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Dedicated to J.S. Bell on the occasion of his sixtieth birthday.

GAMMATRICA

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Equations for strings and traces of strings of γ matrices are summarized.

1. Introduction

In most calculations in high energy physics a certain amount of γ matrix manipulation is unavoidable. The necessary equations were developed long ago by Fubini and Caianiello [1], Chisholm [2], and Kahane [3]. Most of the necessary work is straightforward and trivial but tedious, and is often done by machine. With the advent of dimensional regularization a new dimension has been added to the subject, and not everything is totally trivial or straightforward. In this note we describe the equations and procedures that have been used in a mechanical implementation. The actual application of these methods has been the subject of discussion for some time [4], and will not be considered here.

2. Generalities

Consider a Lorentz-like group in n dimensions. Dimension 4 is timelike, the others $(1-3, 5-n)$ are spacelike. One may construct n matrices γ^μ $(\mu = 1, \dots, n)$ in some space (spinor space) satisfying anticommutation rules*

$$\{\gamma^\mu, \gamma^\nu\} = 2\delta_{\mu\nu} I .$$

Here I is the unit matrix. Neither the dimensionality of the spinors nor the precise form of the matrices is needed for our purposes. Furthermore, the Lorentz group in

* No particular choice of metric is needed for the purposes of this paper.

n-dimensional space has a representation in spinor space such that

$$U^{-1}\gamma^{\mu}U = L_{\mu\nu}\gamma^{\nu},$$

where $L_{\mu\nu}$ is a Lorentz transformation and U is the corresponding transformation in spinor space.

A "string" is a product of γ matrices. The special matrix $\gamma^{\bar{5}}$ is defined by

$$\gamma^{\bar{5}} = \gamma^1\gamma^2\gamma^3\gamma^4.$$

Since the danger of confusion with γ^5, i.e., the matrix corresponding to dimension 5, is minimal we will drop the bar in the rest of this note.

Another matrix that one might consider as the generalization of the usual γ^5 is the matrix γ^N defined as

$$\gamma^N = \gamma^1\gamma^2\gamma^3 \cdots \gamma^n.$$

This matrix satisfies the rule

$$\gamma^N\gamma^N = \pm I$$

with + for $n = 1, 4, 5, 8, 9, \ldots$. The alternating sign as a function of n makes this matrix somewhat unattractive as a generalization of the usual γ^5. Also, consider for example the trace of four γ and γ^N:

$$\mathrm{Tr}\left(\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu}\gamma^{\nu}\gamma^{N}\right).$$

In terms of the dimension n this expression is quite singular, as it is non-zero for $n = 4$ and $n = 2$ only.

The trace of the unit matrix is irrelevant in actual applications; the only necessary property is that it is 4 if $n = 4$. It might well be that there is some advantage in using a particular function of n for this quantity, but so far none is evident. In the following this factor is called λ.

Consider now the trace of any string. That trace is manifestly Lorentz invariant:

$$\begin{aligned}\mathrm{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2} \cdots \gamma^{\alpha_m}\right) &= \mathrm{Tr}\left(U^{-1}\gamma^{\alpha_1}UU^{-1}\gamma^{\alpha_2} \cdots U^{-1}\gamma^{\alpha_m}U\right) \\ &= L_{\alpha_1\beta_1}L_{\alpha_2\beta_2} \cdots L_{\alpha_m\beta_m}\mathrm{Tr}\left(\gamma^{\beta_1} \cdots \gamma^{\beta_m}\right).\end{aligned}$$

Thus the trace is an invariant tensor. It must therefore be a sum of products of Kronecker δ's and ϵ tensors. The latter is denoted by $\epsilon_{\alpha_1 \cdots \alpha_n}$, and is totally antisymmetric in its n indices. If n is even then any such invariant tensor has evidently an even number of indices. Hence, it follows that the trace of a string of an odd number of γ matrices is zero if n is even. In the following we restrict ourselves to even n.

For even n the trace of any string can be determined using the anticommutation rules. First some notations:

$$(\alpha_1 \cdots \alpha_m) \equiv \mathrm{Tr}(\gamma^{\alpha_1} \cdots \gamma^{\alpha_m}),$$

$$(\alpha_1 \cdots \bar{\alpha}_j \cdots \alpha_m) \equiv \mathrm{Tr}(\gamma^{\alpha_1} \cdots \gamma^{\alpha_{j-1}}\gamma^{\alpha_{j+1}} \cdots \gamma^{\alpha_m}).$$

Anticommuting γ^{α_1} to the right we find

$$(\gamma^{\alpha_1} \cdots \gamma^{\alpha_m}) = -(-1)^m(\alpha_2 \cdots \alpha_m\alpha_1) + 2\sum_{j=2}^{m}(-1)^j \delta_{\alpha_1\alpha_j}(\alpha_2 \cdots \bar{\alpha}_j \cdots \alpha_m).$$

Using the trace property $(\alpha_2 \cdots \alpha_m\alpha_1) = (\alpha_1 \cdots \alpha_m)$, we have for m even

$$(\alpha_1 \cdots \alpha_m) = \sum_{j=2}^{m}(-1)^j \delta_{\alpha_1\alpha_j}(\alpha_2 \cdots \bar{\alpha}_j \cdots \alpha_m).$$

This is the trace reduction equation. Repeated application gives the desired expression. For $m = 2$ for example:

$$(\alpha_1\alpha_2) = \delta_{\alpha_1\alpha_2}\mathrm{Tr}(I) = \lambda\,\delta_{\alpha_1\alpha_2}.$$

Let there now be given a string S:

$$S_{\alpha_1 \cdots \alpha_m} = \gamma^{\alpha_1} \cdots \gamma^{\alpha_m}.$$

S is called odd or even depending on whether m is odd or even. The reversed string S^{R} is defined by

$$S^{\mathrm{R}} = S^{\mathrm{R}}_{\alpha_1 \cdots \alpha_m} = \gamma^{\alpha_m} \cdots \gamma^{\alpha_1}.$$

The relevance of S^{R} derives from the fact that the trace of S^{R} is equal to the trace of S. This may be seen by considering the trace of S^{R} and anticommuting γ^{α_1} to the left.

We now define for arbitrary n a set of basis matrices constructed from the γ's:

$$[\mu] \equiv \gamma^{\mu},$$

$$[\mu\nu] = \tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}),$$

$$[\alpha_1 \cdots \alpha_k] = \frac{1}{k!}\sum_{p}(-1)^{S}\gamma^{\alpha_i} \cdots \gamma^{\alpha_j},$$

where the sum is over all permutations of $\alpha_1 \cdots \alpha_k$ and S is 0 or 1 for even or odd permutations respectively. It is fairly evident that the trace of all of these quantities is zero. It is equally evident that any string can be written as a linear combination of these basic matrices and the unit matrix. For example:

$$\gamma^\mu\gamma^\nu = \tfrac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) + \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$
$$= \delta_{\mu\nu} I + [\mu\nu] .$$

In general:

$$S_{\alpha_1 \cdots \alpha_m} = a_0 I + a_\beta[\beta] + a_{\beta_1\beta_2}[\beta_1\beta_2] + \cdots + a_{\beta_1 \cdots \beta_m}[\beta_1 \cdots \beta_m] .$$

Taking the trace of both sides gives a_0:

$$a_0 = \frac{1}{\lambda} \operatorname{Tr}(S) .$$

The other coefficients are totally antisymmetric:

$$a_{\beta_1 \cdots \beta_m} = \frac{1}{\lambda m!} \operatorname{Tr}(S[\beta_m \cdots \beta_1]) .$$

Here we used

$$\operatorname{Tr}([\beta_1 \cdots \beta_m][\mu_m \cdots \mu_1]) = \lambda \sum_P (-1)^P \delta_{\beta_1\mu_i} \cdots \delta_{\beta_m\mu_j} ,$$

where the sum is over all permutations of the μ_j.

The correctness of this equation becomes fairly obvious if one first notes that the right-hand side must have the antisymmetric form given (it must be antisymmetric in all β and also in all μ). That leaves the question of an overall factor which may be obtained by counting the number of times that the first term occurs, i.e., once in every product.

The right-hand side may conveniently be written as a determinant. Define

$$\operatorname{Det}(\beta_1 \cdots \beta_m; \mu_1 \cdots \mu_m) \equiv \text{determinant} \begin{pmatrix} \delta_{\beta_1\mu_1} & \delta_{\beta_2\mu_1} & \cdots & \delta_{\beta_m\mu_1} \\ \delta_{\beta_1\mu_2} & & & \\ \vdots & & & \vdots \\ \delta_{\beta_1\mu_m} & & & \delta_{\beta_m\mu_m} \end{pmatrix} .$$

Then,

$$\mathrm{Tr}([\beta_1 \cdots \beta_m][\mu_m \cdots \mu_1]) = \lambda \,\mathrm{Det}(\beta_1 \cdots \beta_m; \mu_1 \cdots \mu_m).$$

For example:

$$\mathrm{Tr}([\mu\nu],[\beta\alpha]) = \lambda \,\mathrm{Det}(\mu\nu; \alpha\beta)$$

$$= \lambda \,\mathrm{Det}\begin{pmatrix} \delta_{\mu\alpha} & \delta_{\nu\alpha} \\ \delta_{\mu\beta} & \delta_{\nu\beta} \end{pmatrix} = \lambda(\delta_{\mu\alpha}\delta_{\nu\beta} - \delta_{\nu\alpha}\delta_{\mu\beta}).$$

The coefficients of S^{R} are easily obtained from those of S:

$$a^{\mathrm{R}}_{\beta_1 \cdots \beta_m} = a_{\beta_m \cdots \beta_1} = \pm a_{\beta_1 \cdots \beta_m}$$

with + for $m = 0, 1, 4, 5, 8, 9, \ldots$.

Since the coefficients a are completely antisymmetric, and since there are no more than n γ's it follows that the a with more than n indices are 0.

A few more equations. For a given dimension n we have the totally antisymmetric tensor with n indices,

$$\epsilon_{\alpha_1 \cdots \alpha_n}.$$

Reversing the indices:

$$\epsilon_{\alpha_1 \cdots \alpha_n} = (-1)^{n/2} \epsilon_{\alpha_n \cdots \alpha_1},$$

where n is taken to be even. Using the matrix γ^N defined above one has

$$[\mu_1 \cdots \mu_m] = \frac{1}{k!} \epsilon_{\mu_1 \cdots \mu_m \alpha_k \cdots \alpha_1} \gamma^N [\alpha_1 \cdots \alpha_k]$$

$$= \frac{1}{k!} \epsilon_{\alpha_1 \cdots \alpha_k \mu_1 \cdots \mu_m} [\alpha_k \cdots \alpha_1] \gamma^N$$

with $k = n - m$. Using this we have

$$\mathrm{Tr}([\mu_1 \cdots \mu_m][\nu_m \cdots \nu_1])$$

$$= \lambda \,\mathrm{Det}(\mu_1 \cdots \mu_m; \nu_1 \cdots \nu_m)$$

$$= \left(\frac{1}{k!}\right)^2 \epsilon_{\alpha_1 \cdots \alpha_k \mu_1 \cdots \mu_m} \epsilon_{\nu_m \cdots \nu_1 \beta_k \cdots \beta_1} \mathrm{Tr}([\alpha_k \cdots \alpha_1]\gamma^N \gamma^N [\beta_1 \cdots \beta_k])$$

$$= \lambda \left(\frac{1}{k!}\right)^2 (-1)^{n/2} \epsilon_{\alpha_1 \cdots \alpha_k \mu_1 \cdots \mu_m} \epsilon_{\nu_m \cdots \nu_1 \beta_k \cdots \beta_1} \mathrm{Det}(\alpha_1 \cdots \alpha_k; \beta_1 \cdots \beta_m).$$

Reversing the order of the indices in the second ϵ gives another factor $(-1)^{n/2}$. Every term of the determinant gives the same result, and we get finally the equation

$$\mathrm{Det}(\mu_1 \cdots \mu_m; \nu_1 \cdots \nu_m) = \frac{1}{k!} \epsilon_{\alpha_1 \cdots \alpha_k \mu_1 \cdots \mu_m} \epsilon_{\alpha_1 \cdots \alpha_k \nu_1 \cdots \nu_m},$$

which equation can easily be derived directly as well.

3. Index pairs

If all indices in a given string are different then in general, in n dimensions, the trace is just what obtains by repeated application of the trace reduction equation. The number of terms grows rapidly with the number of γ's in the string. For m γ's the number of terms is $(m-1)(m-3)\cdots(1)$, and for example for 14 γ's this is about 135 000. However, the indices are really never all different and pairs to be summed over occur. The problem is now to eliminate such pairs. It can be done quite satisfactory in 4 dimensions (to be discussed later), but in n dimensions the situation remains somewhat cumbersome.

Let us start with some simple cases. Trivially:

$$\gamma^{\mu}\gamma^{\mu} = nI.$$

Also:

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\mu} = -\gamma^{\mu}\gamma^{\mu}\gamma^{\alpha} + 2\,\delta_{\mu\alpha}\gamma^{\mu} = (2-n)\gamma^{\alpha}$$

and

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\beta}\gamma^{\mu} = (n-4)\gamma^{\alpha}\gamma^{\beta} + 4\,\delta_{\alpha\beta}1.$$

The general case ($m \geqslant 3$):

$$\begin{aligned}\gamma^{\mu}\gamma^{\alpha_1}\cdots\gamma^{\alpha_m}\gamma^{\mu} &= (n-4)(-1)^m\gamma^{\alpha_1}\cdots\gamma^{\alpha_m}\\ &\quad+2(-1)^m\gamma^{\alpha_3}\gamma^{\alpha_2}\gamma^{\alpha_1}\gamma^{\alpha_4}\cdots\gamma^{\alpha_m}\\ &\quad+2\sum_{j=4}^{m}(-1)^{m-j}\gamma^{\alpha_j}\gamma^{\alpha_1}\cdots\gamma^{\alpha_{j-1}}\gamma^{\alpha_{j+1}}\cdots\gamma^{\alpha_m}.\end{aligned}$$

This appears to be the most useful and compact form. The equation is obtained simply by anticommuting the last γ^{μ} to the left followed by some work on the last three terms. For this equation to be true it is essential that the summation over μ extends over all values that the in-between indices $\alpha_1 \cdots \alpha_m$ may have. The equation is in general not true if the summation range is 1–4.

To give an idea of the effect of this equation consider a trace of 14 γ's, where the index of the first and seventh are the same. Using this equation the trace reduces to a sum of 5 terms of 12 γ's, which is about 50000 terms. That is really a worst case; if the first and fourth (or less) index are the same the number is 20000 (10000).

In principle it ought to be possible to reduce such a trace to as many terms as would be obtained from a trace without the pair. However, it is quite cumbersome to figure the coefficients of the terms as depending on the location of the pair, and if more than one pair occurs it becomes impractical. The problem is further aggravated by the fact that in practical situations the summation range may not be the full range n; furthermore the occurrence of γ^5 effectively renders such a technique useless. As it turns out, using the methods to be described below, the above equation is very effective in practice, as it is then needed mostly in cases where all or nearly all indices occur in pairs.

4. 4 dimensions

In the case that $n=4$ many simplifications occur. Algorithms due to Caianiello and Fubini [1], Chisholm [2], and Kahane [3] solve the index pair problem satisfactorily. Additional algorithms simplifying further can be established. Here is how.

If $n=4$ then one has

$$[\mu]=\gamma^{\mu},$$

$$[\mu\nu]=\tfrac{1}{2}(\gamma^{\mu}\gamma^{\nu}-\gamma^{\nu}\gamma^{\mu})$$

as before, but

$$[\mu\nu\alpha]=\epsilon_{\mu\nu\alpha\lambda}\gamma^5\gamma^{\lambda},$$

$$[\mu\nu\alpha\beta]=\epsilon_{\mu\nu\alpha\beta}\gamma^5.$$

The resulting expansion for any string S in 4 dimensions is

$$S=a_0 I+a_{\beta}\gamma^{\beta}+a_{\mu\nu}[\mu\nu]+a_{5\beta}\gamma^5\gamma^{\beta}+a_5\gamma^5$$

with

$$a_0=\frac{1}{\lambda}\operatorname{Tr}(S),\qquad a_{\beta}=\frac{1}{\lambda}(S\gamma^{\beta}),$$

$$a_{\mu\nu}=\frac{1}{\lambda}\operatorname{Tr}(S[\nu\mu]),\qquad a_{5\beta}=\frac{1}{\lambda}\operatorname{Tr}(S\gamma^{\beta}\gamma^5),$$

$$a_5=\frac{1}{\lambda}\operatorname{Tr}(S\gamma^5).$$

The expression for the reversed string is

$$S_{\mathrm{R}} = a_0 I + a_\beta \gamma^\beta - a_{\mu\nu}[\mu\nu] - a_{5\beta}\gamma^\beta\gamma^5 + a_5\gamma^5 .$$

If S is odd (even) then a_0, $a_{\mu\nu}$, and a_5 (a_β and $a_{5\beta}$) are zero. A useful equation is

$$S + S_{\mathrm{R}} = 2\left(a_0 I + a_\beta\gamma^\beta + a_5\gamma^5\right),$$

which reduces for even respectively odd S to

$$S + S_{\mathrm{R}} = 2\left(a_0 I + a_5\gamma^5\right), \qquad S \text{ even},$$

$$S + S_{\mathrm{R}} = 2\alpha_\beta\gamma^\beta \qquad S \text{ odd}.$$

The right-hand side of the first equation is minus the brace′ of S_1 to be introduced below. Following Chisholm [2] the least equation can be rewritten as

$$\gamma^\beta \operatorname{Tr}(\gamma^\beta S) = \tfrac{1}{2}\lambda(S + S_{\mathrm{R}}), \qquad S \text{ odd}.$$

This equation can be used to unify a string and a trace. With S_1 and S_2 arbitrary strings:

$$S_1\gamma^\beta S_2 \cdot \operatorname{Tr}(\gamma^\beta S) = \tfrac{1}{2}\lambda S_1(S + S^{\mathrm{R}})S_2 .$$

Concerning γ^5 note the following. Since $\gamma^5 = \gamma^1\gamma^2\gamma^3\gamma^4$ it follows that the reversed γ^5 equals γ^5. Thus, if

$$S = \gamma^\mu\gamma^5\gamma^\nu$$

then

$$S^{\mathrm{R}} = \gamma^\nu\gamma^5\gamma^\mu ,$$

i.e., γ^5 can be treated like any other γ in reversing a string. On the other hand, however, γ^5 is a product of four γ's and is thus even. In the count for odd or even strings γ^5 must not be counted. Thus the last mentioned string here is even.

Consider now a string in-between an index pair:

$$\gamma^\mu S_{\alpha_1 \cdots \alpha_m}\gamma^\mu .$$

To work this out we first evaluate

$$\gamma^\mu[\alpha_1 \cdots \alpha_k]\gamma^\mu$$

for any dimension n. The result must again be antisymmetrical in $\alpha_1 \cdots \alpha_k$. Apart from a factor it must therefore again be $[\alpha_1 \cdots \alpha_k]$. The factor is readily established:

$$\gamma^\mu[\alpha_1 \cdots \alpha_k]\gamma^\mu = (-1)^k(n-2k)[\alpha_1 \cdots \alpha_k] .$$

For example:

$$\gamma^{\mu}\gamma^{\alpha}\gamma^{\mu} = -(n-2)\gamma^{\alpha},$$

$$\gamma^{\mu}[\alpha\beta]\gamma^{\mu} = (n-4)[\alpha\beta],$$

etc.

It follows that

$$\gamma^{\mu}S_{\alpha_1\cdots\alpha_m}\gamma^{\mu} = na_0 I - (n-2)a_{\beta}[\beta] + (n-4)a_{\beta_1\beta_2}[\beta_1\beta_2] - (n-6)a_{\beta_1\beta_2\beta_3}[\beta_1\beta_2\beta_3]\ldots.$$

This is still general. Now specialize to four dimensions, $n = 4$. All a with more than 4 indices are 0, and for an odd string

$$\gamma^{\mu}S_{\alpha_1\cdots\alpha_m}\gamma^{\mu} = -2a_{\beta}[\beta] + 2a_{\beta_1\beta_2\beta_3}[\beta_1\beta_2\beta_3].$$

Apart from a factor -2 the right-hand side is precisely the expression for the reversed string S^{R}. We so obtain the Caianiello–Fubini–Chisholm equation for odd strings:

$$\gamma^{\mu}S\gamma^{\mu} = -2S^{\mathrm{R}}, \qquad \text{odd } S.$$

For even S the following obtains:

$$\gamma^{\mu}S\gamma^{\mu} = 4a_0 I - 4a_{\beta_1\cdots\beta_4}[\beta_1\cdots\beta_4].$$

Unfortunately this is not S^{R}. However, using γ^5 instead of $[\beta_1\cdots\beta_4]$ we have

$$\gamma^{\mu}S\gamma^{\mu} = 4a_0 I - 4a_5\gamma^5$$

with

$$a_0 = \frac{1}{\lambda}\operatorname{Tr}(S),$$

$$a_5 = \frac{1}{\lambda}\operatorname{Tr}(S\gamma^5).$$

With Kahane [3] we will call the expressions

$$\{S\} \equiv -\frac{2}{\lambda}\operatorname{Tr}(S)I + \frac{2}{\lambda}\operatorname{Tr}(S\gamma^5)\gamma^5,$$

$$\{S\}' \equiv -\frac{2}{\lambda}\operatorname{Tr}(S)I - \frac{2}{\lambda}\operatorname{Tr}(S\gamma^5)\gamma^5$$

the brace and brace′ of the string S, and thus

$$\gamma^{\mu} S \gamma^{\mu} = -2\{S\}, \qquad \text{even } S.$$

One more equation is needed. Let S be an odd string. Then

$$S = \frac{1}{\lambda}\,\mathrm{Tr}(S\gamma^{\mu})\gamma^{\mu} + \frac{1}{\lambda}\,\mathrm{Tr}(S\gamma^{\mu}\gamma^{5})\gamma^{5}\gamma^{\mu},$$

or in terms of the brace

$$\begin{aligned} -2\gamma^{\mu}\{S\gamma^{\mu}\} &= -2\{S^{\mathrm{R}}\gamma^{\mu}\}\gamma^{\mu} = -2\{S\gamma^{\mu}\}'\gamma^{\mu} \\ &= -2\gamma^{\mu}\{S^{\mathrm{R}}\gamma^{\mu}\}' = S, \qquad \text{even } S, \end{aligned}$$

where we used $(S\gamma^{\mu}) = (S^{\mathrm{R}}\gamma^{\mu})$ and $(S\gamma^{\mu}\gamma^{5}) = -(S^{\mathrm{R}}\gamma^{\mu}\gamma^{5})$. These equations are sufficient to eliminate all index pairs in a string. The general result is a rather elegant algorithm, the Kahane algorithm. It can best be described in graphic terms, and the proof (by induction) is simple.

Let there be given a string with any number of index pairs separated by strings S_1, S_2, etc. For clarity we will describe the procedure for the case of 2 index pairs.

To begin with anticommute all γ^5 to the left, so that they are out of the way. Now start counting from the left. An index is called odd or even depending on whether it occurs at an odd or even position.

Consider now an index pair. The combinations odd–odd, odd–even, etc. may occur. At the location of any index draw one line up and one line down. If an odd index then the up line is left from the down line, else the other way around (fig. 1). Now connect for an index pair the upper and the lower lines. For example, if both indices are odd (fig. 2). Next read out the γ matrices following the lines starting with the very first γ (fig. 3). The expression

$$S_1 S_2^{\mathrm{R}} S_3$$

is obtained. With the prescription of a factor of -2 for every index pair this is

Fig. 1.

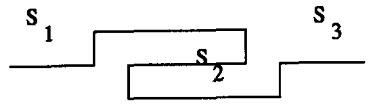

Fig. 2.

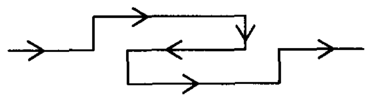

Fig. 3.

precisely what corresponds to

$$S_1\gamma^\mu S_2\gamma^\mu S_3$$

for odd S_2.

If S_2 had been even then fig. 4 results. There is a remaining piece, no more part of the loop. Such a piece is a brace. It may be anticommuted to the very beginning using

$$\gamma^\lambda\{S\} = \{S\}'\gamma^\lambda .$$

One thus may collect the braces to the beginning and use brace or brace′ depending on whether the first index was at an odd or even place. Again, there is a factor -2. The expression is

$$-2S_1\{S\}S_3$$

as corresponding to

$$S_1\gamma^\mu S_2\gamma^\mu S_3 , \qquad S_2 \text{ even} .$$

If there are several index pairs connect them as described above, read from the beginning, collect all remaining pieces and the correct equation results.

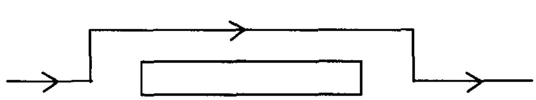

Fig. 4.

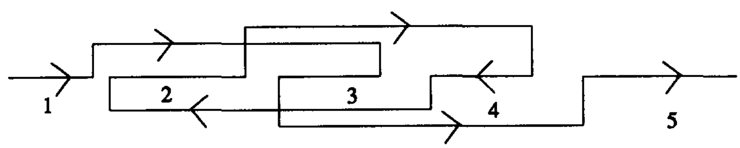

Fig. 5.

Fig. 5 shows an example with 2 pairs, all indices at odd locations. The result is

$$(-2)^2\{S_2S_4^{\mathrm{R}}\}'S_1S_3^{\mathrm{R}}S_5,$$

where all S are odd. Note brace′; this is because the first index of S_2 is at an even location. The result is obtained from

$$\begin{aligned} S_1\gamma^\mu S_2\gamma^\nu S_3\gamma^\mu S_4\gamma^\nu S_5 &= -2S_1S_3^{\mathrm{R}}\gamma^\nu S_2^{\mathrm{R}}S_4\gamma^\nu S_5 \\ &= 4S_1S_3^{\mathrm{R}}\{S_2^{\mathrm{R}}S_4\}S_5 \\ &= 4\{S_2^{\mathrm{R}}S_4\}S_1S_3^{\mathrm{R}}S_4. \end{aligned}$$

Note that for odd S one has $\{S_2^{\mathrm{R}}S_4\} = \{S_2S_4^{\mathrm{R}}\}'$. The proof by induction becomes obvious if we depict the equation

$$-2S_1\{S\gamma^\mu\}'\gamma^\mu S_2 = S_1SS_2$$

in graphic form (S_1 taken to be odd as is S), see fig. 6, showing the validity of the Kahane algorithm for such a case. Note that this algorithm applies also even if the starting expression is not a trace but just a string with index pairs.

After this one is left with strings without index pairs. Often one will have that the γ's are contracted with some momentum, and in a string the same momentum may occur more than once. Further simplification is possible using the equations

$$\not{p}S\not{p} = -p^2S^{\mathrm{R}} + \tfrac{1}{2}\not{p}(\not{p}S^{\mathrm{R}}) + \tfrac{1}{2}\not{p}\gamma^5(\gamma^5\not{p}S^{\mathrm{R}}), \qquad S \text{ odd},$$

$$\not{p}S\not{p} = -p^2S^{\mathrm{R}} + \tfrac{1}{2}\gamma^\mu\not{p}(\not{p}\gamma^\mu S^{\mathrm{R}}), \qquad S \text{ even},$$

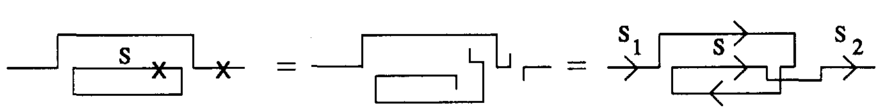

Fig. 6.

where

$$\not p = \gamma^\lambda p_\lambda , \qquad p^2 = (pp) = p_\lambda p_\lambda ,$$

with summation over λ understood.

The final step is to reduce any string to a string containing at most two γ's (and possibly a γ^5). This may be done using the equation (easily verified by inspection)

$$\gamma^\mu \gamma^\alpha \gamma^\beta = \left\{ \delta_{\mu\alpha}\delta_{\beta\lambda} - \delta_{\mu\beta}\delta_{\alpha\lambda} + \delta_{\alpha\beta}\delta_{\mu\lambda} + \epsilon_{\mu\alpha\beta\lambda}\gamma^5 \right\} \gamma^\lambda .$$

Products of ϵ tensors may be reduced to products of δ's:

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\nu\alpha\lambda} = 3!\,\delta_{\beta\lambda} ,$$

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\nu\lambda\kappa} = 2!\left(\delta_{\alpha\lambda}\,\delta_{\beta\kappa} - \delta_{\alpha\kappa}\,\delta_{\beta\lambda}\right) = \mathrm{Det}(\alpha\beta;\,\lambda\kappa) ,$$

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\mu\lambda\kappa\gamma} = \mathrm{Det}(\nu\alpha\beta;\,\lambda\kappa\gamma) ,$$

$$\epsilon_{\mu\nu\alpha\beta}\epsilon_{\lambda\kappa\gamma\delta} = \mathrm{Det}(\mu\nu\alpha\beta;\,\lambda\kappa\gamma\delta) .$$

In all of this work, heavy use is being made of the fact that $n = 4$. It is important to obtain expressions that have the minimal amount in terms, not only for that fact by itself, but also because of the uniqueness problem. Consider, for example, the expression

$$\delta_{\alpha\beta}\epsilon_{\mu\nu\lambda\kappa} .$$

Antisymmetrizing this in $\beta\mu\nu\lambda\kappa$ gives obviously zero because in four dimensions one cannot have antisymmetry in five indices. This results in the equation

$$\delta_{\alpha\beta}\epsilon_{\mu\nu\lambda\kappa} - \delta_{\alpha\mu}\epsilon_{\beta\nu\lambda\kappa} - \delta_{\alpha\nu}\epsilon_{\mu\beta\lambda\kappa} - \delta_{\alpha\lambda}\epsilon_{\mu\nu\beta\kappa} - \delta_{\alpha\kappa}\epsilon_{\mu\nu\lambda\beta} = 0 .$$

Adding this (and other similar expressions) to any expression will not change it but the result may look very different. There is really no foolproof way to get a unique expression; using the techniques described above the problem all but disappears.

5. Sum splitting

In practice one must deal with situations where in a string simultaneously index pairs occur with a range of n as well as four, and where in addition γ^5 may occur. Furthermore, one may have products of strings, with contractions between indices in different strings. For example a product of two traces:

$$(\alpha_1 \cdots \mu \cdots \nu \cdots \alpha_m)\cdot(\beta_1 \cdots \mu \cdots \nu \cdots 5 \cdots \beta_k)$$

with $\mu=1,\ldots,4$ and $\nu=1,\ldots,n$. Another example, involving spinors u:

$$\left(\bar{u}\gamma^{\alpha}\cdots\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\kappa}\cdots\gamma^{\alpha_m}u\right)\cdot\left(\bar{u}\gamma^{\beta}\cdots\gamma^{\mu}\gamma^{\nu}\gamma^{\lambda}\gamma^{\kappa}\cdots\gamma^{\beta_m}u\right)$$

with μ, ν, λ and $\kappa=1,\ldots,n$. Here there is a difficulty. In n dimensions one cannot reduce a product of γ's to the product of only a few γ's. If the spinors u represent particles in 4 dimensions, i.e., if they are subset of u that are possible in the general case, then one can still go ahead using the expansion technique. One again may write a string as a linear combination of the basic antisymmetric quantities $[\alpha_1\cdots\alpha_m]$; in the limit $n=4$ all $[\alpha_1\cdots\alpha_m]$ with $m>4$ are zero. This shows in fact that the u corresponding to a particle in 4-space must be such that

$$\bar{u}[\alpha_1\cdots\alpha_m]u=0 \qquad \text{if } m>4\,.$$

In 4-dimensional space one uses equations for the product $u\bar{u}$ summed over spins, such as

$$\sum u\bar{u}=-i\not{p}+m\,.$$

Strictly speaking this equation is not true if $n\neq 4$, even if p is a vector in 4-space. The correct expression obtains by multiplying the above by the appropriate projection operator, i.e.

$$\sum u\bar{u}=P(-i\gamma p+m)P$$

and

$$P[\alpha_1\cdots\alpha_m]P=0, \qquad \text{if } m>4\,.$$

In practice it is not necessary to introduce this P explicitly as will become clear later. Terms as shown above and combined with other factors as appearing in practical situations always vanish as $n-4$ in the limit $n=4$. That could still be dangerous if poles $1/(n-4)$ appear, but again that appears not to be the case for this kind of term.

To deal with the complications described above we introduce the sum splitting technique. Consider a trace of the form

$$T=(\cdots\mu\cdots\mu\cdots),$$

where the index μ runs from 1 to n, and where γ^5 and other 4-dimensional objects may occur anywhere in the string. We write

$$T=(\cdots\bar{\mu}\cdots\bar{\mu}\cdots)+(\cdots\hat{\mu}\cdots\hat{\mu}\cdots)\,.$$

The index $\bar{\mu}$ takes the values $1,\ldots,4$, and $\hat{\mu}$ runs from 4 to n. Assuming for definiteness that μ is the only n-index in the string it follows that the first term is a purely 4-dimensional trace. The second term is also easily worked out. The matrices $\gamma^{\hat{\mu}}$ anticommute with all other γ's, and commute with γ^5. They can be commuted to the very left. One easily convinces oneself that the second trace can be written as a product of two traces:

$$(\hat{\mu}\hat{\mu}\cdots) = (\mu\mu)\hat{}\,(\cdots),$$

where we define the $n-4$ type trace $(\ )\hat{}$

$$(\mu\mu)\hat{} \equiv \frac{1}{\lambda}(\mu^{\hat{\mu}}\gamma^{\hat{\mu}}) \equiv \frac{1}{\lambda}\operatorname{Tr}(\gamma^{\hat{\mu}}\gamma^{\hat{\mu}}).$$

The other factor is again a purely 4-dimensional trace.

If there are more n-range indices, they can be split likewise. It can also be done if such indices are paired but occur in different strings:

$$(\cdots\mu\cdots)\cdot(\cdots\mu\cdots) = (\cdots\bar{\mu}\cdots)\cdot(\cdots\bar{\mu}\cdots) + (\cdots\hat{\mu}\cdots)(\cdots\hat{\mu}\cdots).$$

In practical cases all n-range indices occur in pairs, and the above technique applies.

This perfectly obvious technique also makes another fact obvious. In reducing the $(n-4)$-type traces using the trace reduction equation, one will in the end always find an expression of the form $(\mu\mu)\hat{}$. The assumption here is that all n-type indices are paired. This last trace is of course just equal to $n-4$:

$$\frac{1}{\lambda}\operatorname{Tr}(\gamma^{\hat{\mu}}\gamma^{\hat{\mu}}) = n-4.$$

One might think that the number of factors $n-4$ is equal to the number of n-range index pairs. This is not so, for example:

$$(\mu\nu\mu\nu)\hat{} = (6-n)(\nu\nu)\hat{} = (6-n)(n-4).$$

A final example:

$$(\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10}\quad \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7\alpha_8\alpha_9\alpha_{10})\hat{}\,,$$

where the range of all indices is from 4 to n. One finds:

$$(n-4)(-n^9 + 126n^8 - 6456n^7 + 178\,416n^6 - 2\,963\,856n^5 + 31\,010\,784n^4$$
$$-206\,139\,904n^3 + 845\,320\,704n^2 - 1\,950\,961\,664n + 1\,938\,948\,096),$$

which is, as should be, equal to

$$(n-4)(n-6)\big[(n-5)(n^7+115n^6-5161n^5+118\,195n^4-1\,508\,881n^3 + 10\,867\,243n^2-4\,133\,801n+64\,631\,603)-1\big]\,.$$

6. Anomalous traces

Traces containing γ^5 and evaluated according to the rules described above are quite unsymmetrical objects. For example:

$$\frac{1}{\lambda}\,\mathrm{Tr}(\gamma^\alpha\gamma^5\gamma^\beta\gamma^5) = -\delta_{\bar\alpha\bar\beta}+\delta_{\hat\alpha\hat\beta}\,.$$

This is the cause of many difficulties in practical applications; among others gauge breaking terms result in case of gauge theories. Such terms must then be explicitly cancelled by counter terms, and the proof that that can be done consistently is not easy. For this reason many authors have tried to define a more suitable γ^5, in particular γ^N mentioned before. These attempts must be considered unsuccessful for the reasons mentioned.

It should be noted that there is no real need for an alternative definition of γ^5. It is equally adequate and much more satisfactory from a mathematical point of view to define a new trace with more pleasant properties.

Let us start with a trace containing two n-range indices α and β and two γ^5 in addition to any number of γ's with four-dimensional indices:

$$\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^\alpha\cdots\gamma^5\cdots\gamma^\beta\cdots)\,.$$

Using sum splitting this becomes

$$\begin{aligned}&\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^{\bar\alpha}\cdots\gamma^5\cdots\gamma^{\bar\beta}\cdots)+\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^{\hat\alpha}\cdots\gamma^5\cdots\gamma^{\hat\beta}\cdots)\\ &\quad=\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^{\bar\alpha}\cdots\gamma^5\cdots\gamma^{\bar\beta}\cdots)\\ &\qquad+(-1)^k\,\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^5\cdots)\mathrm{Tr}(\gamma^\alpha\gamma^\beta)^\wedge\,,\end{aligned}$$

where the number k is determined by the number of γ's (but not the γ^5) between γ^α and γ^β.

The "anomalous trace" for a trace containing an even number of γ^5 matrices is defined by [5]:

$$\begin{aligned}&\mathrm{Tr}'(\cdots\gamma^5\cdots\gamma^\alpha\cdots\gamma^5\cdots\gamma^\beta\cdots)\\ &\quad=\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^{\bar\alpha}\cdots\gamma^5\cdots\gamma^{\bar\beta}\cdots)\\ &\qquad+(-1)^k(-1)^m\,\mathrm{Tr}(\cdots\gamma^5\cdots\gamma^5\cdots)\mathrm{Tr}(\gamma^\alpha\gamma^\beta)^\wedge\,,\end{aligned}$$

where m is the number of γ^5 in between γ^α and γ^β, in this case one.

The anomalous trace differs from the normal one by terms proportional to $n-4$. If a trace contains an odd number of γ^5 matrices then the anomalous trace is defined to be equal to the normal one.

The anomalous trace defined in this manner still has the necessary properties of a trace. In particular, it is invariant for a cyclic permutation of the γ's. This is where the number of γ^5 (odd or even) is essential. For an odd number of γ^5 the number of γ^5 in-between is not cyclic invariant, as can be seen on the following example:

$$\mathrm{Tr}(\gamma^\alpha\gamma^5\gamma^\alpha\gamma^\mu) \leftrightarrow \mathrm{Tr}(\gamma^\alpha\gamma^\mu\gamma^\alpha\gamma^5).$$

For this reason the anomalous trace is defined to be the same as the normal one for traces with an odd number of γ^5.

The definition can be extended easily to the case that there are more n-range indices. After sum splitting the factor in front of each term is $(-1)^s$ with

$$s = s_{12} + s_{34} + s_{56} + \cdots .$$

In here s_{ij} is the number of γ^5 between index i and index j. An example showing only three from the five terms:

$$\begin{aligned}&(\cdots\alpha_1\cdots\alpha_2\cdots\alpha_3\cdots\alpha_4)' \\ &\quad=(\cdots\bar{\alpha}_1\cdots\bar{\alpha}_2\cdots\bar{\alpha}_3\cdots\bar{\alpha}_4)+(-1)^k(-1)^a(\cdots\bar{\alpha}_3\cdots\bar{\alpha}_4)(\hat{\alpha}_1\hat{\alpha}_2) \\ &\qquad+\cdots+(-1)^l(-1)^b(\cdots)(\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3\hat{\alpha}_4).\end{aligned}$$

In here k and l are the normal γ counts, while a is the number of γ^5 between γ^{α_1} and γ^{α_2} and b is the number of γ^5 between γ^{α_1} and γ^{α_2} plus the same between γ^{α_3} and γ^{α_4}. In graphic form, the relevant number of γ^5 for such a count is in the underlined regions:

$$(\cdots\underline{\alpha_1\cdots\alpha_2}\cdots\underline{\alpha_3\cdots\alpha_4}\cdots).$$

If the total number of γ^5 is even then this definition is invariant for cyclic rotation.

Formally this definition of an anomalous trace amounts to the prescription of an anticommuting γ^5 and γ^α also if $\alpha > 4$. Note, however, that here such a γ^5 is not supposed to exist, and in fact it is clear that there are difficulties for trace with an odd number of γ^5.

It should be emphasized that this recipe for an anomalous trace supposes that there are no hidden γ^5. A hidden γ^5 is simply one that is written as $\gamma^1\gamma^2\gamma^3\gamma^4$. Such factors can be recognized on the basis of behaviour under space reflection, and it is clear from this that a consistent definition of an anomalous trace is only possible within the context of a theory. This will not be discussed further here.

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