \leqslant 1\}$, $B = \{[x] \colon x \in L^1,\ \|x\|_1 \leqslant 1,\ x = 0$ on a subset θ_x of $\mathbb{T}$ of measure $(1/2\pi)\int_{\theta_x} dt > \frac{1}{2}\}$. As in Example 2.3, since $H^\infty \simeq (L^1/H_0^1)^*$ and the norm of $u \in H^\infty$ is

$$\operatorname{ess\,max}\{u(e^{it}) \colon t \in [0, 2\pi)\},$$

we readily infer that B super-dominates C. If B were strongly dense in C then $[1] \in B^-$, and hence we can find sequences $\{x_j\}_{j=1}^\infty \subset B$, $\{y_j\}_{j=1}^\infty \subset L^1$, and $\{\varphi_j\}_{j=1}^\infty \subset H_0^1$ such that

$$1 + \varphi_j = x_j + y_j, \qquad j = 1, 2, \ldots, \tag{9.3}$$

and

$$\|y_j\|_1 \to 0 \qquad \text{for} \quad j \to \infty. \tag{9.4}$$

Let

$$\psi_j(z) = \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log|1 + \varphi_j(e^{it})|\, dt\right), \qquad z \in \mathbb{D},$$

be the outer factor of $1 + \varphi_j \in H^1$. Then

$$
\begin{aligned}
1 &\leqslant \exp\left(\frac{1}{2\pi}\int_0^{2\pi} \log |x_j(e^{it}) + y_j(e^{it})|\, dt\right) \\
&\leqslant \left\{\exp\left[(\text{meas } \mathbb{T}\setminus\theta_{x_j}) \log\left(\frac{1}{\text{meas}(\mathbb{T}\setminus\theta_{x_j})}\int_{\mathbb{T}\setminus\theta_{x_j}} |x_j + y_j|\, dt\right)\right]\right\} \\
&\quad \cdot \exp\left(\frac{1}{2\pi}\int_{\theta_{x_j}} \log |y_j|\, dt\right),
\end{aligned}
$$

which tends to 0 for $j \to \infty$, which is a contradiction.

10. H^p-Functional Calculi and Surjectivity of τ_1

Let $T = S(\Theta) \in \mathscr{L}(\mathscr{H} = \mathscr{H}(\Theta)$, be as in Section 9; thus T is an operator of class C_{00}. Fix a number $r \in (1, +\infty)$ and define the space H^r_T as

$$
\begin{aligned}
H^r_T &= H^r(\mathscr{E}) \cap \mathscr{H} && \text{if } \quad r \geqslant 2, \\
H^r_T &= \text{closure of } \mathscr{H} \text{ in } H^r(\mathscr{E}) && \text{if } \quad r \leqslant 2.
\end{aligned}
$$

Denote, as before, by P the orthogonal projection of $H^2(\mathscr{E})$ onto $\mathscr{H}$. We recall that, for $f \in H^2(\mathscr{E})$,

$$
Pf = f - \Theta P_{H^2(\mathscr{E})} g, \qquad \text{where} \quad g(e^{it}) = \Theta(e^{it})^* f(e^{it}) \text{ almost everywhere.}
$$

A famous theorem of M. Riesz, and the fact that ε is a Hilbert space, implies that there exists a constant c_r such that

$$
\|P_{H^2(\mathscr{E})} g\|_r \leqslant c_r \|g\|_r, \qquad g \in L^2(\mathscr{E}) \cap L^r(\mathscr{E}).
$$

We conclude that $P(H^2(\mathscr{E}) \cap H^r(\mathscr{E})) \subset H^r_T$ and

$$
\|Pf\|_r \leqslant (c_r + 1) \|f\|_r, \qquad f \in H^2(\mathscr{E}) \cap H^r(\mathscr{E}).
$$

Now $H^2(\mathscr{E}) \cap H^r(\mathscr{E})$ is dense in $H^r(\mathscr{E})$, and therefore there exists a continuous linear map $P_r\colon H^r(\mathscr{E}) \to H^r_T$ such that P_r coincides with P on $H^2(\mathscr{E}) \cap H^r(\mathscr{E})$. It is easy to check that P_r is actually a projection of $H^r(\mathscr{E})$ onto H^r_T. We remark that the scalar product $(\cdot|\cdot)$ extends to a sesquilinear form

$$
\langle \cdot|\cdot \rangle\colon H^r_T \times H^s_T \to \mathbb{C}
$$

if $1/r + 1/s \leqslant 1$.

10.1. LEMMA. *Let $r, s \in (1, +\infty)$ be such that $1/r + 1/s = 1$. Then we have*

$$(c_s^2 + c_s)^{-1} \|f\|_r \leqslant \sup\{|\langle f \mid g\rangle| : g \in H_T^s \cap \mathscr{H}, \|g\|_s \leqslant 1\} \leqslant \|f\|_r$$

for $f \in H_T^r$.

Proof. Observe first that the sesquilinear form $\langle \cdot | \cdot \rangle$ extends to $L^r(\mathscr{E}) \times L^s(\mathscr{E})$ by the formula

$$\langle h \mid k\rangle = \frac{1}{2\pi} \int_0^{2\pi} (h(e^{it}) \mid k(e^{it}))\, dt, \qquad h \in L^r(\mathscr{E}), \quad k \in L^s(\mathscr{E}),$$

and

$$\|h\|_r = \sup\{|\langle h \mid k\rangle| : k \in L^s(\mathscr{E}), \|k\|_s \leqslant 1\}$$

for all $h \in L^r(\mathscr{E})$. The inequality

$$\sup\{|\langle f \mid g\rangle| : g \in H_T^s \cap \mathscr{H}, \|g_s\| \leqslant 1\} \leqslant \|f\|_r$$

is obvious, and for the first inequality in the lemma it suffices to consider elements $f \in H_T^r \cap \mathscr{H}$. We have, indeed,

$$\begin{aligned}
\|f\|_r &= \sup\{|\langle f \mid g\rangle| : g \in L^s(\mathscr{E}), \|g\|_s \leqslant 1\} \\
&= \sup\{|\langle f \mid g\rangle| : g \in L^s(\mathscr{E}) \cap L^2(\mathscr{E}), \|g\|_s \leqslant 1\} \\
&= \sup\{|(p_{H^2(\mathscr{E})} f \mid g)| : g \in L^s(\mathscr{E}) \cap L^2(\mathscr{E}), \|g\|_s \leqslant 1\} \\
&= \sup\{|(f \mid P_{H^2(\mathscr{E})} g)| : g \in L^s(\mathscr{E}) \cap L^2(\mathscr{E}), \|g\|_s \leqslant 1\} \\
&\leqslant \sup\{|(f \mid k)| : k \in H^s(\mathscr{E}) \cap H^2(\mathscr{E}), \|k\|_s \leqslant c_s\} \\
&= \sup\{|(Pf \mid k)| : k \in H^s(\mathscr{E}) \cap H^2(\mathscr{E}), \|k\|_s \leqslant c_s\} \\
&= \sup\{|(f \mid Pk)| : k \in H^s(\mathscr{E}) \cap H^2(\mathscr{E}), \|k\|_s \leqslant c_s\} \\
&\leqslant \sup\{|(f \mid h)| : h \in H_T^s \cap \mathscr{H}, \|h\| \leqslant c_s(c_s + 1)\}
\end{aligned}$$

and this completes the proof.

We now define the H^p-functional calculi. Let $p \in (1, \infty]$, $r \in [p/(p-1), \infty]$, and let q be defined by $1/q = 1/r + 1/p$. For $\varphi \in H^p$ we define a bounded linear operator $\Gamma_{p,r}(\varphi) : H_T^r \to H_T^q$ by

$$\Gamma_{p,r}(\varphi) f = P_q(\varphi f), \qquad f \in H_T^r.$$

Clearly $\Gamma_{\infty,2}(\varphi) = \varphi(T)$, as defined in (9.1), for $\varphi \in H^\infty$. A density argument based on (9.1) shows that, in fact,

$$\Gamma_{p,r}(\varphi)\, P_r f = P_q(\varphi f), \qquad f \in H^r(\mathscr{E}).$$

Since $\|\varphi f\|_q \leqslant \|\varphi\|_p \|f\|_r$ for $\varphi \in H^p$, $f \in H^r(\mathscr{E})$, we have

$$\|\Gamma_{pr}(\varphi)\| \leqslant (1+c_q)\,\|\varphi\|_p, \qquad \varphi \in H^p.$$

The spaces H_T^r and the functional calculi $\Gamma_{p,r}$ can be defined for arbitrary absolutely continuous contractions, and for $r \in [1, +\infty]$, without the use of functional models. We refer to [9] for details.

We can now state the main result of this section.

10.2. THEOREM. *Fix $p \in (2, +\infty)$ and $r \in [2, 2p/(p-2)]$. If the functional calculus $\Gamma_{p,r}$ is an isomorphism, i.e.,*

$$\|\Gamma_{p,r}(\varphi)\| \geqslant c\,\|\varphi\|_p, \qquad \varphi \in H^p, \tag{10.3}$$

for some $c > 0$, then τ_1 is surjective.

It can be shown (cf. [9]) that the above sufficient condition is also necessary. The conjecture alluded to in Section 9 is equivalent to the limiting case $p = \infty$ in the statement of the above theorem.

The remaining part of this section will be devoted to the proof of Theorem 10.2, which will be broken into a sequence of lemmas. We will consider analogues of the map τ_1 which, unlike τ_1, take values in Banach spaces with a separable dual. More precisely, note that for $p' > 1$ we have a continuous injection $L^{p'}/H_0^{p'} \to L^1/H_0^1$. In order to simplify notation we will identify $L^{p'}/H_0^{p'}$ with a linear manifold in L^1/H_0^1. We define now

$$\tau_{p'}\colon \mathscr{D}(\tau_{p'}) \quad (\subset \mathscr{H} \times \mathscr{H}) \to L^{p'}/H_0^{p'}$$

by setting $\mathscr{D}(\tau_{p'}) = \{(f, g) \in \mathscr{H} \times \mathscr{H} : \tau_1(f, g) \in L^{p'}/H_0^{p'}\}$ and writing $\tau_{p'}(f, g) = \tau_1(f, g)$ for $(f, g) \in \mathscr{D}(\tau_{p'})$. The fact that $\tau_{p'}$ is a closed sesquilinear map (as defined in Section 5) is immediate. It is also clear that

$$(H_T^r \cap \mathscr{H}) \times (H_T^s \cap \mathscr{H}) \subset \mathscr{D}(\tau_{p'})$$

and

$$\|\tau_{p'}(f, g)\| \leqslant \|f\|_r\,\|g\|_s, \qquad f \in H_T^r \cap \mathscr{H}, \quad g \in H_T^s \cap \mathscr{H}, \tag{10.4}$$

provided that $1/r + 1/s \leqslant 1/p'$.

10.5. LEMMA. *Assume that $r, s, p' \in [1, +\infty)$ and $1/r + 1/s \leqslant 1/p'$. Let $f, f_0, f_1, \ldots, \in H_T^r \cap \mathscr{H}$ and $g, g_0, g_1, \ldots, \subset H_T^s \cap \mathscr{H}$ be such that $\sup\{\|f_j\|_r : j \geqslant 0\} < \infty$, $\sup\{\|g_j\|_s : j \geqslant 0\} < \infty$. Then we have*

$$\lim_{j \to \infty} \|\tau_{p'}(T^j f_j, g)\| = \lim_{j \to \infty} \|\tau_{p'}(T^j f, g_j)\| = 0.$$

Proof. We notice first that it suffices to prove the lemma under the more stringent condition $1/r + 1/s = 1/p'$. In order to prove that

$$\lim_{j \to \infty} \|\tau_{p'}(T^j f, g_j)\| = 0$$

it suffices to show that $\lim_{j \to \infty} \|T^j f\|_r = 0$ for all $f \in H^r_T \cap \mathscr{H}$. Since $T \in C_{00}$ we know that $\lim_{j \to \infty} \|T^j f\|_2 = 0$ and hence, if $r \leqslant 2$, we have

$$\lim_{j \to \infty} \|T^j f\|_r \leqslant \lim_{j \to \infty} \|T^j f\|_2 = 0.$$

It suffices therefore to consider the case in which $r > 2$. Moreover, since

$$\|T^j f\|_r \leqslant (c_r + 1) \|f\|_r,$$

we may restrict ourselves to the case where f belongs to the dense (in H^r_T) linear manifold H^ρ_T with $\rho = 2r - 2 > r$. In that case the Schwarz inequality yields

$$\|T^j f\|_r \leqslant \|T^j f\|_2^{1/r} \|T^j f\|_\rho^{1-1/r} \leqslant ((1 + c_\rho) \|f\|_\rho)^{1-1/r} \|T^j f\|_2^{1/r}$$

and again we conclude that $\lim_{j \to \infty} \|T^j f\|_r = 0$ because $T \in C_{00}$. The proof that

$$\lim_{j \to \infty} \|\tau_{p'}(T^j f_j, g)\| = 0$$

is based on the observation that $T^{*j} g \in H^s_T$, $\tau_{p'}(T^j f, g) = \tau_{p'}(f_j, T^{*j} g)$, and $\lim_{j \to \infty} \|T^{*j} g\|_s = 0$. The fact that $T^{*j} g \in H^s_T$ and $\|T^{*j} g\|_s \leqslant \gamma_s \|g\|_s$, $g \in H^s_T \cap \mathscr{H}$, for some $\gamma_s < \infty$ follows by duality from Lemma 10.1. The proof that $\lim_{j \to \infty} \|T^{*j} g\|_s = 0$ is now similar to the above proof for the sequence $\{T^j f : j \geqslant 0\}$. The lemma is proved.

Let us note for future reference the useful relation

$$\langle \varphi, \tau_{p'}(f, g) \rangle = (\Gamma_{p,r}(\varphi) f \mid g) \tag{10.6}$$

for $f \in H^r_T$, $g \in H^s_T$, $s = pr/(r(p-1) - p)$, $1/r + 1/s = 1/p'$, and $\varphi \in H^\infty$. This relation was implicitly used in the above proof.

Let now p and r be as in the statement of Theorem 10.2, and set

$$p' = \frac{p}{p-1}, \qquad s = \frac{pr}{r(p-1) - p}.$$

We define now the set

$$B = \{\tau_{p'}(f, g) : f \in H^r_T \cap \mathscr{H}, g \in H^s_T \cap \mathscr{H}, \|f\|_2 \leqslant 1,$$
$$\|g\|_2 \leqslant 1, \text{ and } \|\tau_{p'}(f, g)\| \leqslant 1\}.$$

10.7. LEMMA. *If* (10.2) *holds then* B *super-dominates the ball* $C=\{h\in L^{p'}/H_0^{p'}:\|h\|\leqslant\gamma\}$, *where*

$$\gamma=\frac{c}{c_r c_s(c_s+1)^2(c_r+1)}.$$

Proof. We use first Lemma 10.1 and relation (10.6) to show that B dominates $C'=\{h:\|h\|\leqslant\gamma'\}$, where $\gamma'=c/(c_s+c_s^2)$. Indeed, for $\varphi\in H^p=(L^{p'}/H_0^{p'})^*$ and $s'=s/(s-1)=pr/(p+r)$ we have

$$\begin{aligned}
&\sup\{|\langle\varphi,x\rangle|:x\in B\}\\
&\quad=\sup\{|\langle\varphi,\tau_{p'}(f,g)\rangle|:f\in H_T^r\cap\mathscr{H},\\
&\qquad\qquad g\in H_T^s\cap\mathscr{H},\ \|f\|_2\leqslant 1,\ \|g\|_2\leqslant 1,\ \|\tau_{p'}(f,g)\|\leqslant 1\}\\
&\quad\geqslant\sup\{|\langle\Gamma_{p,r}(\varphi)f\mid g\rangle|:f\in H_T^r\cap\mathscr{H},g\in H_T^s\cap\mathscr{H},\\
&\qquad\qquad\|f\|_r\leqslant 1,\ \|g\|_s\leqslant 1\}\\
&\quad\geqslant(c_s^2+c_s)^{-1}\sup\{\|\Gamma_{p,r}(\varphi)f\|_{s'}:f\in H_T^r\cap\mathscr{H},\ \|f\|_r\leqslant 1\}\\
&\quad\geqslant\gamma'\ \|\varphi\|_p=\sup\{|\langle\varphi,x\rangle|:x\in C'\},
\end{aligned}$$

where we used the fact that $r\geqslant 2$ and $s\geqslant 2$.

Fix now an element $\varphi\in H^\infty$ and define $\varphi_n\in H^\infty$ by $\varphi_n(\lambda)=\lambda^{2n}\varphi(\lambda)$, $\lambda\in\mathbb{T}$, for $n=0,1,2,\ldots$. By the previous calculation, we can find elements $h_n\in H_T^r\cap\mathscr{H}$ and $k_n\in H_T^s\cap\mathscr{H}$ such that $\|h_n\|_r\leqslant 1$, $\|k_n\|_s\leqslant 1$, and

$$\langle\varphi_n,\tau_{p'}(h_n,k_n)\rangle\geqslant\gamma'\ \|\varphi_n\|_p-1/n=\gamma'\ \|\varphi\|_p-1/n.$$

Observe that

$$\begin{aligned}
\langle\varphi_n,\tau_{p'}(h_n,k_n)\rangle&=\langle\varphi_n,\tau_1(h_n,k_n)\rangle\\
&=(\varphi_n(T)\,h_n\mid k_n)\\
&=(T^{2n}\varphi(T)\,h_n\mid k_n)\\
&=(\varphi(T)\,T^n\,h_n\mid T^{*n}k_n).
\end{aligned}$$

We have $f_n=(1/c_r(c_r+1))\,T^nh_n\in H_T^r\cap\mathscr{H}$, $g_n=(1/c_s)\,T^{*n}k_n\in H_T^s\cap\mathscr{H}$, $\|f_n\|_r\leqslant 1$, $\|g_n\|_s\leqslant 1$, and

$$\langle\varphi,\tau_{p'}(f_n,g_n)\rangle\geqslant\gamma\ \|\varphi\|_p-1/n.$$

Since H^∞ is norm-dense in $H^p=(L^p/H_0^p)^*$, in order to conclude the proof it suffices to show that $\tau_{p'}(f_n,g_n)\in B_F$ eventually, for every finite set $F\subset B$. This will follow at once if we prove that

$$\lim_{n\to\infty}\operatorname{Dist}(\operatorname{aco}\{\tau_{p'}(f_n,g_n),\tau_{p'}(f,g)\},B)=0$$

for every $f \in H_T^r \cap \mathscr{H}$, $g \in H_T^s \cap \mathscr{H}$, such that $\|f\|_2 \leqslant 1$, $\|g\|_2 \leqslant 1$, and $\|\tau_{p'}(f, g)\| \leqslant 1$. It suffices to prove that

$$\lim_{n \to \infty} \operatorname{dist}(\alpha_n \tau_{p'}(f_n, g_n) + \beta_n \tau_{p'}(f, g), B) = 0$$

for any sequence $\{(\alpha_n, \beta_n): n \geqslant 0\} \subset \mathbb{C}$ such that $|\alpha_n| + |\beta_n| \leqslant 1 - \varepsilon$, for some $\varepsilon \in (0, 1)$. Fix square roots $\alpha_n^{1/2}$ and $\beta_n^{1/2}$ and define $u_n = \alpha_n^{1/2} f_n + \beta_n^{1/2} f$, $v_n = \overline{\alpha_n^{1/2}}\, g_n + \overline{\beta_n^{1/2}}\, g$. We have $u_n \in H_T^r \cap \mathscr{H}$,

$$\begin{aligned}\|u_n\|_2^2 &\leqslant |\alpha_n|\ \|f_n\|_2^2 + |\beta_n|\ \|f\|_2^2 + 2\, |\alpha_n|^{1/2}\, |\beta_n|^{1/2}\, |(f \mid f_n)| \\ &\leqslant (|\alpha_n| + |\beta_n|)(1 + |(f \mid f_n)|) \\ &\leqslant (1 - \varepsilon)(1 + |(f \mid f_n)|),\end{aligned}$$

and

$$|(f \mid f_n)| = \frac{1}{c_r(c_r + 1)}\, |(T^{*n}f \mid h_n)| \leqslant \frac{1}{c_r(c_r + 1)}\, \|T^{*n}f\|.$$

We conclude that $\|u_n\|_2 \leqslant 1$ for n sufficiently large. Analogously, $v_n \in H_T^s \cap \mathscr{H}$ and $\|v_n\|_2 \leqslant 1$ for n sufficiently large. Next,

$$\begin{aligned}\|\tau_{p'}(u_n, v_n)\| &\leqslant |\alpha_n|\ \|\tau_{p'}(f_n, g_n)\| + |\beta_n|\ \|\tau_{p'}(f, g)\| \\ &\quad + |\alpha_n^{1/2}|\ |\beta_n^{1/2}| [\|\tau_{p'}(f_n, g)\| + \|\tau_{p'}(f, g_n)\|] \\ &\leqslant (|\alpha_n| + |\beta_n|)(1 + \tfrac{1}{2}\, \|\tau_{p'}(f_n, g)\| + \tfrac{1}{2}\, \|\tau_{p'}(f, g_n)\|) \\ &\leqslant (1 - \varepsilon)(1 + \tfrac{1}{2}\, \|\tau_{p'}(f_n, g)\| + \tfrac{1}{2}\, \|\tau_{p'}(f, g_n)\|),\end{aligned}$$

and an application of Lemma 10.5 shows that $\|\tau_{p'}(u_n, v_n)\| \leqslant 1$ eventually. We conclude that $\tau_{p'}(u_n, v_n) \in B$ eventually. Finally, we show that $\lim_{n \to \infty} \|\alpha_n \tau_{p'}(f_n, g_n) + \beta_n \tau_{p'}(f, g) - \tau_{p'}(u_n, v_n)\| = 0$. Indeed,

$$\begin{aligned}&\|\alpha_n \tau_{p'}(f_n, g_n) + \beta_n \tau_{p'}(f, g) - \tau_{p'}(u_n, v_n)\| \\ &\qquad \leqslant |\alpha_n^{1/2}|\ |\beta_n^{1/2}|\ (\|\tau_{p'}(f_n, g)\| + \|\tau_{p'}(f, g_n)\|),\end{aligned}$$

and the present lemma follows by Lemma 10.5.

10.8. Lemma. *Under the conditions of Lemma* 10.7, $\tau_{p'}$ *has property* $(\varDelta_{0,\gamma})$ *relative to* H_T^r *and* H_T^s.

Proof. The dual H^p of $L^{p'}/H_0^{p'}$ is separable, and therefore the set $D(0)$ (associated with B in Section 3) contains the ball $C = \{x \in L^{p'}/H_0^{p'}: \|x\| \leqslant \gamma\}$ by Corollary 4.6. In particular, the closure of B contains C.

Let $x \in C$ be such that $\|x\| < \gamma$, and choose $\psi \in L^{p'}$ such that $\|\psi\| < \gamma$ and

$\psi + H_0^{p'} = x$. Next, define for $n = 1, 2, 3, \ldots$, $x_n \in C$ by $x_n = \psi_n + H_0^{p'}$, where $\psi_n(e^{it}) = e^{-2int}\psi(e^{it})$ almost everywhere. Since B is dense in C, we can find $h_1, h_2, \ldots \in H_T^r \cap \mathscr{H}$, $k_1, k_2, \ldots \in H_T^s \cap \mathscr{H}$ such that $\|h_n\|_2 \leqslant 1$, $\|k_n\|_2 \leqslant 1$, $\|\tau_{p'}(h_n, k_n)\| \leqslant 1$, and $\|x_n - \tau_{p'}(h_n, k_n)\| < 1/n$ for $n \geqslant 1$. Set now $f_n = T^n h_n$, $g_n = T^{*n} k_n$, $n \geqslant 1$. Fix $\varphi \in H^\infty$, $\|\varphi\|_p \leqslant 1$, and set $\varphi_n(\lambda) = \lambda^{2n}\varphi(\lambda)$, $|\lambda| < 1$. We clearly have $\|\varphi_n\|_p \leqslant \|\varphi\|_p \leqslant 1$, and hence

$$\begin{aligned} |\langle \varphi, x - \tau_{p'}(f_n, g_n) \rangle| &= |\langle \varphi, x\rangle - (\varphi(T) f_n \mid g_n)| \\ &= |\langle \varphi_n, x_n\rangle - (\varphi_n(T)\, h_n \mid k_n)| \\ &= |\langle \varphi_n, x_n - \tau_{p'}(h_n, k_n)\rangle| \\ &\leqslant \|\varphi_n\|_p \, \|x_n - \tau_{p'}(h_n, k_n)\| \leqslant 1/n. \end{aligned}$$

We deduce that

$$\begin{aligned} \|x - \tau_{p'}(f_n, g_n)\| = \sup\{ |\langle \varphi, x - \tau_{p'}(f_n, g_n)\rangle| : \varphi \in H^\infty, \\ \|\varphi\|_p \leqslant 1\} \leqslant 1/n. \end{aligned}$$

By Lemma 10.5 we see that $x \in \mathscr{X}_0(\tau_{p'}, H_T^r, H_T^s)$, where H_T^r and H_T^s are viewed as linear submanifolds of $\mathscr{H}$. Thus $C \subset \mathscr{X}_0(\tau_{p'}, H_T^r, H_T^s)$ and the lemma follows.

Theorem 10.2 follows at once from Theorem 6.8 and the following lemma.

10.9. Lemma. *Under the conditions of Lemma* 10.7, τ_1 *has property* $(\Delta_{0,1})$ *relative to* $\mathscr{H}_0 = \mathscr{H}$ *and* $\mathscr{K}_0 = \mathscr{H}$).

Proof. For each $\lambda \in \mathbb{C}$ such that $|\lambda| < 1$, there exists an element $c_\lambda \in L^1/H_0^1$ such that $\|c_\lambda\| = 1$ and $\langle u, c_\lambda\rangle = u(\lambda)$ for $u \in H^\infty$. Moreover, the closed absolutely convex hull of the elements c_λ is the unit ball of L^1/H_0^1. Thus, by virtue of Proposition 5.11 in order to prove our lemma, it will suffice to show that $c_\lambda \in \mathscr{X}_0 = \mathscr{X}_0(\tau_1, \mathscr{H}, \mathscr{H})$. It is easy to see that $c_\lambda \in L^{p'}/H_0^{p'}$; indeed, the functional $u \to u(\lambda)$ is continuous on H^p. Fix $\lambda \in \mathbb{C}$, $|\lambda| < 1$. By Lemma 10.8 and Theorem 7.2 we can find sequences $\{h_n : n \geqslant 0\}$ and $\{k_n : n \geqslant 0\}$ such that $\tau_{p'}(h_i, k_j) = \tau_1(h_i, k_j) = \delta_{ij} c_\lambda$, $i, j \geqslant 0$. With the notation $u_m(z) = (z - \lambda)^m$, $z \in \mathbb{C}$, $|\lambda| < 1$, we have

$$((T - \lambda)^m h_i \mid k_j) = \langle u_m, \tau_1(h_i, k_j)\rangle = \delta_{ij} u_m(\lambda) = \delta_{ij}\delta_{m0}.$$

These relations imply that $(T - \lambda)\mathscr{M}$ has infinite codimension in $\mathscr{M}$, where

$\mathscr{M} = \bigvee \{T^i h_j : i, j \geqslant 0\}$. Let $e_0, e_1, \ldots$ be an orthonormal system in $\mathscr{M} \ominus ((T-\lambda)\mathscr{M})^-$, and notice that we have

$$\tau_1(e_i, e_i) = c_\lambda, \qquad i \geqslant 0.$$

Since $T \in C_{00}$, we have

$$\lim_{i \to \infty} \|\tau_1(e_i, h)\| = \lim_{i \to \infty} \|\tau_1(h, e_i)\| = 0$$

for every $h \in \mathscr{H}$, and this clearly implies that $c_\lambda \in \mathscr{X}_0$. The lemma is proved.

10.10. *Remark.* It is an easy consequence of Lemma 10.9 and Theorem 7.2 that the operator T of class C_{00} belongs to $\mathrm{A}_{\aleph_0}$ if (10.3) is satisfied. We recall that $T \in \mathbb{A}_{\aleph_0}$ is and only if arbitrary systems of the form $\tau_1(f_i, g_j) = x_{ij}$, $i, j \geqslant 0$, can be solved or, equivalently, if T is a strong dilation of any strict contraction acting on a separable Hilbert space (see [5, 7]).

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