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Symmetry-Breaking for Systems of Nonlinear Elliptic Equations

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1. INTRODUCTION

Consider the Dirichlet problem

$$\begin{aligned} \Delta u(x)+f(u(x))&=0, \qquad x\in D_R^n \\ u(x)&=0, \qquad x\in \partial D_R^n. \end{aligned} \tag{1.1}$$

Here f is a smooth function and $D_R^n \subseteq \mathbb{R}^n$, $n \geqslant 2$, is the open ball of radius R centered at the origin. If one considers radial solutions, then (1.1) becomes an ordinary differential equation. Recently, Smoller and Wasserman [SW 1–SW 3] considered the bifurcation problem for such radial solutions. Specifically, they investigated the ways in which these symmetric solutions can bifurcate into an asymmetric solution. When this happens, we say that the symmetry breaks.

In this paper, we consider the symmetry-breaking problem for the system of elliptic equations

$$\begin{aligned} \Delta u+f(u,v)&=0, \\ \Delta v+g(u,v)&=0, \qquad x\in D_R^n, \\ u(x)=0&=v(x), \qquad x\in \partial D_R^n. \end{aligned} \tag{1.2}$$

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If we are interested in (positive) radial solutions, then $u=u(r)$ and $v=v(r)$ $(r=|x|)$ satisfy

$$\begin{gathered} u''+\frac{n-1}{r}u'+f(u,v)=0 \\ v''+\frac{n-1}{r}v'+g(u,v)=0 \\ u(r)>0, \qquad v(r)>0, \qquad 0<r=|x|<R, \\ u'(0)=0=v'(0), \qquad u(R)=v(R)=0. \end{gathered} \tag{1.3}$$

For general functions f and g, it is quite difficult to even find solutions of (1.3), let alone to prove that the symmetry breaks. We thus consider the following (perturbed) problem

$$\begin{gathered} \Delta u+\lambda^2 f(u)+\varepsilon F(u,v)=0 \\ \Delta v+g(v)+\varepsilon G(u,v)=0, \qquad x\in D_R^n, \\ u(x)=0=v(x), \qquad x\in\partial D_R^n, \end{gathered} \tag{1.4}$$

where $\varepsilon>0$ is a small parameter, $\lambda>0$ is a free parameter to be specified later, F and G are smooth bounded functions, and f and g belong to a class $\mathscr{L}$ to be defined below. Roughly speaking the class $\mathscr{L}$ is defined by the condition that when $\varepsilon=0$, the de-coupled problems are such that the results of [SW 1, SW 2] apply. Thus the system (1.4) is a perturbation of two problems each of which admits a symmetry-breaking bifurcation.

We remark that the parameter λ is inserted in (1.4) for the following reason; namely if v solves $\Delta v+g(v)=0$ on D_R^n, with $v=0$ on ∂D_R^n, then the problem $\Delta u+f(u)=0$ on D_R^n, with $u=0$ on ∂D_R^n, need not be solvable. On the other hand, by adjusting λ, one can often solve the Dirichlet problem for $\Delta u+\lambda f(u)=0$ on the domain D_R^n. Thus, since our technique in this paper is to "perturb off of the de-coupled system," one sees the advantage of inserting λ into the equations.

This paper is organized as follows. In Section 2, we define precisely the set $\mathscr{L}$ and we study the existence of radial solutions of (1.4). In Section 3 we consider the degeneracy problem for these radial solutions. That is, we prove that under the additional assumption $F_v>0$, $G_u>0$ there are radial solutions the spectrum of whose linearized operator contains zero. Such solutions are the only ones on which bifurcation can occur, and in Section 4 we prove that the symmetry actually breaks on these degenerate solutions. Finally, Section 5 is concerned with some concluding remarks. In particular, we show that if $F_v<0$ and $G_u<0$, then there exist positive asymmetric solutions of (1.4). This is to be compared with a result of Troy

[T], who shows that if $f_v \geqslant 0$ and $g_u \geqslant 0$, then any positive solution of (1.2) must be radial.

We note that systems of the form (1.2) arise in several diverse areas of applied mathematics; in particular such systems govern the stationary solutions of the corresponding nonlinear diffusion systems.

2. Existence of Positive Radial Solutions

Before stating our theorems, we must define the class $\mathscr{L}$. Let h be a smooth function, and consider the boundary-value problem

$$\begin{gathered} w'' + \frac{n-1}{r} w' + h(w) = 0, \qquad 0 < r < R, \\ w'(0) = 0 = w(R), \end{gathered} \tag{2.1}$$

and the initial-value problem

$$\begin{gathered} w'' + \frac{n-1}{r} w' + h(w) = 0, \qquad r > 0, \\ w'(0) = 0, \qquad w(0) = p > 0. \end{gathered} \tag{2.2}$$

We denote the unique solution of (2.2) by $w(r, p)$, $r \geqslant 0$.

It was shown by Smoller and Wasserman [SW 1, SW 2] that if h satisfies the conditions

$$h(0) < 0, \qquad \left(\frac{h(w)}{w}\right)' > 0, \qquad h''(w) \leqslant 0, \tag{2.3}$$

then there exists a $\bar{p} > 0$, and a smooth function $T\colon [\bar{p}, \infty) \to \mathbb{R}$, with $T' < 0$, such that if $p > \bar{p}$, w satisfies

$$\begin{aligned} &w(r, p) > 0, \qquad 0 \leqslant r < T(p), \qquad w(T(p), p) = 0, \\ &w'(r, p) < 0, \qquad 0 < r \leqslant T(p). \end{aligned} \tag{2.4}$$

Moreover $w(r, \bar{p})$ satisfies

$$\begin{gathered} w(r, \bar{p}) > 0, \qquad 0 \leqslant r < T(\bar{p}), \qquad w(T(\bar{p}), p) = 0, \\ w'(T(\bar{p}), \bar{p}) = 0. \end{gathered} \tag{2.5}$$

If we assume that

$$w_p(T(\bar{p}), \bar{p}) < 0, \tag{2.6}$$

then the equation

$$w(R, p) = 0 \tag{2.7}$$

defines a function $p = p(R)$ in a neighborhood of the point $(T(\bar{p}), \bar{p})$. We remark that it was shown in [SW 2] that (2.6) holds for "generic"[‡] h.

Let $\bar{R} = T(\bar{p})$. Then the function $p(R)$ is defined on an interval $[R_1, R_2]$ containing $\bar{R}$.

Remark. If h satisfies (2.3) then $R_1 = T(+\infty)$ (cf. [SW 1]).

From (2.7), we have, for $R_1 \leqslant R \leqslant R_2$,

$$w'(R, p(R)) + w_p(R, p(R))\, p'(R) = 0, \tag{2.8}$$

and

$$\begin{aligned} w''(R, p(R)) + 2w_p'(R, p(R))\, p'(R) + w_p(R, p(R))\, p''(R) \\ + w_{pp}(R, p(R))\, p'(R)^2 = 0. \end{aligned} \tag{2.9}$$

For $R < \bar{R}$, $T'(p) < 0$ implies $p'(R) < 0$. Thus using $w'(R, p(R)) < 0$ in (2.8) gives $w_p(R, p(R)) < 0$.

Now at $R = \bar{R}$, since $w'(\bar{R}, p(\bar{R})) = 0$, (2.6) and (2.8) show that

$$p'(\bar{R}) = 0. \tag{2.10}$$

Thus (2.9) gives

$$w''(\bar{R}, p(\bar{R})) + w_p(\bar{R}, p(\bar{R}))\, p''(\bar{R}) = 0.$$

But as

$$\begin{aligned} w''(\bar{R}, p(\bar{R})) &= -\frac{n-1}{\bar{R}} w'(\bar{R}, p(\bar{R})) - h(u(\bar{R}, p(\bar{R}))) \\ &= -h(0) > 0, \end{aligned}$$

it follows that

$$p''(\bar{R}) = h(0)/w_p(\bar{R}, p(\bar{R})) > 0. \tag{2.11}$$

Next, it was also shown in [SW 1] that for $R_1 \leqslant R < \bar{R}$, $w(r, p(R))$ satisfies

$$\begin{aligned} &w(r, p(R)) > 0 \quad \text{for} \quad 0 \leqslant r < R, \qquad w(R, p(R)) = 0, \\ &w'(r, p(R)) < 0 \quad \text{for} \quad 0 \leqslant r < R. \end{aligned} \tag{2.12}$$

[‡] In the sense that any C^2-function h can be composed with an arbitrarily small translation so that (2.6) holds.

Furthermore, $w(r, p(\bar{R}))$ satisfies

$$\begin{aligned} &w(r, p(\bar{R}))>0 \quad \text{for} \quad 0\leqslant r<\bar{R}, \quad w(\bar{R}, p(\bar{R}))=0, \\ &w'(r, p(\bar{R})<0 \quad \text{for} \quad 0<r<\bar{R}, \quad w'(\bar{R}, p(\bar{R}))=0. \end{aligned} \tag{2.13}$$

Finally, for $R<R\leqslant R_2$, $w(r, p(R))$ satisfies

$$\begin{aligned} &w(r, p(R))>0, \quad 0\leqslant r<\tilde{R}(R), \quad w(\tilde{R}(R), p(R))=0, \\ &w(r, p(R))<0, \quad \tilde{R}(R)<r<R, \quad w(R, p(R))=0, \\ &w'(r, p(R))<0, \quad 0<r<\tilde{\tilde{R}}(R), \quad w'(\tilde{\tilde{R}}(R), p(R))=0, \\ &w'(r, p(R))>0, \quad \tilde{\tilde{R}}(R)<r\leqslant R, \end{aligned} \tag{2.14}$$

where $\tilde{R}(R)$ and $\tilde{\tilde{R}}(R)>\tilde{R}(R)$ and are two smooth functions of R.

We can now define the class $\mathscr{L}$ to which f and g will belong (c.f. (1.4)).

DEFINITION 2.1. The function $h: \mathbb{R}_+ \to \mathbb{R}$ belongs to $\mathscr{L}$ provided that the unique solution $w(r, p)$ of (2.2) satisfies

$$w(\bar{R}, \bar{P})=0, \qquad w'(\bar{R}, \bar{P})=0, \qquad w_p(\bar{R}, \bar{P})<0 \tag{2.15}$$

for some $\bar{R}$ and $\bar{p}$. If $p=p(R)$ is the function defined by (2.15) on an interval $[R_1, R_2]$ containing $\bar{R}$, then we require (2.10)–(2.14) to hold.

We note that $\mathscr{L}$ contains a "generic" set of functions which satisfy (2.3) (see [SW 2]).

We can now consider the system (1.4). Since f and g are assumed to belong to the class $\mathscr{L}$, (2.10)–(2.14) are valid for both f and g. Let the corresponding numbers R_1, R_2, $\bar{R}$, and functions $p(\cdot)$ be denoted by

$$R_1^f, R_2^f, \bar{R}^f, p^f(\cdot) \qquad \text{and} \qquad R_1^g, R_2^g, \bar{R}^g, p^g(\cdot),$$

respectively.

We consider positive radial solutions of (1.4). These are functions $u=u(r)$ and $v=v(r)$ which satisfy

$$\begin{aligned} &u''+\frac{n-1}{r}u'+\lambda^2 f(u)+\varepsilon F(u, v)=0, \\ &v''+\frac{n-1}{r}v'+g(v)+\varepsilon G(u, v)=0, \qquad 0<r<R, \\ &u'(0)=0=u(R), \qquad v'(0)=0=v(R) \\ &u(r)>0, \qquad v(r)>0, \qquad 0\leqslant r<R. \end{aligned} \tag{2.16}$$

In order to study this system, we consider the following initial-value problem

$$u'' + \frac{n-1}{r} u' + \lambda^2 f(u) + \varepsilon F(u, v) = 0,$$

$$v'' + \frac{n-1}{r} v' + g(v) + \varepsilon G(u, v) = 0 \qquad (2.16)_{\mathrm{i}}$$

$$u(0) = p > 0, \qquad v(0) = q > 0, \qquad u'(0) = 0 = v'(0).$$

We denote the unique solution of $(2.16)_{\mathrm{i}}$ by $u(r; p, q; \lambda, \varepsilon)$ and $v(r; p, q; \lambda, \varepsilon)$. We can now state the main theorem in this section.

THEOREM 2.1. *Let f and g belong to the class $\mathscr{L}$. Then there exist constants $\varepsilon_0 > 0$, $\delta > 0$, and a function Λ: $[0, \varepsilon_0] \to \mathbb{R}$ such that the following statements hold*:

A. (*Existence*) *For each ε in $[0, \varepsilon_0]$ and each $\lambda \in [\Lambda(\varepsilon) - \delta, \Lambda(\varepsilon) + \delta]$, there exist two constants $R_1(\lambda, \varepsilon)$ and $\bar{R}(\lambda, \varepsilon)$, and there exist two functions $p = p(R; \lambda, \varepsilon)$ and $q = q(R; \lambda, \varepsilon)$, defined on the interval $[R_1(\lambda, \varepsilon), \bar{R}(\lambda, \varepsilon)]$, such that for every $R \in [R_1(\lambda, \varepsilon), \bar{R}(\lambda, \varepsilon)]$, the solutions $u(r; p(R, \lambda, \varepsilon), q(R, \lambda, \varepsilon); \lambda, \varepsilon)$ and $v(r, p(R, \lambda, \varepsilon), q(R, \lambda, \varepsilon); \lambda, \varepsilon)$ of $(2.16)_{\mathrm{i}}$ solve the boundary-value problem* (2.16).

B. (*Properties of solution*)

(i) *For each $\lambda \in [\Lambda(\varepsilon) - \delta, \Lambda(\varepsilon))$ the functions*

$$\begin{aligned} u(r; R) &\equiv u(r; p(R; \lambda, \varepsilon), q(R; \lambda, \varepsilon); \lambda, \varepsilon) \\ v(r; R) &\equiv v(r; p(R; \lambda, \varepsilon), q(R; \lambda, \varepsilon); \lambda, \varepsilon), \end{aligned} \qquad (2.17)$$

for $R \in [R_1(\lambda, \varepsilon), R(\lambda, \varepsilon)]$ satisfy

$$\begin{aligned} &u'(R; R) < 0 \quad \textit{and} \quad v'(R, R) < 0, \quad \textit{for} \quad R < \bar{R}(\lambda, \varepsilon), \\ &u'(\bar{R}(\lambda, \varepsilon); \bar{R}(\lambda, \varepsilon)) < 0 \quad \textit{and} \quad v'(\bar{R}(\lambda, \varepsilon); \bar{R}(\lambda, \varepsilon)) = 0. \end{aligned} \qquad (2.18)$$

(ii) *For $\lambda = \Lambda(\varepsilon)$, the functions*

$$\begin{aligned} u(r; R) &\equiv u(r; p(R; \Lambda(\varepsilon), \varepsilon), q(R; \Lambda(\varepsilon), \varepsilon); \Lambda(\varepsilon), \varepsilon), \\ v(r; R) &\equiv v(r; p(R; \Lambda(\varepsilon), \varepsilon), q(R; \Lambda(\varepsilon), \varepsilon); \Lambda(\varepsilon), \varepsilon), \end{aligned} \qquad (2.19)$$

for $R \in [R_1(\Lambda(\varepsilon), \varepsilon), \bar{R}(\Lambda(\varepsilon), \varepsilon)]$ satisfy

$$\begin{aligned} &u'(R; R) < 0 \quad \textit{and} \quad v'(R, R) < 0 \quad \textit{for} \quad R < \bar{R}(\Lambda(\varepsilon), \varepsilon), \\ &u'(\bar{R}(\Lambda(\varepsilon), \varepsilon); \bar{R}(\Lambda(\varepsilon), \varepsilon)) = 0 = v'(\bar{R}(\Lambda(\varepsilon), \varepsilon); \bar{R}(\Lambda(\varepsilon), \varepsilon)). \end{aligned}$$

(iii) *For* $\lambda \in (\Lambda(\varepsilon), \Lambda(\varepsilon)+\delta]$, *the functions*

$$u(r; R) \equiv u(r; p(R; \lambda, \varepsilon), q(R; \lambda, \varepsilon); \lambda, \varepsilon),$$
$$v(r; R) \equiv v(r; p(R; \lambda, \varepsilon), q(R; \lambda, \varepsilon); \lambda, \varepsilon), \tag{2.20}$$

for $R \in [R_1(\lambda, \varepsilon), \bar{R}(\lambda, \varepsilon)]$ *satisfy*

$$u'(R; R) < 0 \quad \text{and} \quad v'(R, R) < 0 \quad \text{for} \quad R < \bar{R}(\lambda, \varepsilon)$$
$$u'(R(\lambda, \varepsilon); \bar{R}(\lambda, \varepsilon)) = 0, \quad v'(\bar{R}(\lambda, \varepsilon); \bar{R}(\lambda, \varepsilon)) < 0. \tag{2.21}$$

Some comments on these statements should prove helpful to the reader. Thus statement A means that for small $\varepsilon > 0$, there is a corresponding λ for which one can solve (2.15). Part B implies that for λ on either side of a number $\Lambda(\varepsilon)$, these solutions are nondegenerate, while for $\lambda = \Lambda(\varepsilon)$, the corresponding solution is degenerate; see Theorems 3.1 and 3.2 for the precise statements.

In order to prove the theorem, we must first study how solutions of (2.2) behave under perturbation. Thus let $h \in \mathscr{L}$ and consider the problems

$$w'' + \frac{n-1}{r} w' + h(w) + \varepsilon H(w, r) = 0, \qquad 0 < r < R,$$
$$w'(0) = 0 = w(R), \tag{2.22}$$

and

$$w'' + \frac{n-1}{r} w' + h(w) + \varepsilon H(w, r) = 0, \qquad r > 0$$
$$w'(0) = 0, \qquad w(0) = p > 0, \tag{2.23}$$

where H is smooth and bounded; say $|H(w, r)| \leqslant M$. We denote by $w_\varepsilon(r, p)$, the unique solution of (2.23).

THEOREM 2.2. *There exists an* $\varepsilon_0 > 0$ *depending on* M, *and constants* $R_2 > R_1 > 0$, *depending only on* ε_0 *and* M *with the following properties. For every* $\varepsilon, 0 \leqslant \varepsilon \leqslant \varepsilon_0$, *and every smooth bounded function* $H(w, r)$, $|H(w, r)| \leqslant M$, *there exists an* $\bar{R} \in (R_1, R_2)$ *and a function* $p = p(R)$, *depending on* ε *and* H *such that for every* $R \in [R_1, R_2]$, *the function*

$$w(r) \equiv w_\varepsilon(r; p(R))$$

which is a solution of (2.23), *is also a solution of* (2.22). *Furthermore the function* p *satisfies*

$$p'(\bar{R}) = 0, \qquad p''(\bar{R}) > 0, \tag{2.24}$$

and the function w *satisfies conditions as in* (2.12)–(2.14).

Proof. Since $h \in \mathscr{L}$ (cf. Definition 2.1), there exist constants R_1^h, R_2^h, $\bar{R}^h$ and a function p^h, satisfying conditions (2.12)–(2.14). We denote by $w(r, p)$ and $w_\varepsilon(r, p)$ the unique solutions of (2.2) and (2.23), respectively. If $p = p^h(R_1^h)$, we have, from (2.12), $w'(R_1^h, p^h(R_1^h)) < 0$ and $w(R_1^h, p^h(R_1^h)) = 0$. It follows that there is an $\eta > 0$ such that $w(R_1^h + \eta, p^h(R_1^h)) < 0$ and $w'(R_1^h + \eta, p^h(R_1^h)) < 0$. Since for small $\varepsilon > 0$ solutions of (2.23) are close to solutions of (2.2) on $0 \leqslant r \leqslant R_1^h + 2\eta$, we have $w_\varepsilon(R_1^h + \eta, p^h(R_1^h)) < 0$ and $w_\varepsilon'(R_1^h + \eta, p^h(R_1^h)) < 0$ if ε is small. Thus there exists a continuous function $R_1(\varepsilon)$ such that $R_1(0) = R_1^h$,

$$\begin{aligned} w_\varepsilon(R_1(\varepsilon), p^h(R_1^h)) = 0, \qquad & w_\varepsilon'(R_1(\varepsilon), p^h(R_1^h)) < 0, \\ w_\varepsilon(r, p^h(R_1^h)) > 0 \qquad & \text{if} \quad 0 \leqslant r < R_1(\varepsilon). \end{aligned} \tag{2.25}$$

Similarly, if $p = p^h(R_2)$, if ε is sufficiently small, we can find a continuous function $R_2(\varepsilon)$ such that $R_2(0) = R_2^h$, and continuous functions $\tilde{R}(R)$, $\tilde{\tilde{R}}(R)$ satisfying

$$\begin{aligned} w_\varepsilon(R_2(\varepsilon), p^h(R_2^h)) = 0, \qquad & w_\varepsilon'(R_2(\varepsilon), p^h(R_2^h)) > 0, \\ w_\varepsilon(r, p^h(R_2^h)) > 0 \qquad & \text{if} \quad 0 \leqslant r < \tilde{R}((R_2^h)), \\ w_\varepsilon(\tilde{R}(R_2(\varepsilon)), p^h(R_2^h)) = 0, \qquad & \\ w_\varepsilon(r, p^h(R_2^h)) < 0 \qquad & \text{if} \quad \tilde{R}(R_2(\varepsilon)) < r < R_2(\varepsilon), \\ w_\varepsilon'(r, p^h(R_2^h)) < 0 \qquad & \text{if} \quad 0 < r < \tilde{\tilde{R}}(R_2(\varepsilon)), \\ w_\varepsilon'(r, p^h(R_2^h)) > 0 \qquad & \text{if} \quad \tilde{\tilde{R}}(R_2(\varepsilon)) < r \leqslant R_2(\varepsilon). \end{aligned} \tag{2.26}$$

Now since $h \in \mathscr{L}$, h has a smallest positive root, say u_h. For ε small and $0 \leqslant r \leqslant R_2^h$, $h(u) + \varepsilon H(u, r)$ has a root u_h^ε near u_h. For p near u_h^ε, the solution $u_\varepsilon(r, p)$ does not meet $u = 0$, while for p near $p^h(R_1)$, it meets this line transversely. Thus there exists a continuous function $\bar{R}(\varepsilon)$, $\bar{R}(0) = \bar{R}^h$, and a point $\bar{p}^\varepsilon$ near $\bar{p}^h \equiv p^h(\bar{R}^h)$, such that

$$\begin{aligned} w_\varepsilon(r, \bar{p}_\varepsilon) > 0 \quad & \text{if} \quad 0 \leqslant r < \bar{R}(\varepsilon), \qquad w_\varepsilon(\bar{R}(\varepsilon), \bar{p}_\varepsilon) = 0, \\ w_\varepsilon'(r, \bar{p}_\varepsilon) < 0 \quad & \text{if} \quad 0 < r < \bar{R}(\varepsilon), \qquad w_\varepsilon'(\bar{R}(\varepsilon), \bar{p}_\varepsilon) = 0. \end{aligned} \tag{2.27}$$

If ε is small, then from the assumption $w_p(\bar{R}^h, \bar{p}^h) < 0$, we have $(\partial/\partial p)\, w_\varepsilon(\bar{R}(\varepsilon), \bar{p}_\varepsilon) < 0$. Thus from the equation $w_\varepsilon(R, p) = 0$, we can solve for p as a function of R on an interval $R_1(\varepsilon) \leqslant R \leqslant R_2(\varepsilon)$; call this function $p = p(R)$. Then as in (2.10) and (2.11), we have $p'(\bar{R}(\varepsilon)) = 0$, and

$p''(\bar{R}(\varepsilon))>0$. It is also easy to see that $w_\varepsilon(r, p(R))$ satisfies conditions as in (2.12)–(2.14). Finally, for $\varepsilon_0>0$ sufficiently small, we set

$$R_1 = \sup_{\substack{0\leqslant\varepsilon\leqslant\varepsilon_0\\ |H(w,r)|\leqslant M}} R_1(\varepsilon) \qquad \text{and} \qquad R_2 = \sup_{\substack{0\leqslant\varepsilon\leqslant\varepsilon_0\\ |H(w,r)|\leqslant M}} R_2(\varepsilon)$$

to complete the proof of the theorem. ∎

Proof of Theorem 2.1. Since f and g belong to $\mathscr{L}$, Definition 2.1 implies that there exist constants $\tilde{R}_1^f<\tilde{R}_2^f$ and $\tilde{R}_1^g<\tilde{R}_2^g$, such that the problems

$$\begin{aligned} &u''+\frac{n-1}{r}u'+f(u)=0, \qquad 0<r<R\\ &u'(0)=0=u(R), \qquad \tilde{R}_1^f\leqslant R\leqslant\tilde{R}_2^f, \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} &v''+\frac{n-1}{r}v'+g(v)=0, \qquad 0<r<R\\ &v'(0)=0=v(R), \qquad \tilde{R}_1^g\leqslant R\leqslant\tilde{R}_2^g, \end{aligned} \tag{2.29}$$

admit solutions $u=u(r, p^f(R))$ and $v=v(r, p^g(R))$, respectively. Now set

$$\lambda_1=\tilde{R}_1^f/\tilde{R}_2^g \qquad \text{and} \qquad \lambda_2=\tilde{R}_2^f/\tilde{R}_1^g,$$

$\lambda_1<\lambda_2$. Next set

$$(\bar{u}, \underline{u})=\Big(\sup_{\substack{0\leqslant r\leqslant R\\ \tilde{R}_1^f\leqslant R\leqslant\tilde{R}_1^f}} u(r, p^f(R)), \inf_{\substack{0\leqslant r\leqslant R\\ \tilde{R}_2^f\leqslant R\leqslant\tilde{R}_2^f}} u(r, p^f(R))\Big),$$

$$(\bar{v}, \underline{v})=\Big(\sup_{\substack{0\leqslant r\leqslant R\\ \tilde{R}_1^g\leqslant R\leqslant\tilde{R}_2^g}} v(r, p^g(R)), \inf_{\substack{0\leqslant r\leqslant R\\ \tilde{R}_1^g\leqslant R\leqslant\tilde{R}_2^g}} v(r, p^g(R))\Big),$$

and then choose $M>0$ such that for all $\lambda\in[\lambda_1, \lambda_2]$, $u\in[\underline{u}-1, \bar{u}+1]$, $v\in[\underline{v}-1, \bar{v}+1]$, we have

$$\frac{1}{\lambda^2}|F(u, v)|\leqslant M \qquad \text{and} \qquad |G(u, v)|\leqslant M.$$

Now choose $\varepsilon_0>0$ so small that Theorem 2.2 holds for h replaced by both f and g, and the corresponding solutions u and v lie in the range $u\in[\underline{u}-1, \bar{u}+1]$, $v\in[\underline{v}-1, \bar{v}+1]$, respectively. The corresponding R_1 and R_2 in Theorem 2.2 will be denoted by R_1^f, R_2^f and R_1^g, R_2^g, respectively. It is easy to see that

$$\tilde{R}_1^f\leqslant R_1^f<R_2^f\leqslant\tilde{R}_2^f \qquad \text{and} \qquad \tilde{R}_1^g\leqslant R_1^g<R_2^g\leqslant\tilde{R}_2^g, \tag{2.30}$$

and for $R_1^f/R_2^g < \lambda < R_2^f/R_1^g$,

$$I_\lambda = \left[\frac{R_1^f}{\lambda}, \frac{R_2^f}{\lambda}\right] \cap [R_1^g, R_2^g] \neq \varnothing.$$

Now we fix ε and λ, $|\varepsilon| \leqslant \varepsilon_0$, $R_1^f/R_2^g < \lambda < R_2^f/R_1^g$. We shall prove that for every $R \in I_\lambda$, we can find p and q, depending continuously on λ, ε, and R such that the functions

$$u(r) \equiv u(r; p, q) \qquad \text{and} \qquad v(r) = v(r; p, q),$$

which are solutions of (2.16), satisfy the boundary-value problem

$$\begin{gathered} u'' + \frac{n-1}{r} u' + \lambda^2 f(u) + \varepsilon F(u, v) = 0, \\ v'' + \frac{n-1}{r} v' + g(v) + \varepsilon G(u, v) = 0, \qquad 0 < r < R, \\ u'(0) = 0 = u(R), \qquad v'(0) = 0 = v(R). \end{gathered} \tag{2.31}$$

To prove this assertion, we define two transformations T_1 and T_2 as follows. Let $v = v(r)$ be such that for $0 \leqslant r \leqslant R$, $R \in I_\lambda$, $v(r)$ lies in $[\underline{v} - 1, \bar{v} + 1]$. Then from Theorem 2.2, there is a unique function u satisfying

$$\begin{gathered} u'' + \frac{n-1}{r} u' + \lambda^2 f(u) + \varepsilon F(u, v) = 0, \qquad 0 < r < R, \\ u'(0) = 0 = u(R). \end{gathered} \tag{2.32}$$

We set $T_1(v) = u$. Similarly, let $u = u(r)$ be such that for $0 \leqslant r \leqslant R$, $R \in I_\lambda$, $u(r)$ lies in $[\underline{u} - 1, \bar{u} + 1]$. Then from Theorem 2.2, we can find a unique function v satisfying

$$\begin{gathered} v'' + \frac{n-1}{r} v' + g(v) + \varepsilon G(u, v) = 0, \qquad 0 < r < R, \\ v'(0) = 0 = v(R). \end{gathered} \tag{2.33}$$

We set $T_2(u) = v$. Now choose $u_0 = 0$ and define

$$\begin{gathered} v_1 = T_2(u_0), \ldots, v_{n+1} = T_2(u_n), \ldots \\ u_1 = T_2(v_1), \ldots, u_{n+1} = T_1(v_{n+1}), \ldots. \end{gathered}$$

Using straightforward arguments concerning a priori bounds on solutions of (2.32) and (2.33), we can show that the sequence of functions $\{(u_i, v_i)\}_{i\geqslant 1}$ is well defined and equicontinuous on $0 \leqslant r \leqslant R$. Hence there exists a subsequence, also denoted by $\{(u_i, v_i)\}_{i\geqslant 1}$, and a function (u, v), such that $\{(u_i, v_i)\}_{i\geqslant 1}$ converges uniformly to (u, v) on $0 \leqslant r \leqslant R$. It is easy to see that (u, v) is a weak solution of (2.15), and using the standard elliptic regularity results, we see that (u, v) is actually a (strong) solution of (2.15).

Now for a given ε, $|\varepsilon| \leqslant \varepsilon_0$, if λ is near R_1^f/R_2^g and $R \in I_\lambda$, then (cf. (2.12) $u'(R) < 0$ and (cf. (2.14)) $v'(R) > 0$. On the other hand, if λ is near R_2^f/R_2^g and $R \in I_\lambda$, then $u'(R) > 0$ and $v'(R) < 0$. See Fig. 1.

For $R_1^f/R_2^g < \lambda < R_2^f/R_1^g$, we define the following sets:

$$I_\lambda^{(1)} = \{R \in I_\lambda : u'(R) < 0, v'(R) > 0\},$$

$$I_\lambda^{(2)} = \{R \in I_\lambda : u'(R) > 0, v'(R) > 0\},$$

$$I_\lambda^{(3)} = \{R \in I_\lambda : u'(R) < 0, v'(R) < 0\},$$

$$I_\lambda^{(4)} = \{R \in I_\lambda : u'(R) > 0, v'(R) < 0\}.$$

The $I_\lambda^{(i)}$, $i = 1, 2, 3, 4$, are continuous set-valued functions of λ (in any reasonable topology). For λ near R_1^f/R_2^g, $I_\lambda = I_\lambda^{(1)}$, $I_\lambda^{(i)} = \phi$, $i = 2, 3, 4$. We now increase λ from near R_1^f/R_2^g; there then exists $\lambda_a > R_1^f/R_2^g$ such that $I_\lambda = I_\lambda^{(1)}$ for all $\lambda, \lambda_a > \lambda > R_1^f/R_2^g$, and $I_\lambda \neq I_\lambda^{(1)}$ for $\lambda > \lambda_a$. For λ near λ_a, $\lambda > \lambda_a$, either $I_\lambda^{(2)} = \phi$ and $I_\lambda^{(3)} = \phi$, or $I_\lambda^{(2)} = \phi$ and $I_\lambda^{(3)} \neq \phi$. We continue to increase λ. Then there exists $\lambda_b > \lambda_a$ such that for $\lambda_b > \lambda > \lambda_a$, either $I_\lambda^{(3)} = \phi$ or $I_\lambda^{(2)} = \phi$, and for $\lambda > \lambda_b$, λ near λ_b, $I_\lambda^{(1)} \neq \phi$, $I_\lambda^{(2)} \neq \phi$, and $I_\lambda^{(3)} \neq \phi$. If we continue to increase λ, we find $\Lambda(\varepsilon) > \lambda_b$ such that $I_\lambda^{(1)} \neq \phi$ for $\lambda < \Lambda(\varepsilon)$ and $I_\lambda^{(1)} = \phi$ for $\lambda > \Lambda(\varepsilon)$. Thus $\Lambda(\varepsilon)$ is the unique value of λ such that

$$I_\lambda^{(1)} \neq \phi, \qquad I_\lambda^{(4)} = \phi \qquad \text{for} \quad \lambda < \Lambda(\varepsilon)$$

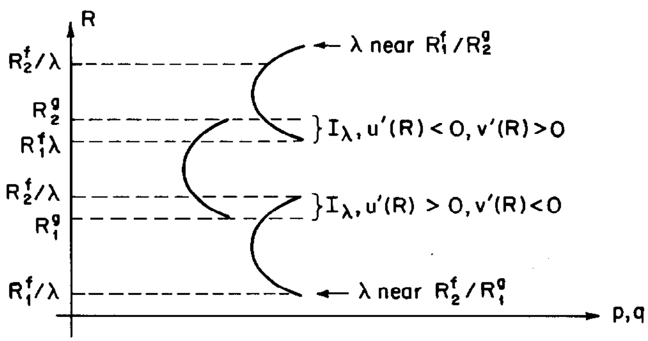

FIGURE 1.

and

$$I_\lambda^{(1)} = \phi, \qquad I_\lambda^{(4)} \neq \phi \qquad \text{for} \quad \lambda > \Lambda(\varepsilon).$$

It is easy to see that there exists $\bar{R}(\varepsilon) \in I_{\Lambda(\varepsilon)}$, such that $I_{\Lambda(\varepsilon)} = I^{(2)}_{\Lambda(\varepsilon)} \cup I^{(3)}_{\Lambda(\varepsilon)} \cup \{\bar{R}(\varepsilon)\}$ and $I^{(3)}_{\Lambda(\varepsilon)} = \{R \in I_{\Lambda(\varepsilon)}: R > \bar{R}(\varepsilon)\}$, $I^{(3)}_{\Lambda(\varepsilon)} = \{R \in I_{\Lambda(\varepsilon)}: R < \bar{R}(\varepsilon)\}$. Since $I_\lambda^{(3)}$ is a continuous set-valued function of λ, we see that we can choose a sufficiently small $\delta > 0$, such that $I^{(3)}_{\Lambda(\varepsilon)} \neq \phi$ for all $\lambda \in [\Lambda(\varepsilon) - \delta, \Lambda(\varepsilon) + \delta]$. Now we set $\bar{R}(\lambda, \varepsilon) = \sup I_\lambda^{(3)}$ and $R_1(\lambda, \varepsilon) = \inf I_\lambda^{(3)}$ for $\lambda \in [\Lambda(\varepsilon) - \delta, \Lambda(\varepsilon) + \delta]$ to complete the proof of Theorem 2.1. ∎

For later use, we need the fact that for the solutions constructed in Theorem 2.1, the following assertions are valid; namely

$$\frac{\partial v}{\partial p} = O(\varepsilon) \qquad \text{and} \qquad \frac{\partial u}{\partial p} = O(\varepsilon). \tag{2.34}$$

To prove these, we shall show that they are valid for each of the above constructed approximants $v_{n+1} = T_2(u_n)$, $u_{n+1} = T_1(v_{n+1})$, $n = 0, 1, 2, \ldots$. Thus, since v_1 satisfies (2.33), where $u = u_1(r, p)$, $v_1(0, p, q) = q$, $v_{1,p}(0, p, q) = 0$, $v'_{1,p}(0, p, q) = 0$, we have $v''_{1,p}(0, p, q) = O(\varepsilon)$ so that $v_{1,p} = O(\varepsilon)$. Similarly, $u_{1,p} = O(\varepsilon)$, and by repeating this argument, we see that (2.34) is indeed valid.

3. Existence of Degenerate Radial Solutions

A solution (u, v) of (1.4) is called *nondegenerate* provided that the only solution of the corresponding linearized problem

$$\begin{aligned} &\Delta U + \lambda^2 f'(u) U + \varepsilon[F_u(u, v) U + F_v(u, v) V] = 0, \\ &\Delta V + g'(v) V + \varepsilon[G_u(u, v) U + G_v(u, v) V] = 0, \qquad x \in D_R^n, \\ &\qquad (U(x), V(x)) = (0, 0), \qquad x \in \partial D_R^n, \end{aligned} \tag{3.1}$$

is $U(x) \equiv 0 \equiv V(x)$. Thus, (u, v) is nondegenerate if and only if zero is not in the spectrum of the associated linearized operator. If this is not the case, then (u, v) is called a *degenerate* solution.

For general functions F and G, the degeneracy problem is quite difficult. Thus we consider in detail only the case

$$\frac{\partial F}{\partial v} > 0 \qquad \text{and} \qquad \frac{\partial G}{\partial u} > 0, \qquad (u, v) \in \eta, \tag{3.2}$$

where η is a small neighborhood of $[\underline{u} - 1, \bar{u} + 1] \times [\underline{v} - 1, \bar{v} + 1]$. We shall then make some remarks regarding the other cases. We shall refer to (3.2) as the "positive interaction" assumption. Here is our first result (cf. Theorem 2.1).

THEOREM 3.1. *Assume that f and g belong to the class $\mathscr{L}$, and that* (3.2) *holds. If* $0 \leqslant \varepsilon \leqslant \varepsilon_0$, $\lambda \in [\Lambda(\varepsilon) - \delta, \Lambda(\varepsilon) + \delta]$, and $\lambda \neq \Lambda(\varepsilon)$, *then the solution*

$$\begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = \begin{pmatrix} u(r; p(R; \lambda, \varepsilon), q(R; \lambda, \varepsilon); \lambda, \varepsilon) \\ v(r; p(R; \lambda, \varepsilon), q(R; \lambda, \varepsilon); \lambda, \varepsilon) \end{pmatrix}$$

of (1.4) *is nondegenerate if ε is sufficiently small.*

Proof. We shall only give the details for $\lambda > \Lambda(\varepsilon)$; the case $\lambda < \Lambda(\varepsilon)$ is entirely analogous. Thus from Theorem 2.1B, part (iii), we have

$$u'(R) \leqslant 0 \qquad \text{and} \qquad v'(R) < 0, \tag{3.3}$$

for each $R \in [R_1(\lambda, \varepsilon), \bar{R}(\lambda, \varepsilon)]$. Our goal is to prove that the only solution of (3.1) is the trivial solution $(U, V) = (0, 0)$. To this end, we write U and V in their spherical harmonic decompositions

$$\begin{aligned} U &= \sum_{N \geqslant 0} a_n(r)\, \Phi_N(\theta) \\ V &= \sum_{N \geqslant 0} a_n(r)\, \Phi_N(\theta), \qquad \theta \in S^{n-1}, 0 \leqslant r \leqslant R, \end{aligned} \tag{3.4}$$

where for each N, Φ_N lies in the Nth eigenspace of the Laplacian on the $(n-1)$-sphere S^{n-1}, corresponding to the eigenvalue $\lambda_N = -N(N+n-2)$. It follows easily that for each $N \geqslant 0$, the functions a_N and b_N satisfy the equations

$$\begin{gathered} a_N'' + \frac{n-1}{r} a_N' + \left[\frac{\lambda_N}{r^2} + \lambda^2 f'(u) + \varepsilon F_u \right] a_N + \varepsilon F_v b_N = 0, \\ b_N'' + \frac{n-1}{r} b_N' + \left[\frac{\lambda_N}{r^2} + g'(v) + \varepsilon G_v \right] b_N + \varepsilon G_u a_N = 0, \\ 0 < r < R, \end{gathered} \tag{3.5}$$

and the boundary conditions

$$\begin{gathered} a_0'(0) = b_0'(0) = 0, \qquad a_0(R) = b_0(R) = 0, \\ a_N(0) = b_N(0) = 0, \qquad a_N(R) = b_N(R) = 0, \qquad \text{for} \quad N \geqslant 1. \end{gathered} \tag{3.5a}$$

We shall show that for each N, $(a_N(r), b_N(r)) \equiv (0, 0)$. We begin with (a_0, b_0). They satisfy the equations

$$
\begin{aligned}
&a_0'' + \frac{n-1}{r} a_0' + [\lambda^2 f'(u) + \varepsilon F_u]\, a_0 + \varepsilon F_v b_0 = 0 \\
&b_0'' + \frac{n-1}{r} b_0' + [g'(v) + \varepsilon G_v]\, b_0 + \varepsilon G_u a_0 = 0, \qquad 0 < r < R, \\
&a_0'(0) = b'(0) = 0 \qquad a_0(R) = b_0(R) = 0.
\end{aligned} \tag{3.6}
$$

For any fixed constants α and β, and $0 \leqslant r \leqslant R$, set

$$
\begin{aligned}
w(r) &= \alpha \frac{\partial u}{\partial p} + \beta \frac{\partial u}{\partial q} \\
z(r) &= \alpha \frac{\partial v}{\partial p} + \beta \frac{\partial v}{\partial q}.
\end{aligned}
$$

An easy calculation shows that w and z satisfy

$$
\begin{aligned}
&w'' + \frac{n-1}{r} w' + [\lambda^2 f'(u) + \varepsilon F_u] w + \varepsilon F_v z = 0 \\
&z'' + \frac{n-1}{r} z' + [g'(v) + \varepsilon G_v] z + \varepsilon G_u w = 0, \qquad 0 < r < R, \\
&w'(0) = z'(0) = 0, \qquad w(0) = \alpha, \qquad z(0) = \beta.
\end{aligned} \tag{3.7}
$$

Comparing (3.6) with (3.7), we conclude that

$$
\begin{aligned}
a_0(r) &= a_0(0) \frac{\partial u}{\partial p} + b_0(0) \frac{\partial u}{\partial q} \\
b_0(r) &= a_0(0) \frac{\partial v}{\partial p} + b_0(0) \frac{\partial v}{\partial q}.
\end{aligned} \tag{3.8}
$$

Using the boundary conditions $a_0(R) = b_0(R) = 0$, we get

$$
\begin{aligned}
a_0(0) \frac{\partial u}{\partial p}(R) + b_0(0) \frac{\partial u}{\partial q}(R) &= 0 \\
a_0(0) \frac{\partial v}{\partial p}(R) + b_0(0) \frac{\partial u}{\partial q}(R) &= 0.
\end{aligned} \tag{3.9}
$$

Now using the assumption that $f, g \in \mathscr{L}$, we know that (see (2.32)), for small $\varepsilon > 0$,

$$\det \begin{bmatrix} \frac{\partial u}{\partial p}(R) & \frac{\partial u}{\partial q}(R) \\ \frac{\partial v}{\partial p}(R) & \frac{\partial v}{\partial q}(R) \end{bmatrix} > 0. \tag{3.10}$$

Thus the only solution of (3.9) is $a_0(0) = b(0) = 0$, and so (3.8) gives $a_0(r) \equiv 0 \equiv b_0(r)$.

Next we shall show that $(a_1(r), b_1(r)) \equiv (0, 0)$. For this, note that (a_1, b_1) satisfies

$$\begin{gathered} a_1'' + \frac{n-1}{r} a_1' + \left[\lambda^2 f'(u) + \varepsilon F_u + \frac{\lambda_1}{r^2} \right] a_1 + \varepsilon F_v b_1 = 0 \\ b_0'' + \frac{n-1}{r} b_1' + \left[g'(v) + \varepsilon G_v + \frac{\lambda_1}{r^2} \right] b_1 + \varepsilon G_u a_1 = 0, \qquad 0 < r < R, \\ a_1(0) = b_1(0) = 0, \qquad a_1(R) = b_1(R) = 0. \end{gathered} \tag{3.11}$$

Let $w(r) = u'(r)$ and $z(r) = v'(r)$. Then w and z satisfy (cf. Theorem 2.1B, part (iii))

$$\begin{gathered} w'' + \frac{n-1}{r} w' + \left[\lambda^2 f'(u) + \varepsilon F_u + \frac{\lambda_1}{r^2} \right] w + \varepsilon F_v z = 0 \\ z'' + \frac{n-1}{r} z' + \left[g'(v) + \varepsilon G_v + \frac{\lambda_1}{r^2} \right] z + \varepsilon G_u w = 0, \qquad 0 < r < R, \\ w(0) = z(0) = 0, \qquad w(r) < 0, \qquad z(r) < 0, \qquad 0 < r < R, \\ w(R) \leqslant 0, \qquad v(R) < 0. \end{gathered} \tag{3.12}$$

Consider now the following equations

$$\begin{gathered} A'' + \frac{n-1}{r} A' + \left[\lambda^2 f'(u) + \varepsilon F_u + \frac{\lambda_1}{r^2} \right] A + \varepsilon F_v B = 0 \\ B'' + \frac{n-1}{r} B' + \left[g'(v) + \varepsilon G_v + \frac{\lambda_1}{r^2} \right] B + \varepsilon G_u A = 0, \qquad r > 0, \\ A(0) = B(0) = 0. \end{gathered} \tag{3.13}$$

Let (A_1, B_1) and (A_2, B_2) denote the solutions of (3.13) which satisfy the initial conditions $(A'(0), B'(0)) = (w'(0), 0)$ and $(A'(0), B'(0)) = (0, z'(0))$, respectively. Now as (A_1, B_1) and (A_2, B_2) form a basis for the set of smooth solutions of (3.11), we can write

$$\begin{pmatrix} w \\ z \end{pmatrix} = \alpha \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} + \beta \begin{pmatrix} A_2 \\ B_2 \end{pmatrix},$$

and differentiating this equation and evaluating at $r=0$ gives $\alpha=\beta=1$. Thus

$$A_1(r) - w(r) = -A_2(r)$$

and (3.14)

$$B_1(r) = z(r) - B_2(r).$$

We assert that

$$\begin{aligned} A_1(r) &= w(r) + O(\varepsilon) \\ B_1(r) &= O(\varepsilon), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} A_2(r) &= O(\varepsilon) \\ B_2(r) &= z(r) + O(\varepsilon). \end{aligned} \tag{3.16}$$

To see these, note first that (3.16) follows from (3.14) and (3.15), so it suffices to prove (3.15). Now as $A_1 - w$ satisfies

$$(A_1 - w)'' + \frac{n-1}{r}(A_1 - w)' + \left[\lambda^2 f' + \varepsilon F_u + \frac{\lambda_1}{r^2}\right](A_1 - W)$$
$$+ \varepsilon F_v (z - B_1) = 0, \qquad 0 < r < R,$$
$$(A_1 - w)(0) = 0 = (A_1 - w)'(0),$$

it follows easily that $(A_1 - w)''(0) = O(\varepsilon)$, and thus $A_1 - w = O(\varepsilon)$. Similarly, as $(z - B_2)$ satisfies

$$(z - B_2)'' + \frac{n-1}{r}(z - B_2)' + \left[g' + \varepsilon G_v + \frac{\lambda_1}{r^2}\right](z - B_2)$$
$$+ \varepsilon G_u (w - A_2) = 0, \qquad 0 < r < R,$$
$$(z - B_2)(0) = 0 = (z - B_2)'(0),$$

we have $B_1 = z - B_2 = O(\varepsilon)$. Thus our assertions hold.

Next, we shall require more detailed information on $B_1(r)$ and $A_2(r)$.

Let $K(r) = B_1(r)/\varepsilon$. Then K satisfies

$$K'' + \frac{n-1}{r} K' + \left[g'(v) + \varepsilon G_v + \frac{\lambda_1}{r^2} \right] K + G_u A_1 = 0, \qquad 0 < r < R,$$

$$K(0) = 0,\ K'(0) = 0. \tag{3.17}$$

Now as $A_1'(0) = w'(0) = u''(0) \approx f(0)/n < 0$ (by (2.3)), then for small ε it follows from the positive interaction assumption (3.2), that for r near $0, r > 0, (G_u A_1)(r) < 0$. Thus as before, we see from (3.17) that for small $r > 0$, $K(r) > 0$. We claim that for $0 < r < R$,

$$K(r) = \frac{B_1(r)}{\varepsilon} > 0. \tag{3.18}$$

To see this let R_0 be the first positive zero of K. Using (3.17) together with the second equation in (3.13), we have

$$\frac{d}{dr} [r^{n-1}(Kz' - K'z)] = r^{n-1}[G_u A_1 z - \varepsilon G_u w K]$$

$$= G_u A_1 z r^{n-1} + O(\varepsilon).$$

If we integrate this from $r = 0$ to R_0, we get

$$-R_0^{n-1} K'(R_0)\, z(R_0) = \int_0^{R_0} G_u A_1 z r^{n-1} dr + O(\varepsilon). \tag{3.19}$$

Now as $K'(R_0) \leqslant 0, z(R_0) \leqslant 0$, and the integral in (3.19) is positive, we obtain a contradiction if ε is sufficiently small. Thus $K(r) > 0$ on $0 < r \leqslant R$. Hence (3.18) is valid. In a like manner, we can show that on $0 < r \leqslant R$

$$J(r) = \frac{A_2(r)}{\varepsilon} > 0. \tag{3.20}$$

We can now complete the proof that $(a_1(r), b_1(r)) \equiv (0, 0)$. Thus comparing (3.11) and (3.13) gives

$$\begin{aligned} a_1(r) &= \frac{a_1'(0)}{w'(0)} A_1(r) + \frac{b_1'(0)}{z'(0)} A_2(r) \\ b_1(r) &= \frac{a_1'(0)}{w'(0)} B_1(r) + \frac{b_1'(0)}{z'(0)} B_2(r), \end{aligned} \tag{3.21}$$

and comparing (3.12) and (3.13) gives

$$w(r) = A_1(r) + A_2(r)$$
$$z(r) = B_1(r) + B_2(r).$$

It follows that

$$A_1(R) = w(R) - A_2(R) = w(R) - \varepsilon J(R)$$
$$B_2(R) = z(R) - B_1(R) = z(R) - \varepsilon K(R),$$

so

$$\begin{aligned} A_1(R)\,B_2(R) - A_2(R)\,B_1(R) &\\ = (w(R) - \varepsilon J(R))(z(R) - \varepsilon K(R)) - \varepsilon^2 J(R)\,K(R) &\\ = w(R)\,z(R) - \varepsilon(J(R)\,z(R) + K(R)\,w(R)). & \qquad (3.22) \end{aligned}$$

Now from (3.12), $w(R) \leqslant 0$ and $z(R) < 0$; hence

$$A_1(R)\,B_2(R) - A_2(R)\,B_1(R) > 0. \qquad (3.23)$$

Thus from (3.21) and (3.23) we conclude that $a_1'(0) = b_1'(0) = 0$, so (3.21) shows that $a_1'(r) \equiv 0 \equiv b_1(r)$.

Finally, we show $(a_N(r), b_N(r)) \equiv (0, 0)$ for $N \geqslant 2$. We have

$$\begin{aligned} & a_N'' + \frac{n-1}{r} a_N' + \left[\lambda^2 f' + \varepsilon F_u + \frac{\lambda_N}{r^2} \right] a_N + \varepsilon F_v b_N = 0 \\ & b_N'' + \frac{n-1}{r} b_N' + \left[g' + \varepsilon G_v + \frac{\lambda_N}{r^2} \right] b_N \\ & \quad + \varepsilon G_u a_N = 0, \qquad 0 < r < R, \\ & a_N(0) = 0 = b_N(0), \qquad a_N(R) = 0 = b_N(R). \end{aligned} \qquad (3.24)$$

Consider the equations

$$\begin{aligned} & S'' + \frac{n-1}{r} S' + \left[\lambda^2 f' + \varepsilon F_u + \frac{\lambda_N}{r^2} \right] S + \varepsilon F_v T = 0 \\ & T'' + \frac{n-1}{r} T' + \left[g' + \varepsilon G_v + \frac{\lambda_N}{r^2} \right] T + \varepsilon G_u S = 0 \\ & S(0) = 0 = T(0). \end{aligned} \qquad (3.25)$$

We denote by (S_1, T_1) and (S_2, T_2) the solutions of (3.24) satisfying $(S'(0), T'(0)) = (w'(0), 0)$ and $(S'(0), T'(0)) = (0, Z'(0))$, respectively. Then from (3.24) and (3.25), we have

$$a_N(r) = \frac{a'_N(0)}{w'(0)} S_1(r) + \frac{b'_N(0)}{z'(0)} S_2(r)$$
$$b_N(r) = \frac{a'_N(0)}{w'(0)} T_1(r) + \frac{b'_N(0)}{z'(0)} T_2(r). \tag{3.26}$$

Using (3.25), it is easy to see (as before) that

$$T_1(r) = O(\varepsilon) \qquad \text{and} \qquad S_2(r) = O(\varepsilon). \tag{3.27}$$

Now let $H = A_1 - S_1$. Then using (3.13) and (3.25), we obtain

$$H'' + \frac{n-1}{r} H' + \left[\lambda^2 f' + \varepsilon F_u + \frac{\lambda_1}{r^2} \right] H$$
$$+ \frac{\lambda_1 - \lambda_N}{r^2} S_1 + \varepsilon F_v (B_1 - T_1) = 0, \tag{3.28}$$
$$H(0) = H'(0) = 0.$$

Now from the first equation in (3.12) and (3.28), we have

$$\frac{d}{dr} [r^{n-1}(Hw' - H'w)] = r^{n-3}(\lambda_1 - \lambda_N) S_1 w + O(\varepsilon). \tag{3.29}$$

Now $S_1(0) = 0$, and since as before $S'_1(0) = w'(0) = u''(0) \approx f(0)/n < 0$, we have $S_1(r) < 0$ for small $r > 0$. Thus from (3.28), $H(r) > 0$ for small $r > 0$. We assert that $H(R) > 0$. If not, let R_0 be the first positive zero of H. Then integrating (3.29) from $r = 0$ to R_0 gives

$$-R_0^{n-1} H'(R_0)\, w(R_0) = \int_0^{R_0} r^{n-3}(\lambda_1 - \lambda_N) S_1 w \, dr + O(\varepsilon). \tag{3.30}$$

But as $H'(R_0) \leqslant 0$, $w(R_0) \leqslant 0$, and the integral in (3.30) is positive, we arrive at a contradiction. Thus $H(R) > 0$, as asserted. Moreover, it is easy to see that $H(R)$ is bounded away from zero, uniformly in ε, for small $\varepsilon > 0$. (This follows from (3.28) if we set $\varepsilon = 0$.)

In a similar manner, we can prove that $\tilde{H}(R) = B_2(R) - T_2(R) > 0$, and that $\tilde{H}(R)$ is bounded away from zero, uniformly in ε, for small $\varepsilon > 0$. Now we compute (cf. (3.22))

$$\begin{aligned}
S_1(R)\,T_2(R) - S_2(R)\,T_1(R) \\
&= (A_1(R) - H(R))(B_1(R) - \tilde{H}(R)) + O(\varepsilon^2) \\
&= (w(R) - \varepsilon J(R) - H(R))(z(R) - \varepsilon K(R) - \tilde{H}(R) + O(\varepsilon^2) \\
&= (w(R) - H(R))(z(R) - \tilde{H}(R)) \\
&\quad - \varepsilon[J(R)(z(R) - \tilde{H}(R)) + K(R)(w(R) - H(R))] \\
&\quad + \varepsilon^2 J(R)\,K(R) + O(\varepsilon^2),
\end{aligned}$$

so that $S_1(R)\,T_2(R) - S_2(R)\,T_1(R) > 0$ for small $\varepsilon > 0$. Now if we set $r = R$ in (3.26) and recall that $a_N(R) = 0 = b_N(R)$, we see that $a'_N(0) = 0 = b'_N(0)$, so from (3.26) it follows that $(a_N(r), b_N(r)) \equiv (0, 0)$, $N \geqslant 2$. This completes the proof of Theorem 3.1. ▮

We also have

THEOREM 3.2. *Assume that f and g belong to the class $\mathscr{L}$, and that* (3.2) *holds. If $\lambda = \Lambda(\varepsilon)$, $R < \bar{R}(\Lambda(\varepsilon), \varepsilon)$, and $\varepsilon > 0$ is sufficiently small, then the solution*

$$\begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = \begin{pmatrix} u(r;\, p(R);\, \Lambda(\varepsilon), \varepsilon),\, q(R;\, \Lambda(\varepsilon), \varepsilon);\, \Lambda(\varepsilon), \varepsilon) \\ v(r,\, p(R);\, \Lambda(\varepsilon), \varepsilon),\, q(R;\, \Lambda(\varepsilon), \varepsilon);\, \Lambda(\varepsilon), \varepsilon) \end{pmatrix}$$

of (1.4) *is nondegenerate.*

We omit the proof since it is essentially the same as the last proof.

Finally, we have the following degeneracy result.

THEOREM 3.3. *Assume that f and g belong to the class $\mathscr{L}$, and that* (3.2) *holds. If $\lambda = \Lambda(\varepsilon)$, $R = \bar{R}(\Lambda(\varepsilon), \varepsilon) \equiv \bar{R}$, and $\varepsilon > 0$ is sufficiently small, then the solution*

$$\begin{pmatrix} u(r) \\ v(r) \end{pmatrix} = \begin{pmatrix} u(r;\, p(\bar{R});\, \Lambda(\varepsilon), \varepsilon),\, q(\bar{R};\, \Lambda(\varepsilon), \varepsilon);\, \Lambda(\varepsilon), \varepsilon) \\ v(r,\, p(\bar{R});\, \Lambda(\varepsilon), \varepsilon),\, q(\bar{R};\, \Lambda(\varepsilon), \varepsilon);\, \Lambda(\varepsilon), \varepsilon) \end{pmatrix}$$

of (1.4) *is degenerate. The kernel consists of elements of the form $(U(x), V(x)) = (u'(r), v'(r))\,\Phi_1(\theta)$.*

Proof. If we expand U and V in terms of spherical harmonics, as in (3.4), then the same proof as in Theorem 3.1 applies to prove that $(a_N(r), b_N(r)) \equiv (0, 0)$ for $N = 0$, and $N \geqslant 2$. For $N = 1$, $(a_1(r), b_1(r)) \equiv (u'(r), v'(r))$ solves (3.5) and (3.5)a (cf. Theorem 2.1B, part ii). Moreover, we claim that the solution $(u', v')\Phi_1$ is unique in the sense that it spans the solution space of (3.1). To see this, we consider (3.21) at $r=\bar{R}$. Using (3.18), (3.20), (3.22), we obtain

$$0=\frac{a_1'(0)}{w'(0)}(-\varepsilon J(\bar{R}))+\frac{b_1'(0)}{z'(0)}\varepsilon J(\bar{R})$$

$$0=\frac{a_1'(0)}{w'(0)}(-\varepsilon K(\bar{R})+\frac{b_1'(0)}{z'(0)}(-\varepsilon K(\bar{R})).$$

But as above we can show that (3.18) and (3.20) hold, at $r=\bar{R}$, so it follows that $a_1'(0)/w'(0)=b_1'(0)/z'(0)$. Thus (3.21) takes the form

$$a_1(r)=\frac{a_1'(0)}{w'(0)}[A_1(r)+A_2(r)]$$

$$b_1(r)=\frac{a_1'(0)}{z'(0)}[B_1(r)+B_2(r)].$$

Thus every solution of (3.11) satisfies these equations. It is clear from this that the solution $(a_1, b_1)\equiv(u', v')=(A_1+A_2, B_1+B_2)$ spans the solution space of (3.5), $N=1$. It follows at once that $(u', v')\Phi_1$ is the unique solution of (3.1). This completes the proof. ∎

4. Symmetry Breaking

In view of Theorems 3.1, 3.2, and 3.3, the positive radial solution

$$u=u(r;\, p(R;\lambda,\varepsilon),\, q(R;\lambda,\varepsilon);\lambda,\varepsilon)$$

$$v=v(r;\, p(R;\lambda,\varepsilon),\, q(R;\lambda,\varepsilon);\lambda,\varepsilon)$$

of the system (1.4) is degenerate only at $\lambda=\Lambda(\varepsilon)$ and $\bar{R}=R(\Lambda(\varepsilon),\varepsilon)$. Since we are alwlays assuming that f and g are in the class $\mathscr{L}$, it follows that (cf. (2.32)) for small $\varepsilon>0$,

$$\det\begin{bmatrix}\frac{\partial u}{\partial p} & \frac{\partial u}{\partial q}\\ \frac{\partial v}{\partial p} & \frac{\partial v}{\partial q}\end{bmatrix}(\bar{R}(\Lambda(\varepsilon),\varepsilon);\, p(\bar{R};\Lambda(\varepsilon),\varepsilon),\, q(\bar{R};\Lambda(\varepsilon),\varepsilon);\Lambda(\varepsilon),\varepsilon)>0. \tag{4.1}$$

Thus the equations

$$\begin{aligned}u(R;p,q;\Lambda(\varepsilon),\varepsilon)&=0\\ v(R;p,q;\Lambda(\varepsilon),\varepsilon)&=0\end{aligned} \tag{4.2}$$

uniquely define two functions $p=p_\varepsilon(R)$ and $q=q_\varepsilon(R)$ on an interval $[R_1(\varepsilon), R_2(\varepsilon)]$ containing the point $\bar{R}(\Lambda(\varepsilon), \varepsilon)$. Thus for every $R\in[R_1(\varepsilon), R_2(\varepsilon)]$, system (1.4) has a unique radial solution.

In order to avoid cumbersome notation, we shall suppress the dependence of our functions on ε; this should cause no confusion. Thus we write $R_1=R_1(\varepsilon)$, $R_2=R_2(\varepsilon)$, $\bar{R}=\bar{R}(\Lambda(\varepsilon),\varepsilon)$, $\Lambda=\Lambda(\varepsilon)$, $p(R)=p_\varepsilon(R)$, $q(R)=q_\varepsilon(R)$, and

$$u(r; p(R), q(R)) = u(r; p(R; \Lambda(\varepsilon), \varepsilon), q(R; \Lambda(\varepsilon), \varepsilon); \Lambda(\varepsilon), \varepsilon)$$

$$v(r; p(R), q(R)) = v(r; p(R; \Lambda(\varepsilon), \varepsilon), q(R; \Lambda(\varepsilon), \varepsilon); \Lambda(\varepsilon), \varepsilon).$$

Let $I=[R_1, R_2]$ and define the operator

$$M\colon \{(u,v)\in C^2(D_1^n)^2\colon (u,v)=(0,0) \text{ on } \partial D_1^n\}\times I\to C^0(D_1^n)^2,$$

by

$$M(z_1,z_2,R)\equiv\begin{pmatrix} M_1(z_1,z_2,R)\\ M_2(z_1,z_2,R)\end{pmatrix}$$

$$=\begin{pmatrix}\Delta(z_1+u)+R^2[\Lambda^2 f(z_1+u)+\varepsilon F(z_1+u, z_2+v)]\\ \Delta(z_2+v)+R^2[g(z_2+v)+\varepsilon G(z_1+u, z_2+v)]\end{pmatrix}, \tag{4.3}$$

where $z_i=z_i(x)$, $u=u(|x|R; p(R), q(R))$, and $v=v(|x|R; p(R), q(R))$. It follows that

$$M(0,0,R)=\begin{pmatrix}0\\0\end{pmatrix}, \qquad \text{for all} \quad R\in I,$$

and that the equation

$$M_z(0,p,R)\begin{pmatrix}U\\V\end{pmatrix}$$

$$\equiv\begin{pmatrix}\Delta U+\Lambda^2 f'(u)U+\varepsilon(F_u U+F_v V)\\ \Delta V+g'(v)V+\varepsilon[G_u U+G_v V]\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix} \tag{4.4}$$

has an n-dimensional solution set spanned by the functions (cf. Theorem 3.3)

$$\frac{x_i}{|x|}\begin{pmatrix}u'\\v'\end{pmatrix}, \qquad i=1,2,\ldots,n.$$

THEOREM 4.1. *Assume that f and g belong to the class $\mathscr{L}$, and that* (3.2) *holds. If* $\lambda = \Lambda(\varepsilon)$, $R = \bar{R}(\Lambda(\varepsilon), \varepsilon)$, *then the symmetry breaks on the degenerate solution*

$$\begin{pmatrix} u(r; p(\bar{R}; \Lambda(\varepsilon), \varepsilon), q(\bar{R}; \Lambda(\varepsilon), \varepsilon); \Lambda(\varepsilon), \varepsilon) \\ v(r; p(\bar{R}; \Lambda(\varepsilon), \varepsilon), q(\bar{R}; \Lambda(\varepsilon), \varepsilon); \Lambda(\varepsilon), \varepsilon) \end{pmatrix} \tag{4.5}$$

of (1.4).

Proof. As was shown in [SW2, SW3], in order to prove that the symmetry actually breaks at $R = \bar{R}$, we can verify the following transversality condition

$$\int_{D_1^n} (U, V)\, M_{zR}(0, 0, \bar{R}) \begin{pmatrix} U \\ V \end{pmatrix} \neq 0, \tag{4.6}$$

for all solutions $(U, V)^t$ of (4.4).

In order to verify (4.6), we compute:

$$M_{zR}(0, 0, \bar{R}) = \begin{bmatrix} \dfrac{\partial^2 M_1(0, 0, \bar{R})}{\partial z_1 \partial R} & \dfrac{\partial^2 M_1(0, 0, \bar{R})}{\partial z_2 \partial R} \\ \dfrac{\partial^2 M_2(0, 0, \bar{R})}{\partial z_1 \partial R} & \dfrac{\partial^2 M_2(0, 0, \bar{R})}{\partial z_2 \partial R} \end{bmatrix}, \tag{4.7}$$

$$\begin{aligned} \frac{\partial^2 M_1(0, 0, \bar{R})}{\partial z_1 \partial R} &= 2\bar{R}[\Lambda^2 f'(u) + \varepsilon F_u] + \bar{R}^2 \frac{d}{dR} [\Lambda^2 f'(u) + \varepsilon F_u], \\ \frac{\partial^2 M_2(0, 0, \bar{R})}{\partial z_2 \partial R} &= 2\bar{R}[\varepsilon F_v] + \bar{R}^2 \frac{d}{dR} [\varepsilon F_v], \\ \frac{\partial^2 M_2(0, 0, \bar{R})}{\partial z_1 \partial R} &= 2\bar{R}[\varepsilon G_u] + \bar{R}^2 \frac{d}{dR} [\varepsilon G_u], \\ \frac{\partial^2 M_2(0, 0, \bar{R})}{\partial z_2 \partial R} &= 2\bar{R}[g'(v) + \varepsilon G_v] + \bar{R}^2 \frac{d}{dR} [g'(v) + \varepsilon G_v]. \end{aligned} \tag{4.8}$$

But as $p'(\bar{R}) = q'(\bar{R}) = 0$, we have

$$\begin{aligned} \frac{d}{dR} [\Lambda^2 f'(u) + \varepsilon F_u] = [\Lambda^2 f''(u) + \varepsilon F_{uu}] \left[\frac{r}{\bar{R}} \frac{\partial}{\partial r} u(r\bar{R}; p(\bar{R}), q(\bar{R})) \right] \\ + \varepsilon F_{uv} \left[\frac{r}{\bar{R}} \frac{\partial}{\partial r} v(r\bar{R}; p(\bar{R}), q(\bar{R})) \right], \end{aligned}$$

$$\frac{d}{dR}[\varepsilon F_v] = \varepsilon F_{uv}\left[\frac{r}{\bar{R}}\frac{\partial}{\partial r}u(r\bar{R};\, p(\bar{R}), q(\bar{R}))\right]$$
$$+\varepsilon F_{vv}\left[\frac{r}{\bar{R}}\frac{\partial}{\partial r}v(r\bar{R};\, p(\bar{R}), q(\bar{R}))\right],$$
$$\frac{d}{dR}[\varepsilon G_u] = \varepsilon G_{uu}\left[\frac{r}{\bar{R}}\frac{\partial}{\partial r}u(r\bar{R};\, p(\bar{R}), q(\bar{R}))\right] \tag{4.9}$$
$$+\varepsilon G_{uv}\left[\frac{r}{\bar{R}}\frac{\partial}{\partial r}v(r\bar{R};\, p(\bar{R}), q(\bar{R}))\right],$$
$$\frac{d}{dR}[g'(v)+\varepsilon G_u] = [g''(v)+\varepsilon G_{vv}]\left[\frac{r}{\bar{R}}\frac{\partial}{\partial r}v(rR;\, p(\bar{R}), q(\bar{R}))\right]$$
$$+\varepsilon G_{uv}\left[\frac{r}{\bar{R}}\frac{\partial}{\partial r}u(r\bar{R};\, p(\bar{R}), q(\bar{R}))\right].$$

Now if $a(r) \equiv u'(r\bar{R};\, p(\bar{R}), q(R))$ and $b(r) \equiv v'(r\bar{R};\, p(\bar{R}), q(R))$, then (4.6) holds if (cf. [SW3])

$$A \equiv \int_0^1 (a, b)\, M_{zR}(0, 0, \bar{R}) \begin{pmatrix} a \\ b \end{pmatrix} r^{n-1}\, dr \neq 0.$$

Now

$$A = \int_0^1 \left[\frac{\partial^2 M_1(0,0,\bar{R})}{\partial z_1 \partial R} a^2 + \frac{\partial^2 M_1(0,0,\bar{R})}{\partial z_2 \partial R} ab\right.$$
$$\left. + \frac{\partial^2 M_2(0,0,\bar{R})}{\partial z_1 \partial R} ab + \frac{\partial^2 M_2(0,0,\bar{R})}{\partial z_2 \partial R} b^2\right] r^{n-1}\, dr$$
$$= \int_0^1 \left\{2\bar{R}[A^2 f'(u) + \varepsilon F_u] + \bar{R}^2[A^2 f''(u) + \varepsilon F_{uu}]\frac{r}{\bar{R}}\frac{\partial u}{\partial r}\right\} a^2 r^{n-1}\, dr$$
$$+ \int_0^1 \left\{2\bar{R}[g'(v) + \varepsilon G_v] + \bar{R}^2[A^2 g''(v) + \varepsilon G_{vv}]\frac{r}{\bar{R}}\frac{\partial v}{\partial r}\right\} b^2 r^{n-1}\, dr + O(\varepsilon) \tag{4.10}$$
$$= \int_0^1 \left\{\frac{d}{dr}[\bar{R}r^2(A^2 f'(u) + \varepsilon F_u)]\, a^2(r)\right.$$
$$\left. + \frac{d}{dr}[\bar{R}r^2(g'(v) + \varepsilon G_v)]\, b^2(r)\right\} r^{n-1}\, dr + O(\varepsilon)$$
$$= -\int_0^1 \bar{R}r^2 \left\{(A^2 f'(u) + \varepsilon F_u)\frac{d}{dr}(a^2(r)\, r^{n-2})\right.$$
$$\left. + (g'(v) + \varepsilon G_v)\frac{d}{dr}\right\}(b^2(r)\, r^{n-2})\, dr + O(\varepsilon).$$

But a and b satisfy

$$\begin{aligned}
&a'' + \frac{n-1}{r} a' + \bar{R}^2 [\Lambda^2 f'(u) + \varepsilon F_u] a - \frac{n-1}{r^2} a + \bar{R}^2 \varepsilon F_v b = 0 \\
&b'' + \frac{n-1}{r} b' + \bar{R}^2 [g'(v) + \varepsilon G_v] b \\
&\qquad - \frac{n-1}{r^2} b + \bar{R}^2 \varepsilon G_u a = 0, \qquad 0 < r < 1, \\
&\qquad a(0) = 0 = b(0),\ a(1) = 0 = b(1),\ a'(1) > 0,\ b'(1) > 0,
\end{aligned} \tag{4.11}$$

so that

$$\begin{aligned}
A &= \int_0^1 \frac{1}{\bar{R}} \left[a'' + \frac{n-1}{r} a' - \frac{n-1}{r^2} a \right] [2a' r^n + (n-2) r^{n-1} a] \, dr \\
&\quad + \int_0^1 \frac{1}{\bar{R}} \left[b'' + \frac{n-1}{r} b' - \frac{n-1}{r^2} b \right] [2b' r^n + (n-2) r^{n-1} b] \, dr + O(\varepsilon) \\
&= \frac{1}{\bar{R}} \int_0^1 \frac{d}{dr} [r^n (a')^2 + (n-2) r^{n-1} a a' - (n-1) r^{n-2} a^2] \, dr \\
&\quad + \frac{1}{\bar{R}} \int_0^1 \frac{d}{dr} [r^n (b')^2 + (n-2) r^{n-1} b b' - (n-1) r^{n-2} b^2] \, dr + O(\varepsilon) \\
&= \frac{1}{\bar{R}} [(a'(1))^2 + (b'(1))^2] + O(\varepsilon) \\
&= \bar{R} [(\Lambda^2 f(0))^2 + (g(0))^2] + O(\varepsilon) \\
&> 0,
\end{aligned}$$

for small $\varepsilon > 0$. This proves the transversality condition, and we thus conclude that the symmetry breaks on the solution

$$(u(r; p(\bar{R}), q(\bar{R})), v(r; p(\bar{R}), q(\bar{R})));$$

thus the proof of the theorem is complete. ∎

5. Remarks

1. For systems of the form (1.2), Troy [T] has shown that if $f_v \geq 0$ and $g_u \geq 0$, then all positive solutions of (1.2) must be radial functions. Thus if $F_v \geq 0$ and $G_u \geq 0$, positive solutions of (1.4) must be radial func-

tions. Our Theorem 4.1 asserts that there are asymmetric solutions bifurcating out of the degenerate solution $(u(r; p(\bar{R}), q(\bar{R})), v(r; p(\bar{R}), q(\bar{R}))$. But since $u'(\bar{R}; p(\bar{R}), q(\bar{R}))=0=v'(\bar{R}; p(\bar{R}), q(\bar{R})$, the bifurcating asymmetric solutions cannot be positive for R near $\bar{R}$. Thus Theorem 4.1 is consistent with Troy's result.

2. From our dislcussion of (3.17), we can show in a completely analogous manner that

$$K(R)<0 \quad \text{if} \quad G_u<0$$

and

$$J(R)<0 \quad \text{if} \quad F_v<0. \tag{5.1}$$

We shall use these to show that the result of Troy [T], which we have discussed above, is, in a certain sense, the best possible. In fact, we have the following theorem.

THEOREM 5.1. *Consider the system* (1.4), *where* $f, g \in \mathscr{L}$. *If* $F_v<0$ *and* $G_u<0$, *then this system admits positive asymmetric solutions.*

Proof. Define $\Delta(R)$ (cf. (3.22)) by

$$\Delta(R)=A_1(R)\,B_2(R)-A_2(R)\,B_1(R), \tag{5.2}$$

$R_1(A(\varepsilon), \varepsilon)\leqslant R<\bar{R}(A(\varepsilon), \varepsilon)$. Then as we have shown in the proof of Theorem 3.1 (see (3.22), ff.),

$$\Delta(R)=w(R)\,z(R)-\varepsilon[J(R)\,z(R)+K(R)\,w(R)], \tag{5.3}$$

where $w=u'$ and $z=v'$. We shall first prove that for some ε, $0<\varepsilon\leqslant\varepsilon_0$, there is an R such that

$$\Delta(R)=0. \tag{5.4}$$

To do this, note that for any R in the above range, Theorem 2.1, part (ii) implies that $u'(R)<0$ and $v'(R)<0$. Thus at $\varepsilon=0$, $\Delta(\tilde{R})>0$ for some $\tilde{R}$, $R_1(A(0), 0)\leqslant\tilde{R}<\bar{R}(\bar{A}(0), 0)$. It follows that for small ε, there is an $R_*<\bar{R}(\bar{A}(\varepsilon), \varepsilon)$ with

$$\Delta(R_*)>0. \tag{5.5}$$

Fix such an ε for which (5.4) holds. Now consider (5.3). Since $F_v<0$ and $G_u<0$, we have that (5.1) holds for all R satisfying $R_1(A(\varepsilon), \varepsilon)\leqslant R<\bar{R}(\bar{A}(\varepsilon), \varepsilon)$. Thus $\varepsilon[J(R)\,z(R)+K(R)\,w(R)]>0$. Since $(w(R), z(R))\to(0, 0)$ as $R\to\bar{R}$, and $J(\bar{R})\,K(\bar{R})\neq 0$, we see from (5.3) that $\Delta(R)<0$ for R near $\bar{R}$, $R<\bar{R}(A(\varepsilon), \varepsilon)$. This, together with (5.5) shows that (5.4) holds.

Now consider Eq. (3.21) where $(w(r), z(r)) \equiv (u'(r), v'(r))$, and

$$\begin{pmatrix} u(r) \\ v(R) \end{pmatrix} = \begin{pmatrix} u(r;\, p(R; \bar{\Lambda}(\varepsilon), \varepsilon), q(R; \bar{\Lambda}(\varepsilon), \varepsilon); \bar{\Lambda}(\varepsilon), \varepsilon) \\ v(r;\, p(R; \bar{\Lambda}(\varepsilon), \varepsilon), q(R; \bar{\Lambda}(\varepsilon), \varepsilon); \bar{\Lambda}(\varepsilon), \varepsilon) \end{pmatrix}. \tag{5.6}$$

Then from (3.21), we have, at $r = R$,

$$\frac{a_1'(0)}{w'(0)} A_1(R) + \frac{b_1'(0)}{z'(0)} A_2(R) = 0$$

$$\frac{a_1'(0)}{w'(0)} B_1(R) + \frac{b_1'(0)}{z'(0)} B_2(R) = 0,$$

and since (5.3) holds, this system has a nonzero solution, i.e., $(a_1'(0), b_1'(0)) \neq (0, 0)$. In fact, $a_1'(0) \neq 0$ and $b_1'(0) \neq 0$, as follows easily from (5.1). Hence $a_1(r) \neq 0$ and $b_1(r) \neq 0$. Thus for some $R < \bar{R}(\bar{\Lambda}(\varepsilon), \varepsilon)$, the kernel of the linearized operator about (5.6) is of the form $(a_1(r), b_1(r))\, \Phi_1(\theta)$, as follows easily from the same sort of arguments as in the proof of Theorem 3.3. Now as in the proof of Theorem 4.1, since $a_1'(R) \neq 0$ and $b_1'(R) \neq 0$ (otherwise $a_1(r) \equiv 0 \equiv b_1(r)$ since $a_1(R) = 0 = b_1(R)$), we see that the symmetry breaks on the solution (5.6). Since $u'(R) < 0$ and $v'(R) < 0$, it follows that the bifurcating asymmetric solution is positive on $|x| < R$. This completes the proof. ∎

3. We note that our results hold for systems of n equations of the form

$$\begin{aligned} \Delta u_1(x) + f_1(u_1(x)) + \varepsilon_1 F_1(u(x)) &= 0 \\ \Delta u_2(x) + \lambda_2 f_2(u_2(x)) + \varepsilon_2 F_2(u(x)) &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \\ \Delta u_n(x) + \lambda_n f_n(u_n(x)) + \varepsilon_n F_n(u(x)) &= 0, \qquad |x| < R, \end{aligned}$$

with Dirichlet boundary conditions

$$u(x) \equiv (u_1(x), ..., u_n(x)) = 0, \qquad |x| = R,$$

where each $f_i \in \mathscr{L}$, $1 \leqslant i \leqslant n$. The proofs for this general case are straightforward extensions of those which we have given for the case $n = 2$. For these equations, the analogue of condition (3.2) takes the form $\partial F_i / \partial u_j > 0$ for each $i \neq j$.

References

[GNN] B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^n$, *Comm. Math. Phys.* **68** (1979), 209–243.

[SW1] J. Smoller and A. Wasserman, Existence, uniqueness, and nondegeneracy of positive solutions of semilinear elliptic equations, *Comm. Math. Phys.* **95** (1984), 129–159.

[SW2] J. Smoller and A. Wasserman, Symmetry-breaking for positive solutions of semilinear elliptic equations, *Arch. Rational Mech, Anal.* **95** (1986), 217–225.

[SW3] J. Smoller and A. Wasserman, Bifurcation from symmetry, *Proc. Conf. on Nonlinear Parabolic Equations*, in press.

[T] W. C. Troy, Symmetry properties in systems of semilinear elliptic equations, *J. Differential equations* **42** (1981), 400–413.