Strongly Regular Graphs with Strongly Regular Decomposition

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Dedicated to Alan J. Hoffman on the occasion of his 65th birthday.

Submitted by Alexander Schrijver

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1. INTRODUCTION AND PRELIMINARY RESULTS

The title refers to strongly regular graphs Γ_0 which admit a partition $\{X_1, X_2\}$ of the vertex set such that each of the induced subgraphs Γ_1 and Γ_2 on X_1 and X_2 respectively is strongly regular, a clique, or a coclique. A central role is played by the design D having point set X_1, block set X_2, and incidence given by adjacency in Γ_0. If Γ_1 is a clique or a coclique and Γ_0 is primitive, D must be a quasisymmetric design. If Γ_1 and Γ_2 are both strongly regular, D is a strongly regular design in the sense of D. G. Higman [14], except possibly when Γ_0 is the graph of a regular conference matrix. Conversely, a quasisymmetric or strongly regular design with suitable parameters gives rise to a strongly regular graph with strongly regular decomposition. Moreover, if Γ_0 and Γ_1 are strongly regular with suitable parameters, then Γ_2 must be strongly regular too. We give several examples and some nonexistence results. We include a table of all feasible parameter sets up to 300 vertices. For most of the cases in the table existence or nonexistence is settled. Some of the results in this paper are old, due to M. S. Shrikhande [17], W. G. Bridges and M. S. Shrikhande [3], and W. H. Haemers [13].

We mainly use eigenvalue techniques. We need results on interlacing eigenvalues (see [13]). Two sequences $\rho_1 \geqslant \cdots \geqslant \rho_n$ and $\sigma_1 \geqslant \cdots \geqslant \sigma_m$

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$(n > m)$ are said to *interlace* whenever

$$\rho_i \geqslant \sigma_i \geqslant \rho_{n-m+i} \qquad \text{for} \quad i = 1, \dots, m.$$

Interlacing is *tight* if there exists an integer k such that

$$\rho_i = \sigma_i \qquad \text{for} \quad i = 1, \dots, k,$$

$$\rho_{n-m+i} = \sigma_i \qquad \text{for} \quad i = k+1, \dots, m.$$

RESULT 1.1. *Let A_0 be a symmetric matrix partitioned as follows:*

$$A_0 = \begin{pmatrix} A_1 & C \\ C^\top & A_2 \end{pmatrix}.$$

Let B be the 2×2 matrix whose entries are the average row sums of the blocks of A_0.

(i) *Cauchy interlacing. The eigenvalues of A_1 interlace the eigenvalues of A_0. If the interlacing is tight, then $C = 0$.*

(ii) *The eigenvalues of B interlace the eigenvalues of A_0. If the interlacing is tight, then A_1, A_2, and C have constant row and column sums. Conversely, if A_1, A_2, and C have constant row and column sums, both eigenvalues of B are also eigenvalues of A_0.*

Our main tool is the following lemma. It is a kind of mixture of Theorem 5.1 in [3] and Theorem 1.3.3 in [13] (J denotes the all-one matrix).

LEMMA 1.2. *For $i = 0, 1, 2$ let A_i be a symmetric $v_i \times v_i$ matrix such that*

$$A_0 = \begin{pmatrix} A_1 & C \\ C^\top & A_2 \end{pmatrix} \quad \text{and} \quad A_1 C + C A_2 = \alpha C + \beta J \quad \text{for some} \quad \alpha, \beta \in \mathbb{R}.$$

Let A_1, A_2, C, and $C^\top$ have constant row sums k_1, k_2, r, and k respectively.

For $i = 0, 1, 2$ denote the eigenvalues of A_i by $\rho_{i,1}, \dots, \rho_{i,v_i}$. Denote the singular values of C by $\sqrt{\gamma_1}, \dots, \sqrt{\gamma_m}$, where $m = \operatorname{rank} C$. Then we can order the ρ_{ij}'s and γ_j's so that:

(i) *$\rho_{1,1} = k_1$, $\rho_{2,1} = k_2$ with all-one eigenvector, $\gamma_1 = rk$, $k_1 + k_2 = \alpha + \beta v_1/k$, and $\rho_{0,1}$, $\rho_{0,2}$ are the roots of $(x - k_1)(x - k_2) = rk$.*

(ii) *$\rho_{1,j} + \rho_{2,j} = \alpha$ with eigenvectors in the range of C and $C^{\top}$, respectively, and $\rho_{0,2j}$, $\rho_{0,2j-1}$ are the roots of $(x - \rho_{1,j})(x - \rho_{2,j}) = \gamma_j$ for $j = 2, \dots, m$.*

(iii) *$\rho_{1,j}$ has an eigenvector in the kernel of $C^{\top}$, $\rho_{1,j} = \rho_{0,m+j}$, for $j = m+1, \dots, v_1$; $\rho_{2,j}$ has an eigenvector in the kernel of C, $\rho_{2,j} = \rho_{0,v_1+j}$, for $j = m+1, \dots, v_2$.*

Proof. We have

$$A_1 CC^{\top} = \alpha CC^{\top} + \beta rJ - CA_2C^{\top}.$$

The right-hand side is a symmetric matrix; hence $A_1CC^{\top} = CC^{\top}A_1$. So A_1 and $CC^{\top}$ commute, and therefore they have a common orthonormal bases of eigenvectors $u_1, \dots, u_{v_1}$ (say), ordered so that $A_1u_j = \rho_{1,j}u_j$ for $j = 1, \dots, v_1$, $CC^{\top}u_j = \gamma_j u_j$ for $j = 1, \dots, m$, $CC^{\top}u_j = 0$ for $j = m+1, \dots, v_1$, and u_1 is the all-one vector. Now the first two equations of (i) are obvious. Furthermore

$$A_2C^{\top}u_j = \alpha C^{\top}u_j + \beta Ju_j - C^{\top}A_1u_j = (\alpha - \rho_{1,j})C^{\top}u_j \qquad \text{for} \quad j = 2, \dots, m,$$

proving the first equation of (ii). Define

$$w_j = \begin{pmatrix} \gamma_j u_j \\ (x - \rho_{1,j})C^{\top}u_j \end{pmatrix} \qquad \text{for} \quad j = 1, \dots, m.$$

Then it is easily verified that $A_0w_j = xw_j$ whenever $(x - \rho_{1,j})(x - \rho_{2,j}) = \gamma_j$. Thus (i) and (ii) are proved. Next define

$$w_j = \begin{pmatrix} u_j \\ 0 \end{pmatrix} \qquad \text{for} \quad j = m+1, \dots, v_1.$$

Then $A_0w_j = \rho_{1,j}w_j$, proving the first part of (iii). The second part of (iii) follows by interchanging A_1 and A_2. ∎

We assume the reader to be familiar with the theory of designs and strongly regular graphs. Some references are Beth, Jungnickel, and Lenz [1], Cameron and Van Lint [9], and Seidel [16]. We recall some result about strongly regular designs (see Higman [14]).

DEFINITION 1.3. A design D with v_1 points and v_2 blocks and incidence matrix C is strongly regular whenever there exist graphs Γ_1 and Γ_2 (not complete or void) with adjacency matrices A_1 and A_2 respectively, such that the following hold:

(i) $CC^{\top} = w_1 I + y_1 J + z_1 A_1$ for integers w_1, y_1, and z_1 ($z_1 \neq 0$),
(ii) $C^{\top}C = w_2 I + y_2 J + z_2 A_2$ for integers w_2, y_2, and z_2 ($z_2 \neq 0$),
(iii) $CC^{\top}C = \gamma C + \delta J$ for integers γ and δ.

It is easily seen that C has constant row sum $r = w_1 + y_1$ and column sum $k = w_2 + y_2$, and that $\delta = k(kr - \gamma)/v_1$. The graph Γ_1 is the *point graph* of D, and Γ_2 is the *block graph* of D. It is straightforward that Γ_i $(i = 1,2)$ is strongly regular with eigenvalues

$$k_i = \frac{kr - y_i v_i - w_i}{z_i}, \qquad \rho_i = \frac{\gamma - w_i}{z_i}, \qquad \sigma_i = \frac{-w_i}{z_i}$$

of multiplicity 1, $m-1$, and $v_i - m$, respectively, where $m = \operatorname{rank} C$. The eigenspaces of the eigenvalues σ_1 and σ_2 are the kernels of C and $C^{\top}$, respectively. (The point and block graph are determined up to taking complements. To avoid this ambiguity one often requires that $z_i > 0$. However, for our purposes it is not convenient to do so.) Bose, Bridges, and Shrikhande [2] proved that (iii) may be replaced by:

(iii′) *The singular values* $\sqrt{\gamma_1}, \dots, \sqrt{\gamma_m}$ *of* C *satisfy*

$$\gamma_1 = rk, \qquad \gamma_2 = \cdots = \gamma_m = \gamma.$$

In case $z_1 = 0$, D is a quasisymmetric block design. A strongly regular design is the same as a quasisymmetric special partially balanced incomplete block design (see Shrikhande [18]).

We finish this section with some notation. For a graph Γ_i, v_i denotes the number of vertices, and the adjacency matrix is denoted by A_i. If A_i has eigenvalues $\rho_1, \dots, \rho_n$ with respective multiplicities $\varphi_1, \dots, \varphi_n$, we write

$$\operatorname{spec} \Gamma_i = \{ \rho_1^{\varphi_1}, \dots, \rho_n^{\varphi_n} \}.$$

If Γ_i is regular, the degree is denoted by k_i, and if Γ_i is strongly regular, we write

$$\operatorname{spec}\Gamma_i = \left\{ k_i, r_i^{f_i}, s_i^{g_i} \right\} \qquad \text{with} \quad r_i \geqslant 0 > s_i.$$

Throughout the paper Γ_0 denotes a graph decomposed into subgraphs Γ_1 and Γ_2, that is, the respective adjacency matrices A_0, A_1, and A_2 satisfy

$$A_0 = \begin{pmatrix} A_1 & C \\ C^\top & A_2 \end{pmatrix},$$

where C is the incidence matrix of some structure D (say). For regular Γ_0 the decomposition is called *regular* if also Γ_1 and Γ_2 are regular. For strongly regular Γ_0 the decomposition is *strongly regular* if Γ_1 and Γ_2 are strongly regular, a clique, or a coclique.

2. THEORY

If Γ_0 or the complement is the disjoint union of two or more cliques of equal size, then Γ_0 is a so-called *imprimitive* strongly regular graph. In this case the strongly regular decompositions are obvious. Therefore we restrict ourselves to a primitive Γ_0.

LEMMA 2.1. *If Γ_0 is strongly regular with a regular decomposition, then*

$$CJ = (k_0 - k_1)J, \qquad C^\top J = (k_0 - k_2)J,$$

$$A_1^2 + CC^\top = (r_0 + s_0)A_1 - r_0 s_0 I + (k_0 + r_0 s_0)J,$$

$$A_2^2 + C^\top C = (r_0 + s_0)A_2 - r_0 s_0 I + (k_0 + r_0 s_0)J,$$

$$A_1 C + CA_2 = (r_0 + s_0)C + (k_0 + r_0 s_0)J.$$

Proof. The first line reflects the fact that the decomposition is regular. If Γ_0 is strongly regular, then $A_0^2 - (r_0 + s_0)A_0 + r_0 s_0 I = (k_0 + r_0 s_0)J$. Thus the block structure of A_0 gives the remaining formulas. ∎

Theorem 2.2. *Suppose Γ_0 is strongly regular and Γ_1 is regular. Then*

$$s_0 \leqslant \frac{k_1 v_0 - k_0 v_1}{v_0 - v_1} \leqslant r_0.$$

The decomposition is regular if and only if equality holds on the left- or right-hand side. If the left-hand [right-hand] inequality is met, then

$$k_2 = k_0 - k_1 + s_0 \qquad [k_2 = k_0 - k_1 + r_0].$$

Proof. We apply Result 1.1(ii). The matrix of the average row sums,

$$B = \begin{pmatrix} k_1 & k_0 - k_1 \\ (k_0 - k_1)v_1/v_2 & k_0 - (k_0 - k_1)v_1/v_2 \end{pmatrix},$$

has eigenvalues k_0 (row sum) and ρ (say). From $k_0 + \rho = \operatorname{trace} B$ it follows that $\rho = (k_1 v_0 - k_0 v_1)/(v_0 - v_1)$, which gives the desired inequalities. Equality on either side means that the interlacing is tight, and hence the decomposition must be regular. If the decomposition is regular, the eigenvalues of B are k_0 and $\rho = k_1 + k_2 - k_0$. These are also eigenvalues of A_0; hence $\rho = s_0$ or $\rho = r_0$. ■

It is easily verified that if equality holds on one side, then the corresponding decomposition of the complement of Γ_0 satisfies equality on the other side. If Γ_1 is a coclique (i.e. $k_1 = 0$) the above result gives

$$v_1 \leqslant \frac{-v_0 s_0}{k_0 - s_0}.$$

This is Hoffman's coclique bound. Another bound is the following one.

Theorem 2.3. *If Γ_1 is a coclique and Γ_0 is primitively strongly regular, then*

$$v_1 \leqslant \min\{f_0, g_0\}.$$

Proof. Define $A = A_0 - v_0^{-1}(k_0 - s_0)J - s_0 I$. Then $\operatorname{rank} A = f_0$. Since $A_1 = 0$, A has a submatrix $-v_0^{-1}(k_0 - s_0)J - s_0 I$ of size $v_1 \times v_1$, which is nonsingular ($s_0 \neq 0$, since Γ_0 is primitive). Hence $v_1 \leqslant f_0$. Similarly we get $v_1 \leqslant g_0$. ■

Theorems 2.2 and 2.3 are special cases of theorems of Haemers [13] and Cvetcović [10], respectively.

THEOREM 2.4. *Suppose Γ_0 and Γ_1 are strongly regular, let Γ_0 be primitive, and suppose the decomposition is regular. Put ε equal to 0 or 1, according to whether the left- or the right-hand side is tight in Theorem 2.2 (e.g. $k_2 = k_0 - k_1 + \varepsilon r_0 + (1-\varepsilon)s_0$). Then one of the following holds*:

(i) $s_1 > s_0$, $r_1 < r_0$, $v_1 \leqslant \min\{f_0 + 1 - \varepsilon,\ g_0 + \varepsilon\}$,

$$\operatorname{spec}\Gamma_2 = \left\{k_2, (r_0 + s_0 - r_1)^{f_1}, (r_0 + s_0 - s_1)^{g_1}, r_0^{f_0 - v_1 + 1 - \varepsilon}, s_0^{g_0 - v_1 + \varepsilon}\right\}.$$

(ii) $s_1 = s_0$, $r_1 < r_0$, $v_1 \leqslant g_0 + \varepsilon$,

$$\operatorname{spec}\Gamma_2 = \left\{k_2, (r_0 + s_0 - r_1)^{f_1}, r_0^{f_0 - f_1 - \varepsilon}, s_0^{g_0 - v_1 + \varepsilon}\right\}.$$

(iii) $s_1 > s_0$, $r_1 = r_0$, $v_1 \leqslant f_0 + 1 - \varepsilon$,

$$\operatorname{spec}\Gamma_2 = \left\{k_2, (r_0 + s_0 - s_1)^{g_1}, r_0^{f_0 - v_1 + 1 - \varepsilon}, s_0^{g_0 - g_1 - 1 + \varepsilon}\right\}.$$

Proof. By Lemmas 1.2 and 2.1 it follows that k_2, $r_0 + s_0 - r_1$, $r_0 + s_0 - s_1$, r_0, and s_0 are the only possible eigenvalues of Γ_2, and that $r_0 + s_0 - r_1$ $[r_0 + s_0 - s_1]$ has multiplicity f_1 $[g_1]$ whenever $r_1 \neq r_0$ $[s_1 \neq s_0]$. From $\operatorname{trace} A_2 = 0$ one finds that the multiplicity of s_0 $[r_0]$ equals $g_0 - v_1 + \varepsilon$ $[f_0 - v_1 + 1 - \varepsilon]$, which must be a nonnegative number. The inequalities $s_1 \geqslant s_0$ and $r_1 \leqslant r_0$ follow from Cauchy interlacing [Result 1.1(i)]. What remains to be proved is that $s_1 = s_0$ and $r_1 = r_0$ do not both occur. Suppose they do. Define $\alpha = (k_0 - \varepsilon r_0 - (1-\varepsilon)s_0)/v_0$; then the matrix $A_0 - \alpha J$, which has eigenvalues r_0 and s_0 only, has principal submatrix $A_1 - \alpha J$, having only eigenvalues r_0 and s_0 too. So, by Result 1.1(i), $C - \alpha J = 0$ and hence Γ_0 is imprimitive: a contradiction. ∎

The regular graph Γ_2 is strongly regular, a clique, or a coclique whenever it has at most two distinct eigenvalues, except for the degree k_2. This leads to the following result.

COROLLARY 2.5. *With the hypotheses of Theorem 2.4, the decomposition is strongly regular if and only if one of the following holds*:

(i) $v_1 = f_0 + 1 - \varepsilon = g_0 + \varepsilon$,
(ii) $s_0 = s_1$ *and* $f_0 = f_1 + \varepsilon$,
(iii) $s_0 = s_1$ *and* $v_1 = g_0 + \varepsilon$,
(iv) $r_0 = r_1$ *and* $g_0 = g_1 + 1 - \varepsilon$,
(v) $r_0 = r_1$ *and* $v_1 = f_0 + 1 - \varepsilon$.

A strongly regular decomposition is called *improper* if Γ_1 or Γ_2 is a clique or a coclique. Without loss of generality we may assume then that Γ_1 is a coclique. If Γ_0 is strongly regular and Γ_1 is a coclique, then also Theorem 2.4(i) holds with $r_1 = 0$ and $g_1 = 0$. Thus we find the following result of Haemers [13]:

THEOREM 2.6. *Let Γ_0 be primitively strongly regular, and let Γ_1 be a coclique. Then $v_1 = g_0 = -v_0 s_0/(k_0 - s_0)$ (i.e., both Hoffman's bound and Cvetcović's bound are tight) if and only if Γ_2 is strongly regular.*

Proof. Hoffman's bound is tight if and only if the decomposition is regular. Theorem 2.4(i) gives

$$\operatorname{spec}\Gamma_2 = \left\{ k_2, (r_0 + s_0)^{v_1 - 1}, r_0^{f_0 - v_1 + 1}, s_0^{g_0 - v_1} \right\},$$

since $\varepsilon = 0$ if Γ_1 is a coclique. By Theorem 2.3 we have $f_0 - v_1 + 1 > 0$; hence Γ_2 is strongly regular if and only if $g_0 = v_1$. ■

We call a proper strongly regular decomposition *exceptional* if $s_1 \neq s_0$ and $r_1 \neq r_0$, which is by Theorem 2.4(i) equivalent to $s_2 \neq s_0$ and $r_2 \neq r_0$.

THEOREM 2.7. *If Γ_0 is primitively strongly regular and admits an exceptional strongly regular decomposition, then Γ_0 is the graph of a regular symmetric conference matrix, that is, Γ_0 or its complement satisfies*

$$v_0 = 4r_0^2 + 4r_0 + 2, \quad k_0 = 2r_0^2 + r_0, \quad s_0 = -r_0 - 1 \qquad \textit{for integer } r_0.$$

Moreover, one of the following holds:

(i) *Γ_1 and Γ_2 are so-called conference graphs, that is,*

$$v_1 = v_2 = 2r_0^2 + 2r_0 + 1, \qquad k_1 = k_2 = r_0^2 + r_0,$$

$$r_1 = r_2 = \frac{-1 + \sqrt{v_1}}{2}, \qquad s_1 = s_2 = \frac{-1 - \sqrt{v_1}}{2},$$

and D is a symmetric 2-$(v_1, r_0^2, r_0(r_0 - 1)/2)$ design, or the complement.

(ii) *We have*

$$v_1 = v_2 = 2r_0^2 + 2r_0 + 1, \qquad k_1 = k_2 = r_0^2 + r_0,$$

$$r_2 = \frac{k_1 - r_1}{2r_1 + 1}, \qquad s_1 = -r_2 - 1, \qquad s_2 = -r_1 - 1,$$

$$r_1 \neq r_2, \qquad r_1 < r_0, \qquad r_2 < r_0,$$

and r_1, r_2, *and* $(2k_1^2 + k_1)/(k_1 + 2r_1^2 + 2r_1 + 1)$ *are integers.*

Proof. Take without loss of generality $\varepsilon = 0$. Then Corollary 2.5(i) gives $f_0 + 1 = g_0$, and the remaining parameters of Γ_0 follow straightforwardly (see [9]). Also by 2.5(i) we have $2v_1 = v_0$, so $v_1 = v_2$ and $k_1 = k_2$. By Theorem 2.2 it follows that $k_1 = k_2 = (k_0 + s_0)/2 = r_0^2 + r_0$. If Γ_1 is a conference graph, then so is Γ_2 (by Theorem 2.4), and by Lemma 2.1 we find

$$CC^{\top} = \frac{r_0(r_0+1)}{2} I + \frac{r_0(r_0-1)}{2} J,$$

proving (i). If Γ_1 and Γ_2 are not conference graphs, then $r_1 \neq -1 - s_1$ and the eigenvalues are integers. The remaining formulas of (ii) follow easily from Theorem 2.4 and the well-known identities for strongly regular graphs. ∎

The Petersen graph partitioned into two pentagons is an example for (i). We give another example in the next section. For case (ii) it seems hopeless to find an example: The smallest feasible solution has $r_1 = 554$, $r_0 = 731$, $v_0 = 2{,}140{,}370$.

The next theorem relates strongly regular decompositions to strongly regular designs. The result is due to W. G. Bridges and M. S. Shrikhande [3].

THEOREM 2.8. *Γ_0 is primitively strongly regular with a strongly regular decomposition which is proper and not exceptional, if and only if D is a strongly regular design with point graph Γ_1 and block graph Γ_2, whose parameters satisfy*

(i) $k_1 + r = k_2 + k$,
(ii) $k_1 - k \in \{\sigma_1, \rho_1 + \rho_2 - \sigma_1\}$,
(iii) $\rho_2 = \sigma_1 + z_1$, $\rho_1 = \sigma_2 + z_2$.

Proof. Let D be a strongly regular design. From Definition 1.3 it follows that $A_1C + CA_2 \in \langle C, J\rangle$; hence Lemma 1.2 applies. Clearly Γ_0 is regular (of degree $k_0 = k_1 + r$) whenever $k_1 + r = k_2 + k$. For the remaining eigenvalues of Γ_0 we get $k_1 + k_2 - k_0 = k_2 - r$ [by Lemma 1.2(i)], σ_1 and σ_2 [by Lemma 1.2(iii)], and the roots of

$$(x - \rho_1)(x - \rho_2) = \gamma \qquad (*)$$

(by 1.2(ii)). These five eigenvalues take only two values if and only if σ_1, σ_2, and $k_2 - r$ are roots of $(*)$. By use of $\rho_i - \sigma_i = \gamma_i / z_i$ we find that σ_i is a root of $(*)$ for $i = 1,2$ if and only if (iii) holds. Suppose σ_1 is a root of $(*)$; then $\rho_1 + \rho_2 - \sigma_1$ is the other root; hence $k_2 - r$ is a root of $(*)$ if and only if (ii) holds. The decomposition is clearly proper, and it is not exceptional, since σ_1 and σ_2 are eigenvalues of Γ_0.

Next assume Γ_0 has the required properties. Then $r_0 = r_1$ or $s_0 = s_1$, since the decomposition is not exceptional. Take without loss of generality $r_0 = r_1$. Then Lemma 1.2(ii) gives

$$\gamma_j = (r_0 - s_0)(s_1 - s_0) \qquad \text{for} \quad j = 2, \dots, m.$$

So D satisfies (iii′) of Definition 1.3. By Lemma 2.1 and the strong regularity of Γ_1 and Γ_2 we have

$$CC^{\top} \in \langle A_1, I, J\rangle, \qquad C^{\top}C \in \langle A_2, I, J\rangle.$$

Moreover, the coefficient of A_i equals $r_0 + s_0 - r_i - s_i \neq 0$ for $i = 1,2$. Hence also (i) and (ii) of Definition 1.3 are satisfied, so D is a strongly regular design. ∎

From the above proof we have that a strongly regular Γ_0 has eigenvalues σ_1 and $\rho_1 + \rho_2 - \sigma_1$; one of the two must be equal to σ_2. The following result can be regarded as a special case of the above theorem (therefore a proof is superfluous).

THEOREM 2.9. *Γ_0 is primitively strongly regular with an improper strongly regular decomposition (where Γ_1 is a coclique) if and only if D is a quasisymmetric 2-design, having Γ_2 as block graph, whose parameters satisfy*

$$r = k + k_2, \qquad \sigma_2 + z_2 = 0, \qquad k = -\gamma / z_2.$$

From $k = -\gamma/z_2$ it follows that $z_2 < 0$. This means that if (as usual) adjacency in the block graph corresponds to the larger intersection number, then Γ_2 is the complement of the block graph of D. M. S. Shrikhande [17] (see also [3]) proved that the conditions for D in Theorem 2.9 are equivalent to the following: D is a quasisymmetric 2-$(1 + z_2(k-1)/(k - z_2^2),\ k,\ k(k - z_2^2)/(z_2+1))$ design with intersection numbers $k - z_2^2$ and $k - z_2^2 - z_2$.

3. CONSTRUCTIONS

In this section we give constructions and some nonexistence results for strongly regular graphs with strongly regular decompositions. With the help of the results of the previous section we have made a table of feasible parameters up to 300 vertices (Table 1). For all cases in the table we indicate existence or nonexistence if known (to us).

EXAMPLE 3.1. The vertices of the *triangular graph* $T(m)$ are all pairs of a given set M of cardinality m ($m > 3$); two vertices are adjacent whenever the pairs are not disjoint. $T(m)$ is strongly regular with

$$\operatorname{spec} T(m) = \left\{2(m-2),\ (m-4)^{m-1},\ (-2)^{m(m-3)/2}\right\}.$$

For a fixed $x \in M$, partition the vertices into the pairs containing x and pairs not containing x. It is easily seen that this gives an improper strongly regular decomposition of $T(m)$ into a clique of size $m-1$ and $T(m-1)$.

The next result has often been observed before.

THEOREM 3.2. *The block graph of a quasisymmetric 3-design E admits a strongly regular decomposition. The decomposition is improper if and only if E is the extension of a symmetric 2-design.*

Proof. Fix a point x of E. Partition the blocks of E to the blocks containing x and the blocks not containing x. This gives a partition of the block graph of E into the block graphs of the derived and the residual design of E (with respect to x) respectively. The derived or residual design of E is symmetric whenever E is the extension of a symmetric design; otherwise both designs are quasisymmetric. This proves the result. ∎

The design whose blocks are just all pairs of points can be seen as a degenerate quasisymmetric 3-design. This leads to Example 3.1. We know of

TABLE 1

ALL FEASIBLE PARAMETER SETS FOR PRIMITIVE STRONGLY REGULAR GRAPHS WITH STRONGLY REGULAR DECOMPOSITION UP TO 300 VERTICES[a]

	Case	v_0	k_0	r_0	s_0	f_0	g_0	λ_0	μ_0	v_1 / v_2	k_1 / k_2	r_1 / r_2	s_1 / s_2	f_1 / f_2	g_1 / g_2	λ_1 / λ_2	μ_1 / μ_2
+	1	15	6	1	−3	9	5	1	3	6	3	0	−3	4	1	0	3
										9	4	1	−2	4	4	1	2
+	2	16	5	1	−3	10	5	0	2	8	1	1	−1	3	4	0	0
										8	1	1	−1	3	4	0	0
+	3	63	30	3	−5	35	27	13	15	28	15	1	−5	20	7	6	10
										35	18	3	−3	14	20	9	9
+	4	63	30	3	−5	35	27	13	15	27	10	1	−5	20	6	1	5
										36	15	3	−3	15	20	6	6
−	5	69	20	5	−3	23	45	7	5	24	5	5	−1	3	20	4	0
										45	12	3	−3	20	24	3	3
+	6	70	27	6	−3	20	49	12	9	21	6	6	−1	2	18	5	0
										49	18	4	−3	18	30	7	6
−	7	76	21	2	−7	56	19	2	7	19	0						
										57	14	2	−5	38	18	1	4
+	8	77	16	2	−6	55	21	0	4	21	0						
										56	10	2	−4	35	20	0	2
−	9	85	14	4	−3	34	50	3	2	35	4	4	−1	6	28	3	0
										50	7	2	−3	28	21	0	1

TABLE 1 (*Continued*)

	Case	v_0	k_0	r_0	s_0	f_0	g_0	λ_0	μ_0	v_1 / v_2	k_1 / k_2	r_1 / r_2	s_1 / s_2	f_1 / f_2	g_1 / g_2	λ_1 / λ_2	μ_1 / μ_2
−	10	95	40	2	−10	75	19	12	20	20	10	0	−10	18	1	0	10
										75	32	2	−8	56	18	10	16
−	11	95	40	2	−10	75	19	12	20	19	0						
										76	30	2	−8	57	18	8	14
+	12	100	22	2	−8	77	22	0	6	50	7	2	−3	28	21	0	1
										50	7	2	−3	28	21	0	1
+	13	112	30	2	−10	90	21	2	10	56	10	2	−4	35	20	0	2
										56	10	2	−4	35	20	0	2
−	14	126	60	12	−3	21	104	33	24	21	20						
										105	52	10	−3	20	84	29	22
+	15	162	56	2	−16	140	21	10	24	81	20	2	−7	60	20	1	6
										81	20	2	−7	60	20	1	6
+	16	176	70	2	−18	154	21	18	34	56	10	2	−4	35	20	0	2
										120	42	2	−12	99	20	8	18
−	17	208	75	10	−5	64	143	30	25	64	30	10	−2	8	55	18	10
										144	55	7	−5	55	88	22	20
−	18	217	66	4	−11	154	62	15	22	63	22	1	−11	55	7	1	11
										154	48	4	−8	98	55	12	16
?	19	220	84	14	−4	44	175	38	28	45	14	14	−1	2	42	13	0
										175	66	11	−4	42	132	29	22

TABLE 1 (*Continued*)

	Case	v_0	k_0	r_0	s_0	f_0	g_0	λ_0	μ_0	v_1 v_2	k_1 k_2	r_1 r_2	s_1 s_2	f_1 f_2	g_1 g_2	λ_1 λ_2	μ_1 μ_2
+	20	221	64	12	−4	51	169	24	16	52	12	12	−1	3	48	11	0
										169	48	9	−4	48	120	17	12
?	21	236	55	11	−4	59	176	18	11	60	11	11	−1	4	55	10	0
										176	40	8	−4	55	120	12	8
?	22	245	52	3	−13	195	49	3	13	49	0						
										196	39	3	−10	147	48	2	9
−	23	246	85	3	−17	204	41	20	34	41	0						
										205	68	3	−14	164	40	15	26
+	24	253	112	2	−26	230	22	36	60	77	16	2	−6	55	21	0	4
										176	70	2	−18	154	21	18	34
+	25	255	126	7	−9	135	119	61	63	120	63	3	−9	84	35	30	36
										135	70	7	−5	50	84	37	35
+	26	255	126	7	−9	135	119	61	63	119	54	3	−9	84	34	21	27
										136	63	7	−5	51	84	30	28
−	27	261	84	21	−3	29	231	39	21	29	28						
										232	77	19	−3	28	203	36	20
?	28	265	96	6	−10	159	105	32	36	105	32	2	−10	84	20	4	12
										160	54	6	−6	75	84	18	18
?	29	266	45	3	−12	209	56	0	9	56	0						
										210	33	3	−9	154	55	0	6
?	30	287	126	3	−21	245	41	45	63	42	21	0	−21	40	1	0	21
										245	108	3	−18	204	40	39	54
?	31	287	126	3	−21	245	41	45	63	41	0						
										246	105	3	−18	205	40	36	51

[a] The triangular graphs (Example 3.1) and the exceptional decompositions (Theorem 2.7) are omitted. For each case it is indicated whether the decomposition exists (+), does not exist (−), or is not settled (?).

just three other quasisymmetric 3-designs (up to taking complements and except for the Hadamard 3-designs, which have imprimitive block graphs): the famous 4-(23, 7, 1) design (see [9]), its derived design, and its residual design. These three 3-designs are cases 8, 16, and 24 in the table. The first one provides an improper decomposition (D is the extension of the projective plane of order 4). In fact, this decomposition and the ones of Example 3.1 are the only improper decompositions we know.

THEOREM 3.3. *Let Γ_1 and D be as in Theorem 2.7(i). Suppose their matrices A_1 and C commute, and let Γ_2 be the complement of Γ_1. Then Γ_0 is strongly regular with an exceptional strongly regular decomposition.*

Proof. We have

$$A_i^2 = -A_i + \tfrac{1}{2}r_0(r_0+1)J + \tfrac{1}{2}r_0(r_0+1)I \qquad \text{for} \quad i=1,2,$$

$$CC^{\top} = C^{\top}C = \tfrac{1}{2}r_0(r_0-1)J + \tfrac{1}{2}r_0(r_0+1)I,$$

$$A_1C + CA_2 = A_1C - CA_1 + CJ - C = r_0^2J - C.$$

This yields

$$A_0^2 = -A_0 + r_0^2J + r_0(r_0+1)I,$$

which proves the result. ∎

If $r_0 = 1$, then Γ_1 is the pentagon, D is the degenerate 2-(5, 1, 0) design, and Γ_0 is the Petersen graph. For $r_0 = 2$, the desired graph and design are known:

$$A_1 = \text{cycle}\,(1\quad 0\quad 1\quad 1\quad 0\quad 0\quad 0\quad 0\quad 1\quad 1\quad 0\quad 1\quad 0),$$

$$C = \text{cycle}\,(1\quad 1\quad 0\quad 1\quad 0\quad 0\quad 0\quad 0\quad 0\quad 1\quad 0\quad 0\quad 0).$$

Since A_1 and C are both cyclic, they commute. Thus by the above theorem we find a strongly regular Γ_0 with $(v_0, k_0, r_0, s_0) = (26, 10, 2, -3)$, decomposed into the strongly regular Γ_1 and Γ_2 with $(v_1, k_1, r_1, s_1) = (v_2, k_2, r_2, s_2) = (13, 6, (-1+\sqrt{13})/2, (-1-\sqrt{13})/2)$. These are all the exceptional strongly regular decompositions we know. More graphs and designs with suitable parameters are known, but it is not known whether there exists a pair with commuting matrices.

THEOREM 3.4. *If q is the order of a projective plane, and if there exist $q-1$ mutually orthogonal Latin squares of order q^2+q+1, then there exists a strongly regular decomposition with*

$$(v_0, k_0, r_0, s_0) = \left((q^2+q+1)(q^2+2q+2), (q+1)^3, q^2+q, -q-1\right),$$

$$(v_1, k_1, r_1, s_1) = \left((q^2+q+1)(q+1), q(q+1), q(q+1), -1\right),$$

$$(v_2, k_2, r_2, s_2) = \left((q^2+q+1)^2, q(q+1)^2, q^2, -q-1\right).$$

Proof. A set of $q-1$ mutually orthogonal Latin squares is the same as a transversal design with $q+1$ groups of size q^2+q+1. Let

$$B_2 = \begin{pmatrix} N_1^{\mathrm{T}} & N_2^{\mathrm{T}} & \cdots & N_{q+1}^{\mathrm{T}} \end{pmatrix}^{\mathrm{T}}$$

be the incidence matrix of the transversal design, where the N_i's correspond to the groups. Let M be the incidence matrix of a projective plane of order q, and define $B_1 = I \otimes M$ ($\otimes$ denotes the tensor product), and $B = (B_1 \ B_2)$. Then B is the incidence matrix of a 2-$((q^2+q+1)(q+1), q+1, 1)$ design (which is obviously quasisymmetric) with block graph Γ_0. Clearly the block graph Γ_1 of B_1 is imprimitively strongly regular. Also the block graph of a transversal design is strongly regular. So the decomposition is strongly regular and the eigenvalues readily follow. ■

For many values of q the conditions of Theorem 3.4 are fulfilled—for instance, if q and q^2+q+1 are both prime powers (e.g. $q = 1,2,3,5,8$), but also (see Brouwer [4]) if q and $q+1$ are both prime powers (e.g. $q = 1,2,3,4,7,8$). Cases 1, 6, and 20 in the table can be constructed in this manner. We do not know if the theorem provides an infinite family. The following example, however, does give infinitely many proper strongly regular decompositions.

EXAMPLE 3.5. For every integer $m > 1$, the symplectic graph Γ_0 with

$$(v_0, k_0, r_0, s_0) = (2^{2m}-1, 2^{2m-1}-2, 2^{m-1}-1, -2^{m-1}-1)$$

admits two strongly regular decompositions: one with

$$(v_1, k_1, r_1, s_1) = (2^{2m-1} + 2^{m-1} - 1, 2^{2m-2} + 2^{m-1} - 2, 2^{m-1} - 1, -2^{m-2} - 1),$$

$$(v_2, k_2, r_2, s_2) = (2^{2m-1} - 2^{m-1}, 2^{2m-2} - 1, 2^{m-2} - 1, -2^{m-1} - 1),$$

and one with

$$(v_1, k_1, r_1, s_1) = (2^{2m-1} - 2^{m-1} - 1, 2^{2m-2} - 2^{m-1} - 2, 2^{m-2} - 1, -2^{m-1} - 1),$$

$$(v_2, k_2, r_2, s_2) = (2^{2m-1} + 2^{m-1}, 2^{2m-2} - 1, 2^{m-1} - 1, -2^{m-2} - 1).$$

In both cases Γ_1 is the orthogonal graph, defined on the points of an orthogonal quadric in PG$(2m-1, 2)$. The symplectic and orthogonal graphs are described in Seidel [15]. For $m = 2$ the decompositions coincide with Theorem 3.4 ($q = 1$) and Example 3.1 ($m = 6$), respectively. For larger m, the decompositions are proper and Γ_1 and Γ_2 are both primitive. Cases 1, 3, 4, 25, and 26 in the table are of this type.

Next we shall give some sporadic examples (making use of the table).

EXAMPLE 3.6. Case 2 in the table exists, that is, the Clebsch graph has a strongly regular decomposition with

$$A_1 = A_2 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} I & J-I \\ J-I & I \end{pmatrix}.$$

EXAMPLE 3.7. Case 12 in the table exists, that is, the Higman-Sims graph admits a strongly regular decomposition into two Hoffman-Singleton graphs (see Sims [19]).

EXAMPLE 3.8. A hemisystem (see Cameron, Delsarte, and Goethals [8]) is a strongly regular decomposition of the point graph of a generalized quadrangle of order (q^2, q). The only known hemisystem has $q = 3$, where the point graph is decomposed into two Gewirtz graphs. This produces case 13 of the table.

EXAMPLE 3.9. Goethals and Seidel [11] give a construction of Γ_0 of case 15 from which the strongly regular decomposition is obvious.

Finally some nonexistence results are considered. Case 7 in the table is impossible, since Wilbrink and Brouwer [22] showed that Γ_2 does not exist. For cases 17 and 18, Γ_1 does not exist because of the absolute bound. By Theorem 2.9 the existence of an improper strongly regular decomposition is equivalent to the existence of a quasisymmetric 2-design with suitable parameters. For quasisymmetric designs many nonexistence results are known. These results lead to nonexistence of cases 11, 14, 23 (due to Calderbank [6, 7]), and 27 (due to Haemers [12]; see also Tonchev [20]) in the table. The remaining cases are more complicated.

THEOREM 3.10. *No strongly regular graphs with strongly regular decomposition exist for the parameter sets numbered* 5 *and* 9 *in Table* 1.

Proof. In both cases Γ_1 is imprimitive. Therefore D is a group divisible design. Take C in canonical form, that is,

$$C = \begin{pmatrix} C_1^{\top} & \cdots & C_n^{\top} \end{pmatrix}^{\top},$$

where the C_i's correspond to the groups. For case 5 we define

$$B_1 = \text{cycle}\,(J - I \quad I \quad I \quad I),$$

wherein the blocks are 6×6 matrices. Then by straightforward verification it follows that $(C \ B_1)$ is the incidence matrix of a quasi-symmetric 2-(24, 8, 7) design with intersection numbers 4 and 2. Brouwer and Calderbank [5] showed that such a design does not exist. Similarly, for number 9 we define

$$B_1 = \text{cycle}\,(J - I \quad I \quad I \quad 0 \quad I \quad 0 \quad 0),$$

wherein the blocks have size 5×5. Then $(C \ B_1)$ is the incidence matrix of a quasisymmetric 2-(35, 7, 3) design with intersection numbers 3 and 1. Calderbank [6] has proved the nonexistence of such a design. ∎

THEOREM 3.11. *Strongly regular decomposition* 10 *in Table* 1 *does not exist.*

Proof. We use Seidel switching (see [9] or [16]). Since $k_0 = -2r_0 s_0$, we obtain a strong graph by extending Γ_0 with an isolated vertex. Next we isolate, by switching, a vertex of Γ_1. The graph on the remaining vertices is strongly regular with the same parameters as the original Γ_0. However, the remaining vertices of Γ_1 now induce a coclique of size 19. Therefore, by Theorem 2.6 we have constructed strongly regular decomposition 11, which is impossible. ∎

The smallest unsettled case in the table is 19. Tonchev constructed a strongly regular graph with the parameters of Γ_0 (see [21]), but it has no cliques of size 15, so it does not admit a strongly regular decomposition. (In fact, Tonchev's graph has maximal clique and coclique size equal to 10.)

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