Asymptotic Stability of Rarefaction Waves for 2 × 2 Viscous Hyperbolic Conservation Laws—The Two-Modes Case

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In this paper, we continue our study on the asymptotic behavior toward rarefaction waves of a general 2 × 2 system of hyperbolic conservation laws with positive viscosity matrix. It is shown that when the initial data is a small perturbation of a weak rarefaction wave (a linear superposition of a 1-rarefaction wave and a 2-rarefaction wave) for the corresponding inviscid hyperbolic conservation laws, then the solution of the Cauchy problem for the viscous system globally exists and tends to the rarefaction wave. The result is proved by using an energy method, combining the technique in [Z. P. Xin, *J. Differential Equations* **73** (1988), 45–77], and using the characteristic-energy method of T. P. Liu [*Mem. Amer. Math. Soc.* **328** (1975), 1–108]. © 1989 Academic Press, Inc.

1. INTRODUCTION

In this paper, we continue our study on the asymptotic behavior toward rarefaction waves of the solution for the initial value problem for 2 × 2 viscous hyperbolic conservation laws of the form

$$u_t + (f(u))_x = (B(u)u_x)_x, \qquad -\infty < x < +\infty, \quad t > 0, \qquad (1.1)$$

with initial data

$$u(x, 0) = u_0(x), \qquad x \in \mathbb{R}^1, \qquad (1.2)$$

satisfying

$$\lim_{x \to \pm\infty} u_0(x) = u_\pm. \qquad (1.3)$$

The corresponding inviscid problem is

$$u_t + (f(u))_x = 0, \qquad x \in \mathbb{R}^1, \quad t \geq 0, \qquad (1.4)$$

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with Riemann initial data

$$u(x, 0) = u_0^r(x) = \begin{cases} u_-, & x < 0, \\ u_+, & x > 0, \end{cases} \qquad (1.5)$$

where $u = (u_1, u_2) \in \Omega$, Ω is some region in $\mathbb{R}^2$, $f(u) = (f_1(u), f_2(u))$ is a smooth vector-valued function of $u \in \Omega$, $B(u)$ is a smooth matrix which is positive definite under some normalization which we will specify later, and u_+ and u_- are two constant states which can be connected by rarefaction waves of the associated inviscid system (1.3).

We assume that (1.4) is strictly hyperbolic, i.e., the 2×2 matrix $(\partial f/\partial u)$ has real and distinct eigenvalues: $\lambda_1(u) < \lambda_2(u)$, with corresponding right and left eigenvectors $r_i(u)$ and $L_i(u)$ satisfying

$$\frac{\partial f}{\partial u} r_i(u) = \lambda_i(u) r_i(u),$$

$$L_i(u) \frac{\partial f}{\partial u} = \lambda_i(u) L_i(u), \qquad (1.6)$$

$$L_i(u) r_j(u) = \delta_{ij}, \qquad i, j = 1, 2.$$

We will use the same notations as in [1]; thus we define the 2×2 matrices I, Λ, L, R, and A, respectively, by $I =$ identity, $\Lambda = \text{diag}(\lambda_1, \lambda_2)$, $L = (L_1, L_2)'$, $R = (r_1, r_2)$, and $A = (\partial f/\partial u)$. In these terms, then (1.6) says that $LAR = \Lambda$, $LR = I$. Also, we denote by δ the distance between u_+ and u_- in $\mathbb{R}^2$.

We require that the system (1.4) is genuinely nonlinear in the sense of Lax [3], i.e.,

$$\nabla \lambda_k(u) r_k(u) = 1, \qquad k = 1, 2. \qquad (1.7)$$

Our requirement on the viscosity matrix $B(u)$ is that LBR is positive definite. Another main technical hypothesis is that (1.1) is strongly coupled in the sense of [1], i.e.,

$$\frac{\partial f_1}{\partial u_2} \cdot \frac{\partial f_2}{\partial u_1} \neq 0, \qquad \forall u \in \Omega. \qquad (1.8)$$

Under the hypothesis that (1.3) is strictly hyperbolic and genuinely nonlinear it is well known [4] that for a fixed state u_-, there exists a region $RR(u_-)$ such that for any state $u_+ \in RR(u_-)$, the Riemann problems (1.4) and (1.5) have a unique solution denoted by $u^r(x, t)$ which can be constructed as follows (see [4]): we can find a unique state on the 1-rarefaction wave curve $R_1(u_-)$, i.e., $\bar{u} \in R_1(u_-)$, such that u_+ is on the 2-rarefaction wave curve $R_2(\bar{u})$ (see Fig. 1.1).
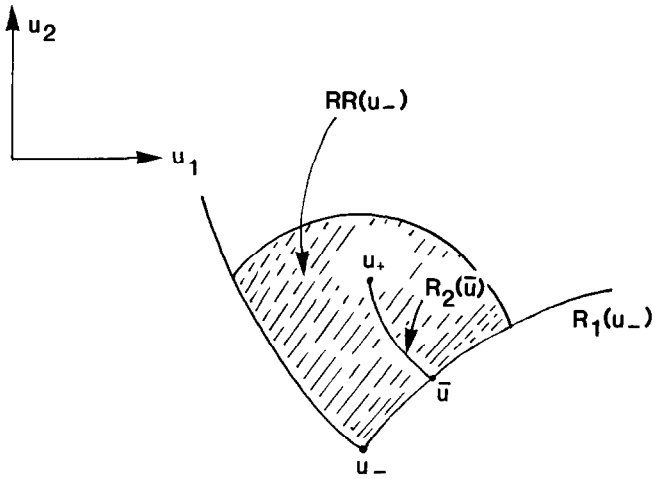
FIGURE 1.1

Let $u_1^r(x, t)$ denote the 1-rarefaction wave connecting u_- to $\bar{u}$ and $u_2^r(x, t)$ denote the 2-rarefaction wave connecting $\bar{u}$ to u_+; then the rarefaction wave $u^r(x, t)$ is a linear superposition of $u_1^r(x, t)$ and $u_2^r(x, t)$ (see Fig. 1.2); i.e.,

$$u^r(x, t) = u_1^r(x, t) + u_2^r(x, t) - \bar{u}. \tag{1.9}$$

Our main result in this paper asserts that $u^r(x, t)$ is a global attractor for the viscous system (1.1); i.e., the following theorem holds.
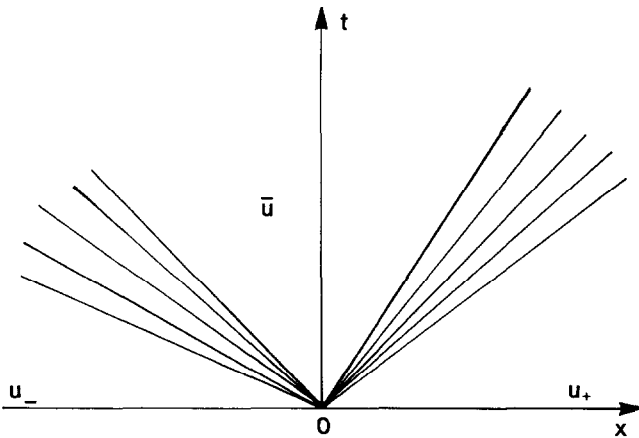

FIGURE 1.2

THEOREM. *Suppose that the system* (1.2) *is strictly hyperbolic, genuinely nonlinear, and strongly coupled. Assume that* $u_+ \in RR(u_-)$ *and that* δ *is small. Then the unique rarefaction wave* $u^r(x, t)$ *constructed in* (1.8) *is non-linearly stable; that is, there exists a positive constant* ε *such that if*

$$\|u_0(x) - u_0^r(x)\|_{L^2}^2 + \|u_{0x}\|_{H^1}^2 \leqslant \varepsilon,$$

then there exists a unique smooth global solution (in time) $u(x, t)$ *to* (1.1) *and* (1.2) *which tends to the rarefaction wave* $u^r(x, t)$ *uniformly in* x *as* t *tends to infinity.*

This theorem generalizes our previous result in [1]. In [1] we proved a similar result for the single mode case, that is, the asymptotic state consists only of a single rarefaction wave. However the technique used in [1] cannot be adapted directly to the two-mode case since, among other things, a major technique in [1] in estimating the transverse wave fields—the "vertical estimate"—is not effective enough to deal with the complifications cause by a linear superposition of a 1-rarefaction wave and a 2-rarefaction wave. We will employ Liu's idea to use a weighted characteristic-energy method to overcome this difficulty. The use of the characteristic-energy method is natural since for expansion waves, the viscous terms in (1.1) decay faster than each term on the left hand side, so the hyperbolic nature of the system (1.1) becomes important. In our analysis, we need a somewhat deeper understanding of the expansive nature of the characteristic field of rarefaction waves.

Our result shows that the viscous system (1.1) is time asymptotically equivalent to the inviscid hyperbolic system (1.4) on the level of rarefaction waves. The nonlinear stability of expansion waves for some special systems can be found in Refs. [5–9].

The rest of the paper is organized as follows: in Section 2, we will construct a smooth approximation $U(x, t)$ of the rarefaction wave $u^r(x, t)$ by making use of the inviscid Burger equation and give some preliminaries. Then, since all expansion waves with the same end data are time asymptotically congruent (i.e., they tend to each other in L_∞ norm as t tends to infinity), we need only show that the smooth expansion wave $U(x, t)$ is nonlinearly stable. In Section 3, we carry out the energy estimate for the perturbation of $U(x, t)$ and reduce the problem to the estimates on the transverse wave fields. In Section 4, we estimate the transverse waves by a weighted characteristic-energy method; this is the essential part of this paper. After finishing the stability estimates, the theorem can be proved in a way similar to that in the single mode case [1].

2. Construction of the Smooth Expansion Wave

For the rarefaction wave $u^r(x, t)$ constructed in (1.9), its smooth approximation $U(x, t)$ can be defined by

$$U(x, t) = U_1(x, t) + U_2(x, t) - \bar{u}, \tag{2.1}$$

where $U_i(x, t)$ are defined by the following relations

$$U_1(x, t) \in R_1(u_-), \qquad \lambda_1(U_1(x, t)) = w_1(x, t); \tag{2.2}$$

$$U_2(x, t) \in R_2(\bar{u}), \qquad \lambda_2(U_2(x, t)) = w_2(x, t), \tag{2.3}$$

where $w_i(x, t)$ are the solutions of the following initial value problems for the inviscid Burger equation,

$$
\begin{aligned}
&(w_i)_t + (\tfrac{1}{2}w_i^2)_x = 0, \qquad x \in \mathbb{R}^1, \quad t \geqslant 0, \\
&w_i(x, 0) = w_0^i(x),
\end{aligned}
\qquad i = 1, 2, \tag{2.4}
$$

with

$$w_0^1(x) = \tfrac{1}{2}(\lambda_1(\bar{u}) + \lambda_2(u_-)) + \tfrac{1}{2}(\lambda_1(\bar{u}) - \lambda_2(u_-)) \tanh x, \tag{2.5}$$

$$w_0^2(x) = \tfrac{1}{2}(\lambda_1(u_+) + \lambda_2(\bar{u})) + \tfrac{1}{2}(\lambda_1(u_+) - \lambda_2(\bar{u})) \tanh x. \tag{2.6}$$

By using the implicit function theorem and the characteristic method, it can be easily shown that the above formulations give unique smooth functions,

$$U_i(x, t) = U_i(x_0^i(x, t)), \qquad i = 1, 2, \tag{2.7}$$

where

$$x_0^i = x - w_0^i(x_0^i)t, \qquad i = 1, 2. \tag{2.8}$$

Furthermore, it can be shown that

$$U_{it} + (f(U_i))_x = 0, \qquad i = 1, 2, \tag{2.9}$$

and

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^1} |U_i(x, t) - u_i^r(x, t)| = 0, \qquad i = 1, 2. \tag{2.10}$$

We will call $U(x, t)$ in (2.1) a *smooth expansive wave*, and we will now give some properties of $U(x, t)$. First, we remark that by a change of coordinates (if necessary), without loss of generality, we may assume that

$$\lambda_1(u) < 0 < \lambda_2(u), \qquad \forall u \in \Omega. \tag{2.11}$$

We divide the half plane $t \geq 0$ by the t-axis and denote the first and second quadrant by Ω_2 and Ω_1, respectively, i.e.,

$$\Omega_i = \{(x, t) \mid x \geq 0, t \geq 0\}, \qquad \Omega_2 = \{(x, t) \mid x \geq 0, t \geq 0\}. \qquad (2.12)$$

Then the specific choices of the initial data (2.5), (2.6) and the explicit solution formulas (2.7), (2.8) combined with (2.11) imply that there exist a positive number α and some smooth bounded function $O(1)$ such that

$$\begin{aligned} U_1 - \bar{u} &= O(1)\,\delta \exp[-\alpha(t + x)], & &\text{on } \Omega_2; \\ U_2 - \bar{u} &= O(1)\,\delta \exp[-\alpha(t - x)], & &\text{on } \Omega_1, \end{aligned} \qquad (2.13)$$

where δ, defined as the distance between u_+ and u_- in $\mathbb{R}^2$, is the strength of the rarefaction wave $u^r(x, t)$. Therefore we get from (2.13) that

$$U(x, t) = \begin{cases} U_1(x, t) + E(x, t), & \text{on } \Omega_1; \\ U_2(x, t) + E(x, t), & \text{on } \Omega_2, \end{cases} \qquad (2.14)$$

where

$$E(x, t) = O(1)\,\delta \exp[-\alpha(t + |x|)]. \qquad (2.15)$$

These show that $U_1(x, t)$ and $U_2(x, t)$ are dominant on Ω_1 and Ω_2, respectively. It follows from (2.9) that

$$U_t + (f(U))_x = (E(U))_x, \qquad x \in \mathbb{R}^1, \quad t \geq 0; \qquad (2.16)$$

here $E(U) = f(U) - f(U_1) - f(U_2) + f(\bar{u})$. Making use of (2.14) and Taylor's formula, we can estimate $E(U)$ directly to obtain

$$(E(U))_x = O(1)\,\delta \exp[-\alpha(t + |x|)], \qquad x \in \mathbb{R}^1, \quad t \geq 0; \qquad (2.17)$$

this asserts that a linear superposition of smooth rarefaction waves satisfies the inviscid hyperbolic equation asymptotically. Analogously to (2.14) and (2.17), we also have

$$\frac{\partial U}{\partial x} = \begin{cases} \dfrac{\partial U_1}{\partial x} + E(x, t), & \text{on } \Omega_1, \\[2mm] \dfrac{\partial U_2}{\partial x} + E(x, t), & \text{on } \Omega_2, \end{cases} \qquad (2.18)$$

and for $L \geq 2$,

$$\frac{\partial^L}{\partial x^L} E(U) = O(1)\,\delta \exp[-\alpha(t + |x|)], \qquad x \in \mathbb{R}^1, \quad t \geq 0. \qquad (2.19)$$

It follows from the above discussion and the estimates for the single-mode case [1] that we have the following estimates (which we will need later).

LEMMA 2.1. *For the smooth functions $U(x, t)$ in (2.1), we have:*

(1) $\dfrac{\partial}{\partial x}(\lambda_k(U_k(x, t))) > 0, \qquad \forall x \in \mathbb{R}^1, \quad t \geqslant 0;$ \hfill (2.20)

(2) $\forall p \in [1, +\infty], \; \exists$ *constant* $c_p > 0,$ *such that for all* $t \geqslant 0$

$$\left\| \frac{\partial U}{\partial x} \right\|_{L_p} \leqslant c_p \, \delta^{1/p}(1+t)^{-1+1/p}, \qquad \left\| \frac{\partial U}{\partial x} \right\|_{L_\infty} \leqslant c_\infty \delta; \qquad (2.21)$$

(3) *For any integer* $L \geqslant 2$ *and* $p \in [1, +\infty), \; \exists$ *constant* $c_{p,L} > 0,$ *such that*

$$\left\| \frac{\partial^L U}{\partial x^L} \right\|_{L_p} \leqslant c_{p,L} \min(\delta, (1+t)^{-1}), \qquad \text{for all} \quad t \geqslant 0; \qquad (2.22)$$

(4) *There exists a positive constant* α, *such that for any* $L \geqslant 1$ *and* $p \in [1, +\infty], \; \exists$ *constant* $c'_{p,L}$ *satisfying*

$$\left\| \frac{\partial^L}{\partial x^L} E(U) \right\|_{L_p} \leqslant c'_{p,L} \, \delta e^{-\alpha t}, \qquad \forall t \geqslant 0; \qquad (2.23)$$

(5) *There exists a positive constant* c *independent of* t *such that*

$$\left\| \frac{\partial}{\partial t} U(t) \right\|_{L_\infty} \leqslant c \left\| \frac{\partial}{\partial x} U(t) \right\|_{L_\infty}, \qquad \forall t \geqslant 0. \qquad (2.24)$$

For the details of the proof, see Xin [1] and Matsumura [6].

3. ENERGY ESTIMATES

In what follows, $\|\phi\|_p$, $p = 0, 1, 2, \ldots,$ denotes the usual Sobolev norm of ϕ in H^p, $\|\phi\| \equiv \|\phi\|_0$, $|\phi| \equiv \|\phi\|_{L_\infty}$, and $\|\phi\|_{L_p}$ is the L_p norm of ϕ, $1 \leqslant p < \infty$. Let $u(x, t)$ be a solution of the Cauchy problem (1.1) and (1.2). With $U(x, t)$ given by (2.1), we set

$$u(x, t) = U(x, t) + \phi(x, t); \qquad (3.1)$$

then using (2.16), we see that the Cauchy problem ((1.1) and (1.2)) is equivalent to the following initial value problem:

$$\phi_t + [f'(U)\phi]_x + [Q(U, \phi)]_x$$

$$= [B(U+\phi)\phi_x]_x + [B(U+\phi) U_x]_x - E(U)_x, \qquad x \in \mathbb{R}^1, \quad t > 0; \quad (3.2)$$

$$\phi(x, 0) \equiv \phi_0(x) = u_0(x) - U(x, 0) \in H^2, \qquad\qquad (3.3)$$

where $Q(U, \phi) = f(U+\phi) - f(U) - f'(U)\phi$ satisfies $|Q(U, \phi)| \leqslant c |\phi|^2$ for all x and t if $|\phi|$ is small enough; here c is a positive constant independent of t. From now on, we will use c to denote such a constant. Since (3.2) is a parabolic system, the local existence and uniqueness of solution for (3.2) and (3.3) in the Sobolev space H^2 is standard. In order to get a global existence and stability estimate, we need an a priori estimate on the solution of (3.2) and (3.3). Thus we set the solution space for (3.2) by

$$X(0, T) = \{\phi \in C^0(0, T; H^2); \phi_x(0, T; H^2)\} \qquad\qquad (3.4)$$

with $0 \leqslant T \leqslant +\infty$, and $\phi \in X(0, T)$ is a solution of (3.2) and (3.3). We put

$$N(t) = \sup_{0 \leqslant \tau \leqslant t} \|\phi(\tau)\|_{H^2}, \qquad \forall t \in (0, T], \qquad\qquad (3.5)$$

and we assume $N(T) \leqslant \varepsilon_0$ for some positive constant ε_0. From now on, for ease of notations, we will set

$$L \equiv L(x, t) \equiv L(U) = (L_{ij}), \qquad R \equiv R(x, t) \equiv R(U) = (r_{ij}),$$

$$\Lambda \equiv \Lambda(x, t) \equiv \Lambda(U), \qquad A \equiv A(x, t)\left(\frac{\partial f}{\partial u}(U)\right),$$

$$M \equiv M(x, t) \equiv M(U) \equiv L_x(x, t) R(x, t) = (m_{ij}), \qquad (3.6)$$

$$N \equiv N(x, t) \equiv N(U) \equiv L_1(x, t) R(x, t) = (n_{ij}),$$

$$B \equiv B(x, t) \equiv B(U), \qquad \Delta B \equiv B(U+\phi) - B(U);$$

then (1.5) says that

$$LAR = \Lambda, \qquad LR = I, \qquad\qquad (3.7)$$

With this diagonalization, our requirement on the viscosity matrix is

$$\bar{B} \equiv \bar{B}(x, t) \equiv LBR > 0, \qquad \forall x \in \mathbb{R}^1, \quad t \geqslant 0. \qquad (3.8)$$

Furthermore, one important observation here is that the diagonal elements of the matrix $M\Lambda + N$ are small in the corresponding region Ω_k ($k = 1, 2$) for weak waves, and precisely, we have the following lemma:

LEMMA 3.1. *There is a constant c_0 such that*

$$|\lambda_1 m_{11} + n_{11}| \leqslant c_0 \delta |U_{1x}| + c_0 |U_{2x}|, \qquad \forall (x, t) \in \Omega_1; \qquad (3.9)$$

$$|\lambda_2 m_{22} + n_{22}| \leqslant c_0 \delta |U_{2x}| + c_0 |U_{1x}|, \qquad \forall (x, t) \in \Omega_2. \qquad (3.10)$$

Proof. By (3.6), we have that $M\Lambda + N = L_x R\Lambda + L_t R$, so

$$\lambda_1 m_{11} + n_{11}$$

$$= \left(\frac{\partial}{\partial x} (L_{11}) r_{11} + \frac{\partial}{\partial x} (L_{12}) r_{21} \right) + \left(\frac{\partial}{\partial t} L_{11} \right) r_{11} + \left(\frac{\partial}{\partial t} L_{12} \right) r_{21}$$

$$= \left(\lambda_1 \frac{\partial}{\partial x} L_{11} + \frac{\partial}{\partial t} L_{11} \right) r_{11} + \left(\lambda_i \frac{\partial}{\partial x} L_{12} + \frac{\partial}{\partial t} L_{12} \right) r_{21}; \qquad (3.11)$$

using (2.1), (2.7), and (2.8), we see that

$$\left(\lambda_1 \frac{\partial}{\partial x} L_{11} + \frac{\partial}{\partial t} L_{11} \right) r_{11}$$

$$= r_{11} \frac{dL_{11}}{dU} \left(\lambda_1 \frac{\partial U_1}{\partial x} + \lambda_1 \frac{\partial U_2}{\partial x} + \frac{\partial U_1}{\partial t} + \frac{\partial U_2}{\partial t} \right)$$

$$= r_{11} \frac{dL_{11}}{dU} \left[(\lambda_1 - w_0^1(x_0^1)) \frac{\partial U_1}{\partial x} + (\lambda_1 - w_0^2(x_0^2)) \frac{\partial U_2}{\partial x} \right], \qquad (3.12)$$

It follows from (2.5) that $|\lambda_1 - w_0^1(x_0^1)| \leqslant c\delta$ for some positive constant c, and therefore we may obtain from (3.12) that

$$\left| \lambda_1 \frac{\partial}{\partial x} L_{11} + \frac{\partial}{\partial t} L_{11} \right| \leqslant c_0 \delta |U_{1x}| + c_0 |U_{2x}|, \qquad \text{on } \Omega_1.$$

As we can get a similar estimate for the second term on the right hand side of (3.11), we see that (3.9) follows. In a similar way, one can show that (3.10) holds. ∎

This lemma is quite crucial in our analysis. Now defining $\phi = RV$, we have $V = L\phi$. We multiply (3.2) by L on the left to get

$$V_t + [\Lambda V]_x - (M\Lambda + N) V + L(Q(U, RV))_x$$

$$= L \frac{\partial}{\partial x} [B(U) RV_x] - L \frac{\partial}{\partial x} (BRMV) + L \frac{\partial}{\partial x} (\Lambda BRV_x)$$

$$- L \frac{\partial}{\partial x} [\Lambda BRMV] + L \frac{\partial}{\partial x} \left[B(U + RV) \frac{\partial U}{\partial x} \right] - L[E(U)]_x. \qquad (3.13)$$

We now begin to estimate V. By multiplying (3.13) on the left by V^t and then integrating the resulting equations over $^1 \times [0, t]$, one obtains after integrating by parts several times,

$$\frac{1}{2} \| V(t) \|^2 - \frac{1}{2} \| V(0) \|^2 + \frac{1}{2} \sum_{i=1}^{2} \iint \frac{\partial}{\partial x} (\lambda_1(U)) V_i^2 \, dx \, d\tau$$

$$- \iint V^t(MA + N) V \, dx \, d\tau + \iint V^t L[Q(U, RV)]_x \, dx \, d\tau$$

$$\leqslant - \iint V_x^t(LBR) V_x \, dx \, d\tau - \iint V^t L_x BR V_x \, dx \, d\tau + \iint V_x^t \, LBRMV \, dx \, d\tau$$

$$+ \iint V^t L_x BRMV \, dx \, d\tau - \iint V_x^t L(AB) VV_x \, dx \, d\tau$$

$$+ \iint V^t L(AB) RMV \, dx \, d\tau$$

$$- \iint V^t L_x(AB) RV_x \, dx \, d\tau + \iint V^t L_x(AB) RMV \, dx \, d\tau$$

$$+ \iint V^t L[B(U + RV) U_x]_x \, dx \, d\tau - \iint V^t L[E(U)]_x \, dx \, d\tau. \qquad (3.14)$$

We now estimate each term in (3.14) separately. First, by the hypothesis (3.8), we can find a positive constant B_0 such that

$$- \iint V_x^t(LBR) V_x \, dx \, d\tau \leqslant - B_0 \int_0^t \| V_x(\tau) \|^2 \, d\tau. \qquad (3.15)$$

Next, we estimate the third term on the left hand side of (3.14). For notational convenience, from now on we will denote any smooth function which satisfies the estimate (2.15) by E; then from (2.19), we have

$$\frac{\partial \lambda_1}{\partial x}(x, t) = \begin{cases} \dfrac{\partial}{\partial x} \lambda_i(U_1) + E(x, t), & (x, t) \in \Omega_1; \\[2mm] \dfrac{\partial}{\partial x} \lambda_i(U_2) + E(x, t), & (x, t) \in \Omega_2. \end{cases} \qquad (3.16)$$

We will also set

$$\cdot \Omega_j(t) = \Omega_j \cap \{(x, \tau) \mid 0 \leqslant \tau \leqslant t\}. \qquad (3.17)$$

Using (3.16), we have

$$\left| \frac{1}{2} \sum_{i=1}^{2} \iint \left[\frac{\partial}{\partial x} (\lambda_i(U)) - \frac{\partial}{\partial x} (\lambda_i(U_i)) \right] V_i^2 \, dx \, d\tau \right|$$

$$= \left| \frac{1}{2} \sum_{i=1}^{2} \sum_{j=1}^{2} \iint_{\Omega_j(t)} \left[\frac{\partial}{\partial x} (\lambda_j(U)) - \frac{\partial}{\partial x} (\lambda_i(U_i)) \right] V_i^2 \, dx \, d\tau \right|$$

$$= O(1) \iint_{\Omega_1(t)} \left| \frac{\partial U_1}{\partial x} \right| V_2^2 \, dx \, d\tau + O(1) \iint_{\Omega_2(t)} \left| \frac{\partial U_2}{\partial x} \right| V_1^2 \, dx \, d\tau$$

$$+ O(1) \iint |V|^2 E(x, t) \, dx \, dt; \tag{3.18}$$

thus,

$$\frac{1}{2} \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U)}{\partial x} V_i^2 \, dx \, d\tau$$

$$= \frac{1}{2} \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau$$

$$+ \frac{1}{2} \sum_{i=1}^{2} \iint \left[\frac{\partial}{\partial x} (\lambda_i(U)) - \frac{\partial}{\partial x} (\lambda_i(U_i)) \right] V_i^2 \, dx \, d\tau$$

$$= \frac{1}{2} \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + O(1) \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau$$

$$+ O(1) \iint |V|^2 E(x, t) \, dx \, dt. \tag{3.19}$$

From (2.23), we see that the first sum on the right hand side of (3.19) is a positive term; this is a principal term which we will use to absorb terms involving $U_{ix} V_i^2$ coming from other integrals. Now, we estimate the third term on the left hand side of (3.15). Since

$$\left| \iint V^t (M\Lambda + N) V \, dx \, d\tau \right| \leqslant \left| \sum_{k=1}^{2} \iint (m_{kk} \lambda_k + n_{kk}) V_k^2 \, dx \, d\tau \right|$$

$$+ \left| \sum_{i \neq j} \iint (m_{ij} \lambda_j + n_{ij}) V_i V_j \, dx \, d\tau \right|,$$

we see, using Lemma 3.1, that the first term on the right hand side can be estimated as

$$\left| \sum_{k=1}^{2} \iint (m_{kk}\lambda_k + n_{kk}) V_k^2 \, dx \, d\tau \right|$$

$$= \left| \sum_{k=1}^{2} \sum_{j=1}^{2} \iint_{\Omega_j(t)} \iint (m_{kk}\lambda_k + n_{kk}) V_k^2 \, dx \, d\tau \right|$$

$$\leqslant c_0 \delta \sum_{i=1}^{2} \iint_{\Omega_i(t)} \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + c_0 \iint_{\Omega_1(t)} |U_{2t}| V_2^2 \, dx \, d\tau$$

$$+ O(1) \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau + c_0 \iint_{\Omega_2(t)} |U_{1t}| V_1^2 \, dx \, d\tau$$

$$\leqslant c_0 \delta \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + O(1) \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau$$

$$+ O(1) \iint |V|^2 E(x, t) \, dx \, dt,$$

where we have used (2.19). In the same way, making use of (2.19) and the Cauchy–Schwartz inequality, we obtain

$$\left| \sum_{i \neq j} \iint (m_{ij}\lambda_j + n_{ij}) V_i V_j \, dx \, d\tau \right|$$

$$\leqslant \beta \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + O(1) \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau$$

$$+ O(1) \iint |V|^2 E(x, t) \, dx \, dt,$$

where β is small positive number to be chosen later. Therefore, we have

$$\iint V'(M\Lambda + N) V \, dx \, d\tau$$

$$\leqslant (\beta + c_0 \delta) \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau$$

$$+ O(1) \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau$$

$$+ O(1) \iint |V|^2 E(x, t) \, dx \, dt. \tag{3.20}$$

Next, it follows from Lemma 2.1 that

$$\left| \iint V(E(U))_x \, dx \, d\tau \right| \leqslant c_0 N(t)\delta. \tag{3.21}$$

We denote by χ the sum of the second through seventh terms on the right hand side of (3.13). Then using the facts that $|L_x| \leqslant c_1 |U_x|$, $|L_x BRM| \leqslant c_1 |U_x|^2$, and $|\Delta B| \leqslant c_1 N(t)$, we may conclude by using the Cauchy–Schwartz inequality that

$$|\chi| \leqslant (\beta + c_1 N(t)) \int_0^t \|V_x(\tau)\|^2 \, d\tau + O(1) \iint |U_x|^2 |V|^2 \, dx \, d\tau, \quad (3.22)$$

where β is as before. From (2.24), we have

$$\iint |U_x|^2 |V|^2 \, dx \, d\tau$$

$$\leqslant O(1) \delta \iint |U_x| |V|^2 \, dx \, d\tau$$

$$\leqslant O(1) \delta \left\{ \sum_{i=1}^2 \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau \right\}$$

$$+ O(1) \, \delta N^2(t); \quad (3.23)$$

combining this with (3.22) shows that

$$|\chi| \leqslant (\beta + c_1 N(t)) \int_0^t \|V_x(\tau)\|^2 \, d\tau$$

$$+ O(1) \delta \left\{ \sum_{i=1}^2 \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau \right\}$$

$$+ O(1) \, \delta N^2(t). \quad (3.24)$$

Straightforward calculations, using the Cauchy–Schwartz inequality, give

$$\iint V' L [B(U + RV) U_x]_x \, dx \, d\tau$$

$$\leqslant \beta \int_0^t \|V_x(\tau)\|^2 \, d\tau + O(1) \iint |U_x|^2 |V|^2 \, dx \, d\tau$$

$$+ O(1) \iint |V'| (|U_x|^2 + |U_{xx}|^2) \, dx \, d\tau. \quad (3.25)$$

By using the Sobolev and Young inequalities together with Lemma 2.1, it can be shown that (see [1])

$$\iint |V'| (|U_x|^2 + |U_{xx}|^2) \, dx \, d\tau$$

$$\leqslant cN(t)^{1/2} \int_0^t \|V_x(\tau)\|^2 \, d\tau + cN(t)^{1/2} \delta^{1/6}. \quad (3.26)$$

Combining (3.25), (3.26), and (3.23) yields

$$\iint V'L[B(U + RV) U_x]_x \, dx \, d\tau$$

$$\leqslant (\beta + cN(t)^{1/2}) \int_0^t \|V_x(\tau)\|^2 \, d\tau$$

$$+ O(1)\delta \left\{ \sum_{i=1}^2 \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau \right\}$$

$$+ O(1)\, \delta N^2(t) + cN(t)^{1/2}\, \delta^{1/6}. \tag{3.27}$$

Concerning the nonlinear term on the left hand side of (3.13), we may follows the same argument for the single-mode case (see [1]), with a few modifications, to show the following lemma:

LEMMA 3.2. *There exist positive constants* $\varepsilon_1 (\leqslant \varepsilon_0)$ *and* $O(1)$ *which are independent of* T *and* δ, *such that if* $N(t) + \delta \leqslant \varepsilon_i$, *then*

$$\left| \iint V'L[Q(U, RV)]_x \, dx \, d\tau \right|$$

$$\leqslant O(1) \left\{ N(0) \|V(0)\|^2 + N(t) \|V(t)\|^2 + N(t) \int_0^t \|V_x(\tau)\|^2 \, d\tau \right.$$

$$+ (\delta + N(t)) \sum_{i=1}^2 \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau$$

$$\left. + \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau + \delta N^2(t) + N(t)^{1/2}\, \delta^{1/6} \right\}. \tag{3.28}$$

Proof. Note that in the estimate of the nonlinear terms for the single-mode case (Lemma 3.1 in Xin [1]), we made essential use of the assumption that the system is strongly coupled (1.8), but we did not use the structure of the rarefaction wave; so the proof of Lemma 3.1 in [1] gives that if $N(t) + \delta$ is small, then

$$\left| \iint V'L[Q(U, RV)]_x \, dx \, d\tau \right|$$

$$\leqslant O(1) \left\{ N(0) \|V(0)\|^2 + N(t) \|V(t)\|^2 + ((\delta + N(t)) \iint \left| \frac{\partial U}{\partial x} \right| |V|^2 \, dx \, d\tau \right.$$

$$\left. + N(t) \int_0^t \|V_x(\tau)\|^2 \, d\tau + \delta N^2(t) + N(t)^{1/2}\, \delta^{1/6} \right\}.$$

Also, since

$$\iint \left| \frac{\partial U}{\partial x} \right| |V|^2 \, dx \, d\tau$$

$$\leqslant O(1) \left\{ \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau + \delta N^2(t) \right\},$$

we see that (3.28) follows. ∎

Collecting the various results that we have thus far obtained, we may conclude that for small $N(t) + \delta$, the following inequality holds:

$$\frac{1}{2} \| V(t) \|^2 + \frac{1}{2} \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau + B_0 \int_0^t \| V_x(\tau) \|^2 \, d\tau$$

$$\leqslant \frac{1}{2} \| V(0) \|^2 + O(1) [N(0) \| V(0) \|^2 + N(t) \, UV(t) \|^2]$$

$$+ [2\beta + O(1) N^{1/2}(t) + O(1) N(t)] \int_0^t \| V_x(\tau) \|^2 \, d\tau$$

$$+ [\beta + O(1)\delta + O(1)N(t)] \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{\partial x} V_i^2 \, dx \, d\tau$$

$$+ O(1) \left\{ \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau + \delta N^2(t) + N^{1/2}(t)[\delta^{1/2} + \delta^{1/6}] \right\}.$$

$$(3.29)$$

Now we choose β such that $\beta = \min\{\frac{1}{4}, B_0/8\}$; then (3.29) yields the following main energy estimate:

LEMMA 3.3. *There exist positive constants* $\varepsilon_2 (\leqslant \varepsilon_1)$ *and* $O(1)$ *independent of* T *and* δ, *such that if* $N(t) + \delta \leqslant \varepsilon_2$, *then the following estimate holds:*

$$\| V(t) \|^2 + \sum_{i=1}^{2} \iint \frac{\partial \lambda_i(U_i)}{dx} V_i^2 \, dx \, d\tau + \int_0^t \| V_x(\tau) \|^2 \, d\tau$$

$$\leqslant O(1)\{ \| V(0) \|^2 + \delta^{1/6} \} + O(1) \sum_{i \neq j} \iint_{\Omega_j(t)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau. \qquad (3.30)$$

This lemma gives estimates on each wave in its dominant region. It remains to estimate the transverse wave on each region; i.e., we need to estimate the integrals $\iint_{\Omega_j(t)} |U_{jx}| V_i^2 \, dx \, d\tau$ for $j \neq i$. These will be given in the next section by a weighted characteristic-energy method.

4. CHARACTERISTIC-ENERGY ESTIMATES

In this section, we will estimate integrals $\iint_{\Omega_j(t)} |U_{jx}| V_i^2 \, dx \, d\tau$ for $i \neq j$. Note that in the single-mode case, the smooth rarefaction wave depends on one variable, so that a "vertical" energy analysis can apply to estimate the nonprimary waves (see [1]). However, for the twomode case, due to the complication caused by the linear superposition of two elementary rarefaction waves, we need a more effective method to deal with transverse wave fields; this will be done by using the characteristic-energy method introduced by Liu [2].

First, we study the characteristic fields and associated characteristic curves for the smooth expansive wave constructed in Section 2. Let $\lambda_i(x, t) = \lambda_i(U(x, t))$ be the i-characteristic field; then $\lambda_i(x, t)$ is a smooth and uniformly bounded function together with its derivatives. Also from (2.4)–(2.8) and (2.14) we have

$$\lambda_i(x, t) = \begin{cases} \lambda_i^1(x, t) + E(x, t), & (x, t) \in \Omega_1; \\ \lambda_i^2(x, t) + E(x, t); & (x, t) \in \Omega_2, \end{cases} \tag{4.1}$$

where $\lambda_i^k(x, t) = \lambda_i(w_0^k(x_0^k(x, t)))$ (see (2.5)–(2.8)). The corresponding ith-characteristic curves $x_i = x(\xi, t)$ satisfy the ordinary differential equation

$$\frac{\partial x_i(\xi, t)}{\partial t} = \lambda_i(x_i(\xi, t), t), \qquad t \geqslant 0, \ \xi \in \mathbb{R}^1;$$

$$x_i(\xi, t_0 = \xi, \qquad\qquad\qquad \xi \in \mathbb{R}^1, \tag{4.2}$$

where t_0 is any conveniently chosen nonnegative constant. It follows from (4.2) that

$$\frac{\partial x_i}{\partial \xi} = \exp \left\{ \int_{t_0}^t \frac{\partial \lambda_i(x_i(\xi, \tau), \tau)}{\partial x} \, d\tau \right\}. \tag{4.3}$$

Now, we rewrite (3.12) as differential equations along characteristics:

$$\frac{d}{dt_i} V_i = [(M\Lambda - N) V]_i - \lambda_{ix} V_i - [L(Q(U, RV))_x]_i [\text{R.H.S.}]_i, \quad i = 1, 2.$$

$$\frac{d}{dt_i} \equiv \frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x}, \tag{4.4}$$

where [R.H.S.] denotes the sum of all terms on the right hand side of

(3.12). We now multiply (4.4) by V_i and integrate the resulting equations along the ith-characteristic curve to obtain

$$V_i^2(x_i(\xi, t), t)$$

$$= V_i^2(x_i(\xi, t'), t')$$

$$+ \int_{t'}^t 2V_i\{[M\Lambda - N]_i - \lambda_{ix}V_i - [LQ(U, RV)_x]_i + [\text{R.H.S.}]\} \, dx \, d\tau. \tag{4.5}$$

Given a domain Ω, we denote by Γ_1 (or Γ_2) the part of the boundary where the ith-characteristic curve enters (or leaves) Ω; see Fig. 4.1 below for $i = 2$. Also along any given ith-characteristic curve $(x_i(\xi, t), t)$, we denote by $t = t_i(\xi)$ the time it enters Ω through Γ_i, and by $t = t_2(\xi)$ the time it leaves Ω through Γ_2. Now, we multiply (4.5) by a piecewise smooth positive function K defined on Γ_2 as $(x_i(\xi, t'), t')$ moves along Γ_i and integrate to obtain

$$\int_{\Gamma_2} K(x_i(\xi, t_2(\xi)), t_2(\xi)) V_i^2(x_i(\xi, t_2(\xi)), t_2(\xi)) \, ds(x_i(\xi, t_2(\xi)), t_2(\xi))$$

$$= \int_{\Gamma_1} K(x_i(\xi, t_2(\xi)), t_2(\xi)) V_i^2(x_i(\xi, t_1(\xi)), t_1(\xi))$$

$$\times \frac{ds(x_1(\xi, t_2(\xi)), t_2(\xi))}{ds(x_1(\xi, t_1(\xi)), t_1(\xi))} \, ds(x_1(\xi, t_1(\xi)), t_1(\xi))$$

$$+ \iint_\Omega K(x_i(\xi, t_2(\xi)), t_2(\xi))[RH_1 + RH_2 + RH_3](x_i(x, t), t)$$

$$\times \frac{ds(x_i(\xi, t_2(\xi)), t_2(\xi))}{dx_i(\xi, t)} \, dx_i(\xi, t) \, dt, \tag{4.6}$$

where ds is the arc length element, and

$$RH_1 \equiv 2V_i[(M\Lambda - N)V]_i + 2\lambda_{ix}V_i^2, \tag{4.7}$$

$$RH_2 \equiv 2V_i[LQ(U, RV)_x]_i, \tag{4.8}$$

$$RH_3 \equiv 2V_i[\text{R.H.S.}]_i. \tag{4.9}$$

We also set

$$\bar{L} \equiv \bar{L}(x_i(\xi, t)) = K(x_i(\xi, t_2(\xi)), t_2(\xi)) \frac{ds(x_i(\xi, t_2(\xi)), t_2(\xi))}{dx_i(\xi, t)}. \tag{4.10}$$
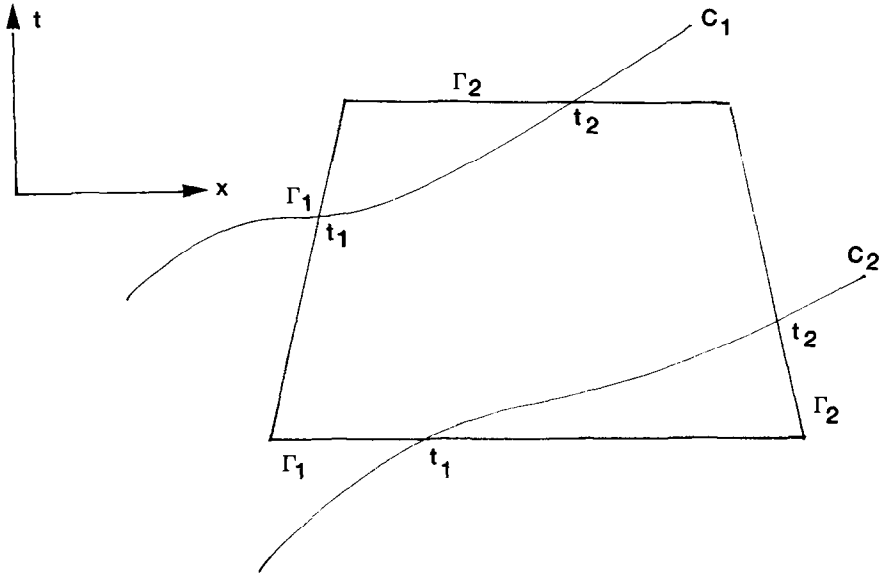
FIGURE 4.1

In the application of formula (4.6), the major step is to choose an appropriate weighted function $K(x, t)$ to simplify the formula. In our analysis, we will see that we can choose $K(x, t)$ such that $\bar{L}$ in (4.10) does not contain terms which depend on the variable $(x_i(\xi, t_2(\xi)), t_2(\xi))$, and this will enable us quite easily to estimate integrals in (4.6).

Now we are ready to estimate the transverse wave fields. First, we procede to bound integral $\iint_{\Omega_1(t)} |U_{1x}| V_2^2 \, dx \, d\tau$. For this, we apply formula (4.6) with Ω chosen as follows: let $T_1 > 0$ be fixed, and set

$$X = \{x_0 | \{(x, t); x = x_0 + w_0^1(x_0)t\} \cap \Omega_1(T_1) \neq \emptyset\}. \qquad (4.11)$$

Now fix $x_0 \in X$, and let $T_0(\geq 0)$ be the time when the line $x = x_0 + w_0^1(x_0)t$ intersects with $x = 0$; then we define (see Fig. 4.2)

$$\Omega = \{(x, t) | T_0 \leq t \leq T_1; x \leq x_0 + w_0^1(x_0)t\} \subset \Omega_1(T_1). \qquad (4.12)$$

Recalling that $\lambda_1 < 0 < \lambda_2$ (see (2.11)), we have that a 2-characteristic curve enters Ω through

$$\Gamma_1 = \{(x, T_0) \mid x \leq x_0 + w_0^1(x_0) T_0\} \qquad (4.13)$$

and leaves Ω through two line segments Γ_{21} and Γ_{22}:

$$\Gamma_{21} = \{(x, T_1) \mid x \leq x_0 + w_0^1(x_0) T_1\};$$
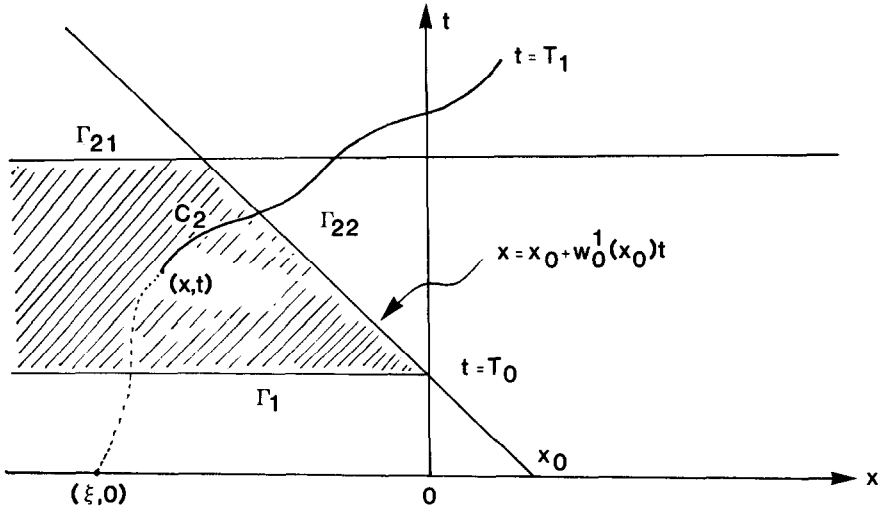$$\Gamma_{22} = \{(x, t) \mid x = x_0 + w_0^1(x_0)t\}. \qquad (4.14)$$

FIGURE 4.2

We now consider a 2-characteristic curve $C_2 = \{(x_2(\xi, \tau), \tau); t \leqslant \tau \leqslant t_2(\xi)\}$ with $x_2(\xi, t) = x$ and $(x, t) \in \Omega$. We first compute

$$\frac{ds(x_2(\xi, t_2(\xi)), t_2(\xi))}{dx_2(\xi, t)}.$$

Since $\Omega \subset \Omega_1(T_1)$, from (4.1) and (4.3), we can get

$$\frac{\partial x_2(\xi, t)}{\partial \xi} = \exp\left\{\int_{t_0}^t \frac{\partial \lambda_2(x_2(\xi, \tau), \tau)}{\partial x} \, d\tau\right\}$$

$$= \exp\left\{\int_{t_0}^t \frac{\partial E(x_2, \tau)}{\partial x} \, d\tau\right\} \exp\left\{\int_{t_0}^t \frac{\partial \lambda_2^1(x_2, \tau)}{\partial x} \, d\tau\right\}$$

$$= \exp\left\{\int_{t_0}^t \frac{\partial E(x_2, \tau)}{\partial x} \, d\tau\right\} \exp\left\{\int_{t_0}^t \frac{\partial \lambda_2(w_0^1(x_0^1))}{\partial x} \frac{1}{1 + w_0^1(x_0^1)\tau} \, d\tau\right\}$$

$$= \frac{\lambda_2(x_0^1(x_2, t)) - w_0^1(x_2, t)}{\lambda_2(w_0^1(x_2, t_0)) - w_0^1(x_2, t_0)} \exp\left\{\int_{t_0}^t \frac{\partial E(x_2, \tau)}{\partial x} \, d\tau\right\}$$

$$\times \exp\left\{\int_{x_0^1(t_0)}^{x_0^1(t)} \frac{[w_0^1(w)]_x}{\lambda_2(w_0^1(x)) - w_0^1(x)} \, dx\right\}, \tag{4.15}$$

where we have used (2.4)–(2.8) and the fact that

$$\frac{\partial x_0^1}{\partial x} = \frac{1}{1 + w_0^1(x_0^1)\tau} = \frac{1}{\lambda_2(w_0^1(x_0^1)) - w_0^1(x_0^1)} \frac{\partial x_0^1}{\partial \tau}. \tag{4.16}$$

Note that on Γ_{21}, $dt_2(\xi) = 0$, so we have along Γ_{21}

$$\frac{ds(x_2(\xi, t_2(\xi)), t_2(\xi))}{dx_2(\xi, t)}$$

$$= \frac{dx_2(\xi, t_2(\xi))}{dx_2(\xi, t)}$$

$$= \frac{(\lambda_2 - w_0^1)(x_2(\xi, t_2(\xi)), t_2(\xi))}{(\lambda_2^1 - w_0^1)(x_2(\xi, t), t)}$$

$$\times \exp\left\{ \int_t^{t_2} \frac{\partial E(x_2, \tau)}{\partial x} d\tau + \int_{x_0^1(t)}^{x_0^1(t_2)} \frac{[w_0^1(x)]_x}{\lambda_2(w_0^1(x)) - w_0^1(x)} dx \right\}. \quad (4.17)$$

For notational convenience, we will denote the first and second factors on the right-hand side of (4.17) by $I_1(x_2, t, t_2, \xi)$ and $I_2(x_2, t, t_2, \xi)$, respectively. Next, on another part of $\partial\Omega$, Γ_{22}, we have from $x_2 = x_0 + w_0^1(x_0) t_2(\xi)$ that

$$\frac{\partial t_2(\xi)}{\partial \xi} = - \frac{1}{\lambda_2(x_2(\xi, t_2(\xi)), t_2(\xi)) - w_0^1(x_0)} \frac{\partial x_2}{\partial \xi}\bigg|_{t = t_2(\xi)}$$

Using this and (4.15), we can show in a way similar to that in (4.17) that along Γ_{22}

$$\frac{ds(x_2(\xi, t_2(\xi)), t_2(\xi))}{dx_2(\xi, t)}$$

$$\frac{(1 + [w_0^1(x_0)]^2)^{1/2}}{\lambda_2(x_2(\xi, t_2(\xi)), t_2(\xi)) - w_0^1(x_0)} \times I_1(x_2, t, t_2, \xi) \times I_2(x_2, t, t_2, \xi). \quad (4.18)$$

Thus we have shown that

$$\bar{L}(x_2(\xi, t), t) = \begin{cases} K(x_2(\xi, t_2(\xi)), t_2(\xi)) \, I_1(x_2, t, t_2, \xi) \, I_2(x_2, t, t_2, \xi) \\[4pt] \qquad \text{for} \quad (x_2(\xi, t_2(\xi)), t_2(\xi)) \in \Gamma_{21}, \\[8pt] K(x_2(\xi, t_2(\xi)), t_2(\xi)) \, I_1(x_2, t, t_2, \xi) \, I_2(x_2, t, t_2, \xi) \\[4pt] \qquad \times \frac{(1 + [w_0^1(x_0)]^2)^{1/2}}{\lambda_2(x_2(\xi, t_2(\xi)), t_2(\xi)) - w_0^1(x_0)} \\[8pt] \qquad \text{for} \quad (x_2(\xi, t_2(\xi)), t_2(\xi)) \in \Gamma_{22}. \end{cases} \quad (4.19)$$

Since $\lambda_1 < 0 < \lambda_2$, and along a 2-characteristic C_2, we have $x_0^1 = x_2 - w_0^1(x_0^1) t$, so as $t \to +\infty$, along C_2, $x_2 \to +\infty$, and $x_0^1 \to +\infty$ (see Fig. 4.2). Therefore, we conclude from (2.15) and (2.5) that

$$I_3(x_2, t_2) \equiv \int_{t_2}^{+\infty} \frac{\partial E(x_2, \tau)}{\partial x} d\tau + \int_{x_0^1(t_2)}^{+\infty} \frac{[w_0^1(x)]_x}{\lambda_2(w_0^1(x)) - w_0^1(x)} dx \quad (4.20)$$

is a smooth and bounded function for $(x_2(\xi, t_2(\xi)), t_2(\xi))$. Now we define the weight function K by

$$
K(x_2(\xi, t_2(\xi)), t_2(\xi)) = \begin{cases} \dfrac{\exp\{I_3(x_2(\xi, t_2(\xi)), t_2(\xi))\}}{(\lambda_2 - w_0^1)(x_2(\xi, t_2(\xi)), t_2(\xi))} \\[2pt] \qquad \text{for} \quad (x_2(\xi, t_2(\xi)), t_2(\xi)) \in \Gamma_{21}, \\[8pt] \dfrac{\lambda_2(x_2(\xi, t_2(\xi)), t_2(\xi)) - w_0^1(x_0)}{(1 + [w_0^1(x_0)]^2)^{1/2}} \\[6pt] \qquad \times \dfrac{\exp\{I_3(x_2(\xi, t_2(\xi)), t_2(\xi))\}}{(\lambda_2 - w_0^1)(x_2(\xi, t_2(\xi)), t_2(\xi))}, \\[6pt] \qquad \text{for} \quad (x_2(\xi, t_2(\xi)), t_2(\xi)) \in \Gamma_{22}. \end{cases}
\tag{4.21}
$$

It follows from (4.21) that K is piecewise smooth, positive, and bounded below. Substituting (4.21) into (4.19) yields

$$
\bar{L}(x_2(\xi, t), t) = \frac{\exp\{I_3(x_2(\xi, t), t)\}}{(\lambda_2 - w_0^1)(x_2(\xi, t), t)}.
\tag{4.22}
$$

Noting that

$$
\frac{\partial}{\partial x_2} \int_t^{+\infty} \frac{\partial E}{\partial x}(x_2(\xi, \tau), \tau)\, d\tau = (\lambda_2)^{-1} \frac{\partial E}{\partial x}(x_2, t),
$$

we can easily establish the following lemma

LEMMA 4.1.   *The piecewise smooth function K defined by (4.21) satisfies the following conditions:*

(1) $|\bar{L}(x_2(\xi, t), t)| = O(1), \qquad |K(x_2(\xi, t_2(\xi)), t_2(\xi))| = O(1),$ (4.23)

(2) $\left|\dfrac{\partial \bar{L}}{\partial x}(x_2(\xi, t), t)\right| = O(1)\left|\dfrac{\partial \lambda_2(U_1)}{\partial x}\right| + E(x_2, t),$ (4.24)

(3)   *there exists a positive constant C_1 such that*

$$
K(x_2(\xi, t_2(\xi)), t_2(\xi)) \geqslant C_1, \qquad \forall (x_2(\xi, t_2(\xi)), t_2(\xi)) \in \Gamma_2,
\tag{4.25}
$$

where $O(1)$ and C_1 are independent of x and t.

Thus, applying formula (4.6) with $i = 2$ and using (4.23), (4.25) and noting that the kernel in the integral along Γ_1 is similar to $\bar{L}$, we can treat this kernel is a manner similar to what we have done before for $\bar{L}$ to arrive at the following estimate:

$$C_1 \int_{\Gamma_{21}} |V_2(x, T_1)|^2 \, dx + C_1 \int_{T_0}^{T_1} |V_2(x_0 + w_0^1(x_0)t, t)|^2 \, d\tau$$

$$\leqslant O(1) \int_{-\infty}^{x_0 + w_0^1(x_0)T_0} |V_2(x, T_0)|^2 \, dx$$

$$+ \iint_\Omega \bar{L}[RH_1 + RH_2 + RH_3](x_2, t) \, dx_2 \, dt$$

$$\leqslant O(1) \int |V_2(x, T_0)|^2 \, dx + \iint_\Omega \bar{L}[RH_1 + RH_2 + RH_3](x_2, t) \, dx_2 \, dt.$$

$$(4.26)$$

We now estimate the last term on the right hand side of (4.26). The treatment of the integrals involving RH_1 is equite easy; in fact, from (4.7), (3.15), and (4.23), we have

$$\left| \iint_\Omega \bar{L}[RH_1](x_2, t) \, dx_2 \, dt \right|$$

$$\leqslant O(1) \iint_\Omega |U_{1x}| \, V_2^2 \, dx_2 \, dt$$

$$+ O(1) \iint_\Omega |U_{1x}| \, V_1^2 \, dx_2 \, dt + O(1) \, N(T_1)^2 \iint_\Omega E(x_2, t) \, dx_2 \, dt$$

$$\leqslant O(1) \left\{ \iint_{\Omega_1(T_1)} |U_{1x}| \, V_1^2 \, dx_2 \, dt + \iint_{\Omega_1(T_1)} |U_{1x}| \, V_1^2 \, dx_2 \, dt \right.$$

$$\left. + N(T_1)^2 \iint E(x_2, t) \, dx_2 \, dt \right\}. \qquad (4.27)$$

Applying integration by parts, Cauchy's inequality, and Lemma 4.1, we can estimate the integrals involving RH_3 as

$$L_1 \equiv \left| \iint_\Omega 2V_2 \bar{L} L_{2j}(B_{jm} R_{mn} V_{nx})_x \, dx \, dt \right|$$

$$\leqslant \left| \int_{\Gamma_{22}} 2\bar{L} L_{2j} B_{jm} R_{mn} V_2 V_{nx} \, dt \right| + \Sigma \iint_\Omega 2(V_2 \bar{L}_{2j})_x B_{jm} R_{mn} V_{nx} \, dx \, dt \right|$$

$$\leqslant \gamma \int_{T_0}^{T_1} V_2^2(x_0 + w_0^1(x_0)t, t) \, dt + O(1) \int_{T_0}^{T_1} V_x^2(x_0 + w_0^1(x_0)t, t) \, dt$$

$$+ O(1) \left\{ \iint_\Omega |V_x|^2 \, dx \, dt + \iint_{\Omega_1(T_1)} |U_{1x}| \, V_2^2 \, dx \, dt \right.$$

$$\left. + N(T_1)^2 \iint E(x, t) \, dx \, dt \right\} \equiv \alpha_1, \qquad (4.28)$$

where γ is a small positive number to be chosen later, Similarly

$$L_2 \equiv \left| \iint_\Omega 2V_2 \bar{L} L_{2j} [B_{jm}(RM)_{mn} V_n]_x \, dx \, dt \right|$$

$$\leqslant \left| \int_{\Gamma_{22}} 2\bar{L} L_{2j} B_{jm}(RM)_{mn} V_2 V_n \, dt \right| + \left| \iint_\Omega 2(V_2 \bar{L} L_{2j})_x B_{jm}(RM)_{mn} V_n \, dx \, dt \right|$$

$$\leqslant O(1) \delta \left\{ \int_{T_2}^{T_1} V_2^2(x_0 + w_0^1(x_0)t, t) \, dt + \int_{T_0}^{T_1} V_1^2(x_0 + w_0^1(x_0)t, t) \, dt \right\}$$

$$+ O(1) \left\{ \iint_\Omega |V_x|^2 \, dx \, dt + \iint_{\Omega_1(T_i)} |U_{1x}| \, V_1^2 \, dx \, dt \right.$$

$$\left. + \iint_{\Omega_1(T_1)} |U_{1x}| \, V_2^2 \, dx \, dt + N(T_1)^2 \iint E(x, t) \, dx \, dt \right\} \equiv \alpha_2, \qquad (4.29)$$

where we have used the fact that $|M| \leqslant C |U_x| \leqslant C_0 \delta$. In a similar way, we have

$$L_3 \equiv \left| \iint_\Omega 2V_2 \bar{L} L_{2j} [\Delta B_{jm}(R_{mn} V_{nx} - (RM)_{mn} V_n)]_x \, dx \, dt \right| \leqslant \alpha_1 + \alpha_2; \qquad (4.30)$$

also

$$L_4 \equiv \left| \iint_\Omega 2V_2 \bar{L} (LE(U)_x)_2 \, dx \, dt \right| \leqslant N(T_1) \iint E(x, t) \, dx \, dt. \qquad (4.31)$$

Using the Cauchy, Sobololev, and Young inequalities as in (3.23) and (3.24), we may obtain

$$L_2 \equiv \left| \iint_\Omega 2V_2 \bar{L} L_{2j} [B(U + RV)_{jm} U_{mx}]_x \, dx \, dt \right|$$

$$\leqslant O(1) \iint_\Omega |V_2| \, (|U_x|^2 + |U_x|^2 |V| + |U_x| |V_x| + |U_{xx}|) \, dx \, dt$$

$$\leqslant O(1) \iint_\Omega [|V_x|^2 + |U_x|^2 |V|^2 + |V| \, (|U_x|^2 + U_{xx}|) \, dx \, dt$$

$$\leqslant O(1) \delta \iint_{\Omega_1(T_1)} |U_{1x}| \, |V|^2 \, dx \, dt$$

$$+ O(1) \iint_\Omega |V_x|^2 \, dx \, dt + N(T_1)^{1/2} \delta^{1/6} + N(T_1)^2 \iint E(x, t) \, dx \, dt \right\} \equiv \alpha_3.$$

Combining this with (4.28)–(4.31) yields

$$\left| \iint_{\Omega} \bar{L}[RH_3](x_2, t)\, dx_2\, dt \right| \leqslant \alpha_1 + \alpha_2 + \alpha_3. \tag{4.32}$$

It remains to estimate the nonlinear terms RH_2. For this, we have the following lemma:

LEMMA 4.2.   *There exists a positive constant $O(1)$ independent of x and t such that $N(T) + \delta$ is small*

$$\left| \iint_{\Omega} 2\bar{L}V_2[LQ(U, RV)_x]_2\, dx\, dt \right|$$

$$\leqslant O(1) \left\{ N(T_0) \, \| V(T_0) \|^2 + N(T_1) \, \| V(T_1) \|^2 \right.$$

$$+ (\delta + N(T_1)) \int_{T_0}^{T_1} | V(x_0 + w_0^1(x_0)t, t)|^2 \, dt$$

$$+ \int_{T_0}^{T_1} | V_x(x_0 + w_0^1(x_0)t, t)|^2 \, dt + \iint_{\Omega_1(T_1)} |U_{1x}| \, |V|^2 \, dx\, dt$$

$$\left. + \iint_{\Omega} |V_x|^2 \, dx\, dt + N(T_1)^{1/2} \delta^{1/6} + N(T_1)^2 \iint E(x, t)\, dx\, dt \right\}. \tag{4.33}$$

Proof.   Integrating by parts leads to

$$\left\| \iint_{\Omega} 2\bar{L}V_2[LQ(U, RV)_x]_2\, dx\, dt \right|$$

$$\leqslant \left| \int_{\Gamma_{22}} 2\bar{L}V_2 L_{2j} Q_j \, dt \right| + \left| \iint_{\Omega} 2(\bar{L}L_{2j})_x \, Q_j V_2 \, dx\, dt \right|$$

$$+ \left| \iint_{\Omega} 2\bar{L}L_{2j} Q_j V_{2x} \, dx\, dt \right|$$

$$\leqslant O(1) N(T_1) \left\{ \int_{T_0}^{T_1} V_2^2(x_0 + w_0^1(x_0)t, t) \, dt + \iint_{\Omega_1(T_1)} |U_{1x}| \, |V|^2 \, dx\, dt \right\}$$

$$+ \left| \iint_{\Omega} 2\bar{L}L_{2j} Q_j V_{2x} \, dx\, dt \right|.$$

By Lemma 4.1, we have that $(2\bar{L}L_{2j})_x = O(1) U_{1x} + E(x, t)$. Therefore, the last term on the right above can be estimated in a way similar to that in the single-mode case (see Xin [1, pp. 55–64]) to obtain (4.33); details are omitted.   ∎

Collecting (4.27), (4.32), and (4.33) gives

$$\left| \iint_\Omega \bar{L}[RH_1 + RH_2 + RH_3](x_2, t) \, dx_2 \, dt \right|$$

$$\leqslant 2\gamma \int_{T0}^T V_2^2(x_0 + w_0^1(x_0)t, t) \, dt + O(1)(\delta + N(T_1))$$

$$\times \int_{T_0}^{T_1} |V(x_0 + w_0^1(x_0)t, t)|^2 \, dt$$

$$+ O(1) \left\{ \iint_\Omega |V_x|^2 \, dx \, dt + \int_{T_0}^{T_1} |V_x(x_0 + w_0^1(x_0)t, t)|^2 \, dt \right.$$

$$+ \iint_{\Omega_1(T_1)} |U_{1x}| \, |V|^2 \, dx \, dt + N(T_1)^{1/2} \delta^{1/6} + N(T_0) \|V(T_0)\|^2$$

$$\left. + N(T_1) \|V(T_1)\|^2 + \delta \right\}. \tag{4.34}$$

Now choose $2\gamma = C_1/4$; then (4.26) and (4.34) yield (discarding the first nonnegative term in (4.26))

$$\int_{T_0}^{T_1} V_2^2(x_0 + w_0^1(x_0)t, t) \, dt$$

$$\leqslant O(1) \left\{ N(T_0) \|V(T_0)\|^2 + N(T_1) \|V(T_1)\|^2 \right.$$

$$+ (\delta + N(T_1)) \int_{T_0}^{T_1} |V(x_0 + w_0^1(x_0)t, t)|^2 \, dt$$

$$+ \int_{T_0}^{T_1} |V_x(x_0 + w_0^1(x_0)t, t)|^2 \, dt$$

$$+ \iint_{\Omega_1(T_1)} |U_{1x}| \, |V|^2 \, dx \, dt + \int_{T_0}^{T_1} \|V_x\|^2 \, dx \, dt$$

$$\left. + N(T_1)^{1/2} \delta^{1/6} + \delta \right\}. \tag{4.35}$$

By a change of variable, we see that

$$\iint_{\Omega_1(T_1)} |U_{1x}| \, V_2^2(x, t) \, dx \, dt$$

$$= \int_{-\infty}^{x_0} dx_0 \left| \frac{\partial U_1(x_0)}{\partial x_0} \right| \int_{T_0(x_0)}^{T_1} V_2^2(x_0 + w_0^1(x_0)t, t) \, dt,$$

where $x_0 = \sup X$, and X is given by (4.11). Thus multiplying (4.35) by $(\partial/\partial x_0) U_1(x_0)$ and integrating the resulting inequality with respect to x_0 over $(-\infty, x_0)$, one obtains

$$\iint_{\Omega_1(T_1)} |U_{1x}| \, V_2^2(x, t) \, dx \, dt$$

$$\leqslant O(1) \, \delta \left\{ \sup_{0 \leqslant T_0 \leqslant T_1} \|V(T_0)\|^2 + N(T_1) \, \|V(T_1)\|^2 \right.$$

$$\left. + \int_0^{T_1} \|V_x\|^2 \, dx \, dt + N(T_1)^{1/2} \, \delta^{1/6} + \delta \right\}$$

$$+ O(1)(\delta + N(T_1)) \iint_{\Omega_1(T_1)} |U_{1x}| \, |V|^2 \, dx \, dt. \qquad (4.36)$$

Similarly, we can show that

$$\iint_{\Omega_2(T_1)} |U_{2x}| \, V_1^2(x, t \, dx \, dt$$

$$\leqslant O(1) \, \delta \left\{ \sup_{0 \leqslant T_0 \leqslant T_1} \|V(T_0)\|^2 + N(T_1) \, \|V(T_1)\|^2 \right.$$

$$\left. + \int_0^{T_1} \|V_x\|^2 \, dx \, dt + N(T_1)^{1/2} \, \delta^{1/6} + \delta \right\}$$

$$+ O(1)(\delta + N(T_1)) \iint_{\Omega_2(T_1)} |U_{2x}| \, |V|^2 \, dx \, dt. \qquad (4.37)$$

Adding (4.36) and (4.37) gives

$$\sum_{i \neq j} \iint_{\Omega_j(T_1)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau$$

$$\leqslant O(1) \, \delta \left\{ \sup_{0 \leqslant T_0 \leqslant T_1} \|V(T_0)\|^2 \right.$$

$$\left. + N(T_1 \, \|V(T_1)\|^2 + \int_0^{T_1} \|V_x(t)\|^2 \, dt + N(T_1)^{1/2} \, \delta^{1/6} + \delta \right\}$$

$$+ O(1)(\delta + N(T_1)) \left\{ \sum_{i \neq j} \iint_{\Omega_j(T_1)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau \right.$$

$$\left. + \sum_{i=1}^{2} \iint_{\Omega_i(T_1)} \left| \frac{\partial U_i}{\partial x} \right| V_i^2 \, dx \, d\tau \right\}. \qquad (4.38)$$

We estimate the first two terms on the right hand side of (4.38) by varying t in (3.30) to get

$$\sup_{0 \leqslant T_0 \leqslant T_1} \|V(T_0)\|^2 + N(T_1)\| V(T_1)\|^2$$

$$\leqslant O(1)\{ \|V(0)\|^2 + \delta^{1/6} \} + O(1) \sum_{i \neq j} \iint_{\Omega_j(T_1)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau = \alpha_4. \quad (4.39)$$

Similarly, it follows from (3.30) that

$$\int_0^{T_1} \|V_x(t)\|^2 \, dt \leqslant \alpha_4 \quad (4.40)$$

and

$$\sum_{i=1}^2 \iint_{\Omega_i(T_1)} \left| \frac{\partial U_i}{\partial x} \right| V_i^2 \, dx \, d\tau \leqslant \alpha_4. \quad (4.41)$$

Putting (4.43)–(4.41) together gives

$$\sum_{i \neq j} \iint_{\Omega_j(T_1)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau$$

$$\leqslant O(1)\{ \|V(0)\|^2 + \delta^{1/6} \}$$

$$+ O(1)(\delta + N(T_1)) \sum_{i \neq j} \iint_{\Omega_j(T_1)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau. \quad (4.42)$$

Since $T_1 \in [0, T]$ is arbitrary, we have shown that

LEMMA 4.4. *There exist positive constants $\varepsilon_4 (\leqslant \varepsilon_3)$ and $O(1)$ independent of T and δ such that for $N(T) + \delta \leqslant \varepsilon_4$, for any t, $0 \leqslant t \leqslant T$, it holds that*

$$\sum_{i \neq j} \iint_{\Omega_j(T_1)} \left| \frac{\partial U_j}{\partial x} \right| V_i^2 \, dx \, d\tau \leqslant O(1)\{ \|V(0)\|^2 + \delta^{1/6} \}. \quad (4.43)$$

This lemma completes the estimates on the transverse wave fields. Combining Lemmas 4.4 and 3.3 yields the following basic a priori stability estimate.

PROPOSITION 4.5. *If $N(T) + \delta \leqslant \varepsilon_4$, then for any t, $0 \leqslant t \leqslant T$, we have*

$$\phi(t)\|^2 + \int_0^t \int |U_x| \, |\phi(x, \tau)|^2 \, dx \, d\tau + \int_0^t \|\phi_x(\tau)\|^2 \, d\tau$$

$$\leqslant C\{ \|\phi(0)\|^2 + \delta^{1/6} \}, \quad (4.44)$$

where $C > 0$ is independent of T and δ.

5. GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

Using the basic stability estimate, Proposition 4.5, the proof of our main theorem can be given in the same way as that in [1]. For completeness, we give an outline here.

First, making use of Proposition 4.5, we may apply the standard energy method for parabolic systems to obtain the following estimates on derivatives:

LEMMA 5.1.   *There exist positive constants $\varepsilon_5(\leqslant \varepsilon_4)$ and $O(1)$ independent of T and δ such that if $N(T) + \delta \leqslant \varepsilon_5$; then for any $t \in [0, T]$,*

$$\|\phi_x(t)\|_1^2 + \int_0^t \|\phi_x(\tau)\|_2^2 \, d\tau \leqslant C\{\|\phi_x(0)\|_1^2 + \delta^{1/6}\}. \tag{5.1}$$

Next, note that the hypothesis that $N(T) + \delta$ is small both in this lemma and Proposition 4.5 follows from (4.44) and (5.1), under the assumption that $N(0)$ and δ are small. Thus, the usual local existence theorem and continuity argument for parabolic systems yield the following global existence result:

PROPOSITION 5.2.   *For each state $u_- \in \Omega$, there exist positive constants ε and C_0 such that if $\|\phi_0\|_2 + \delta \leqslant \varepsilon$, then problems (3.2) and (3.3) have a unique global solution $\phi \in X(0, +\infty)$ satisfying*

$$\sup_{t \leqslant 0} \|\phi(t)\|_2^2 + \int_0^{+\infty} \{\| |U_x|^{1/2} \phi(t)\|^2 + \|\phi_x(t)\|_2^2\} \, dt$$

$$\leqslant C_0\{\|\phi(0)\|_2^2 + \delta^{1/6}\}. \tag{5.2}$$

Now, the asymptotic behavior of the solution is a consequence of (5.2) and Sobolev's inequality, since from (5.2) and Eq. (3.2), we have

$$\int_0^{+\infty} \left(\|\phi_x(t)\|^2 + \left| \frac{d}{dt} \|\phi_x(t)\|^2 \right| \right) dt < +\infty,$$

so that

$$\lim_{t \to +\infty} \|\phi_x(t)\| = 0.$$

It follows from this and the Sobolev inequality

$$\|\phi(t)\|_{L_\infty} \leqslant 2^{1/2} \|\phi(t)\| \, \|\phi_x(t)\|$$

that

$$\lim_{t \to +\infty} \sup_{x \in \mathbb{R}^1} |\phi(x, t)| = 0.$$

This proves our main theorem.

References

1. Z. P. XIN, Asymptotic stability of rarefaction waves for 2 * 2 viscous hyperbolic conservation laws, *J. Differential Equations* **73** (1988), 45–77.

2. T-P. LIU, Nonlinear stability of shock waves for viscous conservation laws, *Mem. Amer. Math. Soc.* **328** (1985), 1–108.

3. P. D. LAX, Hyperbolic system of conservation laws and the mathematical theory of shock waves, *in* "Philadelphia SIAM Regional Conf. Ser. in Appl. Math., No. 11 (1973)."

4. J. SMOLLER, "Shock Waves and Reaction Diffusion Equations," Springer-Verlag, New York/Berlin, 1983.

5. T-P. LIU AND Z. P. XIN, Nonlinear stability of rarefaction waves for compressible Navier–Stokes equations, *Comm. Math. Phys.* **118** (1988), 451–465.

6. A. MATSUMURA AND K. NISHIHARA, Asymptotics toward the rarefaction waves of the solution of a one-dimensional model system for compressible viscous gas, *Japan J. Appl. Math.* **3** (1986), 1–13.

7. T. P. LIU, Hyperbolic conservation laws with relaxation, *Comm. Math. Phys.* **108** (1987), 153–175.

8. A. M. ILIN AND O. A. OLEINIK, Behavior of the solution of the Cauchy problem for cetain quasilinear equations for unbounded increase of the time, *Amer. Math. Soc. Transl.* **42** (1964), 19–23.

9. E. HOPF, The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math* **3** (1950), 201–230.