The Structure of Symmetric Lie Algebras

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1. INTRODUCTION

In Winter [7], a certain class of Lie algebras, symmetric Lie algebras, and a corresponding class of combinatorial structures, Lie root systems, are introduced and studied independently. They are then interrelated, making it possible to study symmetric Lie algebras using Lie root systems. This is undertaken in Winter [4, 6] where further results on Lie root systems are obtained and used to study simple symmetric Lie algebras and two large classes of Lie algebras generalizing the Albert–Zassenhaus Lie algebras and the Kaplansky Lie algebras.

The Lie algebras and structures to which we refer are the symmetric Lie algebras and the Lie root systems which we define in Definitions 1.1–1.4 below. And the system of roots corresponding to a given symmetric Lie algebra to which we refer is the one we then describe in Theorem 1.5.

The purpose of this paper is to use the classification of Lie root systems of low rank to gain access to the structure of the automorphism group of a symmetric Lie algebra $L$; and then we use the theory of algebraic groups and the theory of Lie root systems to prove the following theorem, which expresses the structure of a symmetric Lie algebra $L$ in terms of a classical Lie algebra $L_s$, a semisimple symmetric Lie algebra $L^W$ whose root system is a Witt root system (defined below) and solvable ideals. In the case of a ground field of characteristic 0, the theorem simply says that a symmetric Lie algebra of characteristic 0 is of the form $L = L_R \oplus \text{Solv} L$ with $L_R$ semisimple, which follows from Levi’s Theorem. So, we restrict ourselves in this paper to the much more difficult context of a ground field $k$ of prime characteristic $p > 3$ (and sometimes 7).

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THEOREM 5.1. Let \((L, H)\) be a symmetric Lie algebra over a field of characteristic \(p > 7\). Assume that \((L, H)\) has no section of type \(T_2\). Then \((L, H)\) has a subalgebra \(L^W\) and an ideal \(I\) such that

1. \(L = L^W + I\) and \(L^W/\text{Solv } L^W\) is a symmetric Lie algebra with Witt root system isomorphic to \(R^W\) and \(L^W \cap I \subseteq \text{Solv } L^W\);

2. \(I\) has a "Levi decomposition" \(I = L_S \oplus \text{Solv } I\) where \(S\) is any classical complement of \(R^W\) in \(R\), \(L_S\) is a classical Lie algebra of type \(S\), and \((\text{Solv } I)^2\) is nilpotent.

1.1. DEFINITION. Given a Lie algebra \(L = \sum_{a \in R} L_a\) (Cartan decomposition of \(L\) relative to a specified split Cartan subalgebra \(H = L_0\)), we let \(L^+_a = \{ x \in L_a \mid [h, x] = a(h) x \text{ for all } h \in L_0 \}\).

1.2. DEFINITION. A symmetric Lie algebra with symmetric Cartan subalgebra \(H = L_0\) is a finite dimensional Lie algebra \(L = \sum_{a \in R} L_a\) such that \(a([L^+_a, L^+_a]) \neq 0\) for all nonzero roots \(a \in R\).

We will refer to a symmetric Lie algebra \(L\) with symmetric Cartan subalgebra \(H\) as the symmetric Lie algebra \((L, H)\).

1.3. DEFINITION. A root system is a pair \((V, R)\) where \(V\) is a vectorspace and \(R\) is a finite subset of \(V\) containing \(0\) and spanning \(V\) which has a symmetry \(r_a(v) = v - a^0(v) a\) \((v \in V)\) for each nonzero \(a \in R\), the conditions defining "symmetry" being:

1. \(a^0 \in \text{Hom}_k(V, k)\) and \(a^0(a) = 2\);

2. \(r_a(R_b(a)) = R_b(a)\) for every bounded \(a\)-orbit \(R_b(a)\) \((b \in R)\).

In the above definition, \(R_b(a)\) denotes the maximal string \(\{b - ra, \ldots, b + qa\}\) of \(a\)-consecutive roots containing \(b\), which is unbounded if it has \(p\) (the maximum possible number of) elements and bounded otherwise.

The rank of the root system \(R\) is the dimension of \(V\), whereas the \(\mathbb{Z}\)-rank of \(R\) is the rank of its groupoid dual

\[\text{Hom}(R, \mathbb{Z}) = \{ f : \mathbb{R} \to \mathbb{Z} \mid f(a + b) = f(a) + f(b) \}\]

when all of \(a, b, a + b\) are in \(R\).

The set \(R \cap \mathbb{Z}a\) of elements of \(R\) that are integral multiples of \(a\) is denoted by \(Ra\) and is called the 1-section determined by \(a\). Note that \(\mathbb{Z}a\) has \(p\) elements if \(a\) is not zero, since the characteristic is \(p > 0\).

For a nonzero root \(a\), we introduce the following terminology:

1. If \(Ra = \mathbb{Z}a\), then \(a\) is called a Witt root and \(Ra\) is said to be of type \(W_1\) (since it is isomorphic to the system of roots of a Witt algebra of rank 1).
(2) If \( Ra = \{-a, 0, a\} \), then \( a \) is called a classical root and \( Ra \) is said to be of type \( A_1 \) (since it is isomorphic to the system of roots of a simple Lie algebra of type \( A_1 \)).

This leads us to root systems that play a key role in studying symmetric Lie algebras.

1.4. Definition. A Lie root system is a root system \((V, R)\) such that

1. \( R \) is the union of the subsets

\[ R^W = \{ a \in R | a = 0 \text{ or } a \text{ is a Witt root} \} \]

and

\[ R^C = \{ a \in R | a \text{ is a classical root} \}; \]

2. for every Witt root \( a \in R^W \), the \( a \)-orbit \( R(a) \) of any root \( b \) has 1, \( p - 1 \), or \( p \) elements.

In particular, any nonzero root system in the sense of 1.3 all of whose roots are classical or 0 is a Lie root system called a classical root system, condition (2) of 1.4 being satisfied vacuously in this case. Similarly, a Lie root system all of whose roots are Witt roots is called a Witt root system. Classical root systems play a key role in this paper, because of their frequent appearance in the structure theory and because they are the root systems of the classical Lie algebras (cf. Theorem 2.2). Witt root systems also play a key role, as we see from the main structure theorem, Theorem 5.1.

Using the representation theory for the simple rank 1 symmetric Lie algebras (the classical and Witt simple Lie algebras of rank 1), the following is proved.

1.5. Theorem (Winter [7]). For any symmetric Lie algebra \( L = \sum_{a \in R} L_a \), \( (V, R) = (L^*_0, R) \) is a Lie root system, \( L^*_0 \) being the dual space of \( L_0 \).

In order to study the root system locally, two roots at a time, as a tool to use in studying it globally, the rank two Lie root systems are determined by classifying the possibilities for pairs of roots. These possibilities for root pairs can be represented by the root pair types

\[ A_1 \vee A_1, A_2, B_2, G_2, W_1 \vee A_1, W_2(m, n), W_1 \oplus A_1(m), S_2(m), T_2 = S_2(A_1)(m). \]

Here, the parameters \( m \) and \( n \) are needed to take into account all possibilities for certain Cartan integers of the pairs of roots, as described in
Table I taken from Winter [7]. In the diagrams given in the table, black nodes represent classical roots and white nodes represent Witt roots.

This classification of root pairs induces a classification of Lie root systems of rank 2 (those generated by 2 but no fewer roots). Since the isomorphism type of a rank 2 Lie root system does not change when only the parameters of the 2 roots generating it are changed, the isomorphism types of rank 2 root systems are obtained by simply dropping the parameters. So, the isomorphism types can be represented as

\[ A_1 \vee A_1, A_2, B_2, G_2, W_1 \vee A_1, W_2, W_1 \oplus A_1, S_2, T_2 = S_2(A_1). \]

The last of these, the Lie root system \( T_2 = S_2(A_1) \), deserves special attention at this time because of the role it does not play in this paper:

Henceforth, we assume that there are no 2-sections of type \( T_2 \) in the Lie root system of the symmetric Lie algebra under study.

### Table I

Possibilities for Pairs of Independent Roots \( a, b \), Up to Change of Signs

<table>
<thead>
<tr>
<th>No.</th>
<th>Diagram</th>
<th>Recognition conditions on ( a ) and ( b )</th>
<th>Type of ( \text{Rab} )</th>
<th>( a^0(b) )</th>
<th>( b^0(a) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td><img src="https://example.com/diagram1.png" alt="Diagram" /></td>
<td>( a, b \in R' ), ( R_b(a) = {b} )</td>
<td>( A \vee A )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2.</td>
<td><img src="https://example.com/diagram2.png" alt="Diagram" /></td>
<td>( a, b \in R' ), ( a^<em>(b) b^</em>(a) = 1 )</td>
<td>( A_2 )</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>3.</td>
<td><img src="https://example.com/diagram3.png" alt="Diagram" /></td>
<td>( a, b \in R' ), ( a^<em>(b) b^</em>(a) = 2 )</td>
<td>( B_2 )</td>
<td>-2</td>
<td>-1</td>
</tr>
<tr>
<td>4.</td>
<td><img src="https://example.com/diagram4.png" alt="Diagram" /></td>
<td>( a, b \in R' ), ( a^<em>(b) b^</em>(a) = 3 )</td>
<td>( G_2 )</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>5.</td>
<td><img src="https://example.com/diagram5.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( R_b(a) = {b} )</td>
<td>( W \vee A )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>6.</td>
<td><img src="https://example.com/diagram6.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( R_b(a) = {b} )</td>
<td>( W \vee W )</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>7.</td>
<td><img src="https://example.com/diagram7.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( R_b(a) = a + b )</td>
<td>( W_2 )</td>
<td>-m</td>
<td>n</td>
</tr>
<tr>
<td>8.</td>
<td><img src="https://example.com/diagram8.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( a + b )</td>
<td>( W \oplus A )</td>
<td>-m</td>
<td>0</td>
</tr>
<tr>
<td>9.</td>
<td><img src="https://example.com/diagram9.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( a^<em>(b) b^</em>(a) = 1 )</td>
<td>( W \oplus A )</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>10.</td>
<td><img src="https://example.com/diagram10.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( a^0(b) b^0(a) = 4 )</td>
<td>( S_2 )</td>
<td>-m</td>
<td>-4/m</td>
</tr>
<tr>
<td>11.</td>
<td><img src="https://example.com/diagram11.png" alt="Diagram" /></td>
<td>( a, b \in R^0 ), ( a + b )</td>
<td>( T_2 )</td>
<td>0</td>
<td>-m</td>
</tr>
</tbody>
</table>
The reasons for making this assumption are that no symmetric Lie algebra having $T_2$ as its Lie rootsystem is known to exist, and that we need it for our methods to work.

The Lie rootsystem $T_2 = S_2(A_1)$ that we have just banished is the Lie rootsystem $S_n(R) = S_n + R$ introduced in Winter [7] where $n = 2$ and $R = A_1$. Here $S_n$ is the rootsystem

$$\{0\} \cup \{(r_1, ..., r_n)|r_1 + \cdots + r_n \neq 0\}$$

(where the $r_1, ..., r_n$ Lie in the prime field) and $R$ is any Lie rootsystem contained in

$$\{(r_1, ..., r_n)|r_1 + \cdots + r_n = 0\}$$

(where the $r_1 + \cdots + r_n$ lie in the prime field). So,

$$T_2 = S_2 + \{(1, -1), (0, 0), (-1, 1)\} = S_2 \cup \{(1, -1), (-1, 1)\}.$$

It is a rootsystem having $p(p - 1)$ Witt roots and two classical roots $\pm a$.

Given this classification, which gives the possibilities for the 2-sections of $L$ (rootsystems of the symmetric subalgebras of toral rank 2 of $L$ determined by two roots), general results about symmetric Lie algebras are proved in Winter [7] and then applied in Winter [4, 6] to get the following results:

a. classification of all irreducible Witt rootsystems of ranks 1, 2, and 3;

b. classification of all irreducible Witt rootsystems having no sections $S_2, W_1 \oplus (W_1 \vee W_1);$

c. the rootsystem of any simple nonclassical symmetric Lie algebra having no 2-section $T_2$ is a Witt rootsystem;

d. classification of the rootsystems of the generalized classical Albert–Zassenhaus Lie Algebras for $p > 3$;

e. classification of the rootsystems of the classical Albert–Zassenhaus–Kaplansky Lie algebras for $p > 3$;

f. construction of a Weyl group $W(R)$ of the rootsystem $R$ of a symmetric Lie algebra which, when the Cartan matrix is nonsingular, acts transitively on certain classical complements $S$ (certain maximal classical subrootsystems of $R$).

Now, in this paper, we pursue the structure of symmetric Lie algebras from another vantage point, that of its automorphism group Aut $L$. Although it is elementary that Aut $L$ is an algebraic group for any finite dimensional Lie algebra $L$, it is rare that much more is known about Aut
$L$ for general classes of Lie algebras of characteristic $p$. A notable exception is when $L$ is a classical Lie algebra, in which case the existence of inner automorphisms is assured by the shortness of the lengths of root strings. This is explained in Seligman [2, Chap. III, pp. 50–72] which is devoted to determining the automorphism groups of the classical Lie algebras.

In this paper, we make use of the theory of Lie root systems and their classification for low ranks to prove, subject to the assumption that there be no 2-section of type $T_2$, that the lengths of root strings of classical roots are at most 4. For $p > 7$, this has rather strong implications for $\text{Aut } L$, namely that it is quite large and that much can be said about its structure in terms of certain special subgroups. The full power of the theory of algebraic groups now can be brought into play in studying $L$, since $L$ is built from pieces having counterparts in $\text{Aut } L$. It is harnessing the power of the theory of algebraic groups in this way that lies at the heart of the proof of the structure theorem of Section 5.

2. PRELIMINARIES

Throughout the paper, $k$ is a field of prime characteristic $p$, $L$ is a Lie algebra, and $L = \sum_{a \in R} L_a$ is a Cartan decomposition where the Cartan subalgebra is $H = L_0$. We take the point of view throughout the paper that we are working with a fixed Lie algebra $L$ and a fixed Cartan subalgebra $H$ of it, never to be changed in the discussion. So, we are really studying Lie algebras relative to given Cartan subalgebras.

Assume next that $L$ is a symmetric Lie algebra. We regard the root-system $R$ of $L$ as a subset of the $k$-span $V$ of $R$ in the vector space of functions from $L_0$ to $k$. By Theorem 1.5, $(R, V)$ is a Lie root system. The type of $(L, H)$ is just the isomorphism class or type of its root system $R$. For example, if $R$ is of type $W_2$, then $L$ is of type $W_2$.

If $S$ is a subrootsystem of $R$, then $L_S$, defined below, is a symmetric Lie algebra with Cartan subalgebra $H_S$, also defined below, and root system $S$.

2.1. DEFINITION. $L_S = \sum_{a \in S - \{0\}} ([L_a, L_{-a}] + L_a)$ and $H_S = \sum_{a \in S - \{0\}} [L_a, L_{-a}]$.

The intersection $Ra_1 \cdots a_k$ of $R$ with $\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_k$ is a Lie root system for any roots $a_1, \ldots, a_k$, called a $k$-section. We have seen in Section 1 that the types of possible 2-sections are

$A_1 \lor A_1, A_2, B_2, G_2, W_1 \lor A_1, W_2, W_1 \oplus A_1, S_2, T_2$.

If $S$ is the $k$-section $S = Ra_1 \cdots a_k$, the subalgebra $L_S$ is a $k$-section of $L$. 
Then $L_S$ is a symmetric Lie algebra with root system $S$ and the type of $L_S$ is just the isomorphism type of $S$.

Recall from Section 1 that $R^w$ denotes the set of roots $a \in R$ such that

$$a, 2a, \ldots, (p-1)a$$

are also in $R$, the set of Witt roots of $R$; and $R^C = R - R^w$, the set of classical roots of $R$. The set $R^w$ of Witt roots is a Witt root system as defined in Section 1. So, by Winter [7], the possible types of 1-, 2-, and 3-sections of $R^w$ are

$$W_1, W_2, S_2, W_3, W_1 \oplus (W_1 \vee W_1), W_1 \oplus S_2, S_3, S_3 \oplus (W_1 \vee W_1), S_3(S_2).$$

and joins (unions of pieces sharing exactly one element in common, namely 0) of two or more of these. Moreover, if there are no sections of type $S_2$ or $S_3(W_1 \vee W_1)$, the irreducible components of $R^w$ are finite vector spaces.

We need the following results of Winter [4, 7].

2.2. **Theorem** (Winter[7]). A Lie rootsystem $R$ is isomorphic to the root-system of a classical Lie algebra if and only if it is a classical rootsystem as defined in Section 1 (that is, a nonzero rootsystem all of whose nonzero roots are classical.)

2.3. **Theorem** (Winter[7]). The rootsystem $R$ of a symmetric Lie algebra $L$ is classical if and only if

$$L_R = \sum_{a \in R - \{0\}} ([L_a, L_{-a}] + L_a)$$

is classical with classical Cartan subalgebra $H_R = \sum_{a \in R - \{0\}} [L_a, L_{-a}]$.

2.4. **Theorem** (Winter [4, 7]). Let $R$ be a Lie rootsystem which has no section of type $T_2$, and let $b \in R$ be any classical root of $R$. Then

1. $R$ has a splitting $f$ at $b$, that is, a mapping $f : R \to R$ such that $b \in f(R)$ and
   
   (i) $f(a) = 0$ if and only if $a \in R^w$;
   
   (ii) $f^2 = f$;
   
   (iii) $f(a)(f(c)) = a(c)$ for all $a \in R^C, c \in R$;

2. for any splitting $f$ of $R$, $S = f(R)$ is a classical rootsystem and $R \subseteq R^w + S$;

3. Any two splittings $f, f'$ are uniquely isomorphic; that is, there exists a unique isomorphism $W : f(R) \to f'(R)$ of classical rootsystems such that $W(c) = c \in R^w$ for all $c \in f(R)$. 
If we let $f$ be any splitting for the root system $R$ of a symmetric Lie algebra $L$, we get a classical root system $S = f(R)$. We'll refer to any $S$ obtained in this fashion as a classical complement of $R$. By Theorem 2.3, the corresponding Lie subalgebra $L_S$ defined below is a classical Lie algebra with classical Cartan subalgebra $H_S$.

2.5. Definition. For any classical complement $S$ of $R$, the classical Lie algebra $(L_S, H_S)$ where

$$L_S = \sum_{a \in S} ([L_a, L_{-a}] + L_a)$$

and $H_S = L_S \cap H$ is called a classical complement of $L$.

Although there is a classical complement $L_S$ of $L$ for any classical complement $S$ of its root system, we define only one Witt core, in the sense given below, since $R^W$ is a subrootsystem of the root system of $L$.

2.6. Definition. The Witt core of a symmetric Lie algebra $(L, H)$ is the subalgebra $L_W = \sum_{a \in R^W} ([L_a, L_{-a}] + L_a)$.

3. The Groups $\operatorname{Aut} L$ and $\operatorname{Inner Aut}(L, H)$

In this section, we study the group $\operatorname{Aut} L$ of automorphisms of a symmetric Lie algebra $L$. As a backdrop to this study, we fix a symmetric Cartan subalgebra $H$ and corresponding Lie root system $R$ of $L$. Using $R$ and its properties, we introduce the subgroup $\operatorname{Inner Aut}(L, H)$ of inner automorphisms of $L$ with respect to $H$. We also introduce some important subgroups of it. These are algebraic groups and their Lie algebras come into play during the discussion. By using their Lie algebras, we are able to get the structure of $L$ by going to $\operatorname{Aut} L$, using properties of algebraic groups and then transferring important information back.

In order to be sure that we can construct automorphisms from inner derivations by the method given below, we assume for the remainder of the paper that $R$ has no 2-section $T_2$ and that $p > 7$.

Consider any roots $a \in R^c$, $b \in R$, and the section $Rab$ determined by them. If $Rab$ is reducible, then the $a$-orbit $R_b(a)$ of $b$ is $\{b\}$. If $Rab$ is irreducible, then $Rab$ is isomorphic to one of $A_1, A_2, B_2, G_2, A_1 \oplus W_1$, since $T_2$ is excluded by hypothesis, and all other rank 2 Lie root systems contain no classical roots. In every case, the cardinality $|R_b(a)|$ of the
a-orbit $R_b(a)$ of $b$ is at most 4. This is well known if $R$ is classical, e.g., Jacobson [1, p. 117]. In the remaining case, $R$ must be of type $A_1 \oplus W_1$, so that it is isomorphic to $\{a, 0, -a\} \oplus \{0, c, \ldots, (p - 1) c\}$, by Winter [6]. In this case, it is also true since the only coefficients $m$ of linear combinations $ma + nc \in A_1 \oplus W_1$ are $-1$, $0$, $1$. It follows that $(ad x)^4 = 0$ for all $x \in L_\alpha$ ($a \in R^C$).

Since we've assumed that $p > 7$, we have $4 < (p - 1)/2$ and it follows that $\exp(ad x)$, for $a \in R^C$ and $x \in L_\alpha$, is contained in the automorphism group $G = \text{Aut} L$ of $L$. So, for any classical root $a$ and any $x \in L_\alpha$, the algebraic group $G$ contains the subgroup

$$U_x = \{\exp tx | t \in k\}.$$

As a homomorphic image (as a group and as an algebraic variety) of the closed connected additive group $kx$, $U_x$ is a closed connected subgroup of $G$. It follows from the theory of algebraic groups that for any subset $X$ of $R$, the subgroup $G_X$ of $G$ generated by the union of all $U_x$ for which $x \in L_\alpha$ for some $a \in X \cap R^C$ is a closed connected subgroup of $L$. We now single out certain of these groups of special interest to us here.

3.1. Definition. The group $G_R$ is called the group of inner automorphisms of $L$ with respect to $H$ and is denoted $\text{Inner Aut} (L, H)$. If $S$ is a classical complement of $R$, then $G_S$ is called a classical complement of $\text{Aut} L$.

In Winter [4], we defined $\mathcal{M}(X)$ for any $X \subseteq R$ to be the subgroup of the automorphism group $\text{Aut} R$ of $R$ generated by $\{r_b | b \in X \cap R^C\}$; and we defined $\hat{U}(X)$ to be the subgroup of $\text{Aut} R$ generated by

$$\{r_b r_{a+b} | b \in X \cap R^C, a \in R^W, a + b \in R\}.$$

(Recall from Section 1 that $r_b(a)$ is defined as $a - b^0(a) b$.) We also considered the matrix $(a^0_j(a_i))$, $a_1, \ldots, a_n$ being a base (simple system of roots in the sense of Jacobson [1, pp. 119–121]) for a given classical complement of $R$. Since any two classical complements of $R$ are isomorphic by some automorphism of $R$, by Theorem 2.4, the matrices $(a^0_j(a_i))$, $(b^0_j(b_i))$ relative to bases $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ of any two classical complements are similar. In this sense, we refer to any such matrix $(a^0_j(a_i))$ with respect to any base $a_1, \ldots, a_n$ for any classical complement as a Cartan matrix for $R$. Since $p > 7$, the Cartan matrix for $R$ is nonsingular if and only if no irreducible component of a classical complement of $R$ is of type $A_n$ where $p | n + 1$. Given this, we state the following result on the rootsystem $R$ of $L$.

3.2. Theorem (Winter [4]). Suppose that $R$ has no section of type $T_2$ and let $S$ be a classical complement of $R$ with base of simple roots $\pi$. Then
In any case, the module $\Pi_{[0]}(R)$ is the trivial submodule of $R$, and the module $\Pi_{[(\nu'-1)\beta]}(R)$ is one dimensional. Thus by (3.2) we obtain

(3.7) **Proposition.** Under the hypothesis of (3.1), we have a direct sum decomposition

$$\text{Res}_{(TP,)^0 \cap TP_r}^{TP_r}(\text{Ind}_{TP_r}^{TP_r}(V)) \cong \text{Inf}^0(\text{Ind}_{TP, \cap TL_r}^{TL_r}(V)) \otimes \bigoplus_{\gamma \in \mathbb{Z}^+((\mathfrak{g}' \setminus \mathfrak{g}'))} \Pi_{[\gamma]}(R).$$

In particular, the modules of fixed points given in (3.5) are $TL_r$-module direct summands of $\text{Res}_{(TP,)^0 \cap TP_r}^{TP_r}(\text{Ind}_{TP_r}^{TP_r}(V))$.

(3.9) **Remark.** The ring $R$ is a local ring of the form $k[\{x_i\}]$ where the $\{x_i\}$ are a suitable set of generators, and it is naturally a graded ring. The homogeneous components of $R$ under this grading are invariant under the conjugation action of $TL_r$. This gives rise to a second decomposition of $R$, $R = \bigoplus R_i$, where $R_i$ is the homogeneous component of degree $i$ relative to the above grading. Each $R_i$ has a decomposition analogous to (3.6), and the decomposition of $R$ obtained from the decompositions of the $R_i$'s is a refinement of (3.6). One can use this to improve the decomposition (3.8).

### 4. Lifting, Extensions, and Filtrations

(4.1) **Lifting.** If $TL_r$ is the Levi factor of a parabolic subgroup scheme $TP_r$, then it is also the Levi factor of $TP_r^0$. Thus we have two ways to inflate $TL_r$-modules to modules for a parabolic subgroup scheme of $TG_r$. Denote these inflation functors by $\text{Inf}: \mathcal{M}_{TL_r} \to \mathcal{M}_{TP_r}$ and $\text{Inf}^0: \mathcal{M}_{TL_r} \to \mathcal{M}_{TP_r^0}$, respectively.

The composition of inflation with induction is called lifting. If $TP_r \subset TP'_r$ are two parabolic subgroup schemes of $TG_r$, we obtain two lifting functors $A_{TP_r}: \mathcal{M}_{TL_r} \to \mathcal{M}_{TP_r}$ and $(A^0)_{TP_r}: \mathcal{M}_{TM_r} \to \mathcal{M}_{TP_r}$. When $TP'_r = TG_r$, we denote these functors by $A_{TL_r}$ and $(A^0)_{TL_r}$, respectively.

(4.2) **Extensions.** In this paragraph, we show that a large class of modules for $TP_r^0$ can be extended to rational modules for $TG_r$. The extensions are not unique in general, but in one important special case, they are unique up to isomorphism.

We let $Q^0 = \text{Ind}_{TP_r^0}^{TP_r}(k)$ and recall that $\text{Res}_{U_{TP_r^0}}^{TP_r^0}(Q^0)$ is the injective hull of the trivial $U_{TP_r^0}$-module.
LEMMA. Let $M$ be a rational $TL_r$-module. Then the $TP_r^0$-module, $	ext{Ind}_{TL_r}^{TP_r^0}(M) \cong \text{Inf}^0(M) \otimes Q^0$ can be extended to a rational $TG_r$-module. This extension is not unique in general.

Proof. By (3.1), $\text{Res}_{TP_r^0}^{TG_r}(A_{TL_r}(M)) \cong \text{Ind}_{TL_r}^{TP_r^0}(M)$, so an extension is simply $A_{TL_r}(M)$.

Suppose $I$ is an injective indecomposable $TG_r$-module. The restriction of $I$ to $TP_r^0$ is also injective since $TG_r/TP_r^0$ is affine [CPS 5, (2.1), (4.5)]. In general, it does not remain indecomposable. We write $I = I_1 \oplus I_2 \oplus \ldots \oplus I_n$ where the $I_j$'s are injective indecomposable rational $TP_r^0$-modules.

Suppose $S$ is an irreducible rational $TL_r$-module and $Q_L(S)$ is its injective hull. Then $\text{Ind}_{TL_r}^{TP_r^0}(Q_L(S))$ is a rational injective $TP_r^0$-module. Since the socle of $\text{Ind}_{TL_r}^{TP_r^0}(Q_L(S)) = Q'$ is contained in the space of $U_{TP_r^0}$-fixed points of this module, it follows from (3.5) that $\text{Soc}(Q') = \text{Soc}(Q'_{TP_r^0}) = \text{Soc}(Q_L(S)) = \text{Inf}^0(S)$, hence

$$\text{Ind}_{TL_r}^{TP_r^0}(Q_L(S))$$

is the injective hull of $\text{Inf}^0(S)$.

It follows that each of the modules $I_j$ has an extension to a $TG_r$-module, hence $\text{Res}_{TP_r^0}^{TG_r}(I_j)$ has at least two distinct extensions to a $TG_r$-module whenever $s > 1$. It is clear that examples where $s > 1$ occur in all types of rank at least two, hence the proof of the lemma is complete.

PROPOSITION. Let $M$ be a finite dimensional rational $TL_r$-module. Let $A \in X$ and $Z = A + Z \Sigma_L$. If $M$ is $Z$-generated, then the $TP_r^0$-module $\text{Ind}_{TL_r}^{TP_r^0}(M)$ has an extension to a rational $TG_r$-module which is unique up to isomorphism.

Proof. Suppose $V$ is a rational $TG_r$-module with

$$\text{Res}_{TP_r^0}^{TG_r}(V) \cong \text{Inf}^0(M) \otimes Q^0.$$

By (2.2.3) and (3.7), it follows that $\Pi_Z(V) \cong \text{Inf}^0(M) \otimes \text{Soc}(Q^0)$ and that as an $TL_r$-module we have a direct sum decomposition

$$\text{Res}_{TL_r}^{TG_r}(V) \cong \Pi_Z(V) \oplus \Pi_{Z'}(V),$$

where $Z'$ is the complement of $Z$ in $X$. If $\theta$ is a weight of the module $\Pi_Z(M)$, then the sum of $\theta$ and any non-negative integer linear combination of roots for root subgroups in $U_{TP_r}$ is an element of $Z'$. Hence $\Pi_{Z'}(V)$ is in fact a $TP_r$-submodule of $V$, and the quotient map $V \to V/\Pi_{Z'}(V)$ is a homomorphism of $TP_r$-modules onto $\text{Inf}^0(M) \otimes \text{Soc}(Q^0)$. By universal mapping, we obtain a homomorphism of $TG_r$-modules, $V \to A_{TL_r}(M)$. This homomorphism is injective on the $U_{TP_r}$-socle of $V$, hence it is injective. Now a dimension count shows it is surjective.
On the other hand, applying our above equation for \( r_b(a) \) to \( h \) gives

\[
\begin{align*}
  r_b(a)(h) &= a(h) - 2(a(h_b)/b(h_b)) b(h) \\
  &= a(h) - a(h_b) b(h)
\end{align*}
\]

since \( b(h_b) = 2 \). Comparing, we see that \( w_b^*(a)(h) = r_b(a)(h) \) are equal for all \( h \).

For any subset \( X \) of \( R \), we let \( T(X) \) be the group of elements of \( G_R \) which map \( L_a \) into itself for \( a = 0 \) and all \( a \in X \cap R^C \). Then \( T(R) \) is a closed normal subgroup of \( M(R) \). Moreover, it contains the products \( w_b(s) w_b(t) \) and \( w_b(s)^{-1} w_b(t) \) for any nonzero \( s \) and \( t \), by a straightforward argument using Lemma 3.3 and Theorem 3.4. Taking \( w_b = w_b(1) \), we can express

\[ w_b(t) = w_b v(t), \]

where \( v(t) = w_b^{-1} w_b(t) \in T(R) \). This having been said, for any subset \( X \) of \( R \), we define \( N(X) \) to be the subgroup of \( M(R) \) generated by

\[ \{ w_b | b \in X \cap R^C \} \cup T(R). \]

Then \( N(X) \) contains \( w_b(t) \) for all \( b \in X \cap R^C \). We then let \( W(X) \) be the factor group \( N(X)/T(R) \).

The action of \( w_b^* \) on \( R \) is the same as the action of the reflection \( r_b \) of the Lie rootsystem \( R \), by Theorem 3.4. So, we get the Weyl group \( \mathfrak{W}(R) \) of the rootsystem \( R \) as a homomorphic image of the groups \( N(R) \). Since \( T(R) \) leaves each element of \( R^C \) fixed, any action of \( N(R) \) factors to an action of \( W(R) \) on \( R \). So, we have a surjective homomorphism from \( W(R) \) to \( \mathfrak{W}(R) \). Since an element of the kernel of this mapping is represented by an element leaving invariant each \( L_a \) for \( a \in R^C \), and since such elements are in \( T(R) \), this kernel is trivial. So, our homomorphism is actually an isomorphism from \( W(R) \) to \( \mathfrak{W}(R) \).

Similarly, we get the same type of thing for each classical complement of \( S \), where the discussion goes along the same lines as in the case of complex semisimple Lie algebras. The phenomena for \( R \) amounts to a coherently patched together phenomena for the classical complements. Specifically, for any classical complement \( S \) of \( R \) and any \( b \in S \), \( S \) is mapped into itself by \( w_b^* \) and the action of \( w_b^* \) on \( S \) is the same as the action of the reflection \( r_b \) of the Lie rootsystem \( R \) on \( S \), that is, \( w_b^* |_S = r_b |_S \). So, we get the Weyl group of the classical rootsystem \( S \) as a homomorphic image of the groups \( N(S) \). As in the case of \( R \) above, the action of \( N(S) \) factors to an action of \( W(S) \) on \( S \), since \( T(R) \) leaves each element of \( S \) fixed. So, as in the case of \( R \), we have an isomorphism from \( W(S) \) to \( \mathfrak{W}(S) \).
We also define $V(X)$ as the group generated by the set of products
\[ \{ w_b w_{a+b} | b \in X \cap R^C, a \in R^w, a + b \in R \} \cup T(R). \]
Since $b \in X \cap R^C, a \in R^w, a + b \in R$ implies that $a + b \in R^C$, $V(X)$ is a subgroup of $W(X)$. In particular, it follows that $V(X)$ normalizes $T$, so that $U(X) = V(X)/T(R)$ is a group. Under the isomorphism from $W(R)$ to $\mathfrak{W}(R)$, $U(R)$ is mapped isomorphically to $\hat{U}(R)$. We now can lift Theorem 6.8 of Winter [4] from $\text{Aut } R$ to $\text{Aut } L$ as follows, using Theorem 3.4.

3.5. Theorem. Suppose that $R$ has no section of type $T_2$ and let $S$ be a classical complement of $L$ with base of simple roots $\pi$. Then

1. $W(R)$ and $\mathfrak{W}(R)$ are canonically isomorphic, and this isomorphism induces a canonical isomorphism from $U(R)$ to $\hat{U}(R)$;
2. $W(\pi) = W(S)$;
3. $U(\pi) = U(S)$, $U(S)$ is normalized by $W(S)$, $U(S) \cap W(S) = T(R)$, and $W(R) = W(S) U(S)$.

Although the equation
\[ g(L_a) = L_{g^*(a)} \]
relates the actions of $\mathfrak{W}(R)$ on $R$ and $W(R)$ on $L$, the relation between the action of $\mathfrak{W}(R)$ on $R$ and the inner automorphism action of $W(R)$ on $G = \text{Aut } L$ is not so close. About the best that can be done easily is to exponentiate the relationship $g(L_a) = L_{g^*(a)}$ for a classical complement, using the equation $g(\exp \text{ad } x) g^{-1} = \exp \text{ad } g(x)$. Of course, this means that whenever the Weyl group $\mathfrak{W}(R)$ of $R$ acts transitively on the classical complements $S$ of $R$, the Weyl group $W(R)$ acts transitively by inner automorphisms on the classical complements $G_S$ of $\text{Aut } L$. We know that this is true when the Cartan matrix of $R$ is nonsingular, by Theorem 3.2, which gives us the following version of Levi's Theorem for algebraic groups of characteristic 0.

3.6. Theorem. When the Cartan matrix of $R$ is nonsingular, the classical complements $G_S$ of the algebraic group $\text{Aut } L$ are conjugate.

The $G_S$ are contained in a special part of $\text{Aut } L$ whose Lie algebra has an ideal which we are able to study structurally, in the next section.
4. An Ideal in the Lie Algebra of the Group inner Aut(L, H)

Whereas the group $G_R = \text{Inner Aut } L$ is generated by the closed connected one parameter groups

$$U_a = \{ \exp t \text{ ad } e_a \mid t \in k \} = \exp L_a \text{ (for } a \in R^C, e_a \in L_a - \{0\})$$

we consider the subalgebra $I$ generated by the one-dimensional Lie algebras $L_a$ of these $U_a$ (for $a \in R^C, e_a \in L_a - \{0\}$).

We show first that $I$ is an ideal of $L$. For this, it suffices to prove that $I$ is normalized by the Witt core $L_W$ of $L$, since $L = L_W + I$. So, consider $a \in L_W^C - \{0\}$ and $b \in L^C$. If $a + b \in R$, then $a + b \in R^C$, as one sees from the classification of root systems of rank 2. It follows that $[L_a, L_b] \subseteq I$. On the other hand, $[L_a, L_b] = 0$ if $a + b$ is not in $R$. Using this, we see that $\text{ad } L_W$ maps a generating set for the algebra $I$ back into $I$, from which it follows that $L_W$ normalizes $I$. So, $I$ is an ideal of $L$.

The $U_a$ (for $a \in R^C$) stabilize the ideal $I$, since they consist of exponentials of inner derivations by elements of $I$. So, the group $G_R$ stabilizes $I$. The subalgebra $\text{Lie } G_R$ of the Lie algebra $\text{Der } L$ of derivations of $L$ contains $\text{Lie } U_a = \text{ad } L_a$ for all $a \in R^C$. It follows that, $\text{Lie } G_R$ contains the Lie subalgebra $\text{ad } I$ of $\text{Der } L$ generated by the $\text{ad } L_a$ (for $a \in R^C$). Since the group $G_R$ stabilizes the ideal $I$, its Lie algebra $\text{Lie } G_R$ stabilizes $I$. It then follows that

$$[\text{Lie } G_R, \text{ad } I] \subseteq \text{ad } G_R(I) \subseteq \text{ad } I,$$

so that

1. $\text{ad } I$ is an ideal of $\text{Lie } G_R$.

Since $G_R$ is an algebraic group with a Cartan subgroup $C_G$ containing a maximal torus of $T(R)$ containing the torus $T$ generated by the elements $w_b(s)^{-1} w_b(t)$ (for nonzero scalars $s, t$ and $b \in R^C$), its Lie algebra $\text{Lie } G_R$ has Cartan subalgebra $C$ containing $\text{Lie } T$ and classical Lie algebra quotient $\text{Lie } G_R/(\text{Solv Lie } G_R)$ with classical Cartan subalgebra $C$ containing $\text{Lie } T$. Ideals and quotients of classical Lie algebras are classical, as one easily verifies from the definition of classical Lie algebra, Seligman [2, p. 28]. Since

$$(\text{ad } I + \text{Solv Lie } G_R)/\text{Solv Lie } G_R$$

is an ideal of the classical Lie algebra $\text{Lie } G_R/(\text{Solv Lie } G_R)$, it follows that

$$(\text{ad } I + \text{Solv Lie } G_R)/\text{Solv Lie } G_R$$
is a classical Lie algebra. By the fundamental homomorphism theorem, 
adI/J is a classical Lie algebra where J is the intersection of ad I and 
Solv Lie $G_R$. Since J is solvable, it is contained in Solv ad I. So, since 
(Solv ad I)/J is a solvable ideal in a classical Lie algebra, it is 0 and 
Solv ad I = J. It follows that ad I/J = ad I/Solv ad I, so that ad I/Solv ad I 
is a classical Lie algebra. Since Solv I contains the center of I, the algebras 
I/Solv I and ad I/ad(Solv I) are isomorphic. It follows that ad(Solv I) must 
be the solvable radical of ad I, so that ad(Solv I) = Solv ad I. But then 
I/Solv I and ad I/Solv ad I are isomorphic, from which we conclude that

2. $I$/Solv $I$ is a classical Lie algebra.

Since $(Solv Lie G_R)^2$ is nilpotent, by the Lie–Kolchin theorem for the 
connected solvable radical Solv $G_R$ of the algebraic group $G_R$, it follows 
that

3. $(Solv I)^2$ is nilpotent.

5. The structure of $L$

We now use the ideal ad $I$ of Lie Inner Aut($L$, $H$) and its properties to 
prove

5.1. Theorem. Let $(L, H)$ be a symmetric Lie algebra over a field of 
characteristic $p > 7$. Assume that $(L, H)$ has no section of type $T_2$. Then 
$(L, H)$ has a subalgebra $L^w$ and an ideal $I$ such that

1) $L = L^w + I$ and $L^w$/Solv $L^w$ is a symmetric Lie algebra with Witt 
rootsystem isomorphic to $R^w$ and $L^w \cap I \subseteq$ Solv $L^w$;

2) $I$ has a Levi decomposition $I = L_S \oplus$ Solv $I$ where $S$ is any classical 
complement of $R^w$ in $R$, $L_S$ is a classical Lie algebra of type $S$, and $(Solv I)^2$ 
is nilpotent.

Proof. We take $L^w$ to be the Witt core of $(L, H)$ introduced in 
Section 2, and we let $I$ be the ideal introduced in Section 4. Since 

$$R = R^w \cup R^c,$$

we have

$$L = L^w + I.$$

We now prove that $L^w$/Solv $L^w$ is symmetric with Witt rootsystem $R^w$ 
relative to $H +$ Solv $L^w$. For this, it suffices to show that for each 
$a \in R^w - \{0\}$, $e_a \in L_{-a}$, $e_a \in L_a^1$, $h_a = [e_{-a}, e_a]$ with $a(h_a) \neq 0$ that $h_a$ is not
an element of $\text{Solv}^L W$. Suppose, to the contrary, that $h_\alpha \in \text{Solv}^L W$, under such circumstances. Then $L_\alpha = [h_\alpha, L_\alpha]$ and $L_{-\alpha} = [h_\alpha, L_{-\alpha}]$ are contained in $\text{Solv}^L W$, so that $\text{Solv}^L W$ contains the semisimple algebra $ke_{-\alpha} + kh_\alpha + ke_\alpha$. This is not possible, however, so we must conclude that $h_\alpha$ is not an element of $\text{Solv}^L W$, as we set out to show. Consequently, $L^W/\text{Solv}^L W$ is a symmetric Lie algebra with Witt root system $R^W$.

We next proceed to show, for any classical complement $S$ of $R^W$ in $R$, that $I = L_S + \text{Solv}^I$ where $L_S = \sum_{a \in R - \{0\}} ([L_{-a}, L_a] + L_a$), that $L_S$ is a classical Lie algebra, and that $L^W \cap I$ is solvable. Since the generators $L_a$ (for $a \in R^C$) of the ideal $I$ are one dimensional, by Theorems 2.2 and 2.3, $L^W \cap I, L_S, I$ do not change when we replace $L$ by $L_0 + \sum_{a \in R - \{0\}} L_a^1$. (Recall that $L_a^1 = \{x \in L_a | [h, x] = a(h) x$ for all $h \in L_0\}$.) Also, $L^W \cap I, L_S, I$ do not change when we replace $L$ by $L^2$. By the first of these two observations, we may assume, with no loss of generality, that $\text{ad} L_0$ acts diagonally on $\sum_{a \in R - \{0\}} L_a$. By the second, we may assume, with no loss of generality, that $L = L^2$. Since

$$L = L_0 + \sum_{a \in R - \{0\}} L_a$$

and $L = L^2$, we have

$$L_0 = [L_0, L_0] + \sum_{a \in R - \{0\}} [L_{-a}, L_a].$$

It follows that $\text{ad} L_0$ acts diagonally (as well as by nilpotent transformations) on the subset $\sum_{a \in R - \{0\}} [L_{-a}, L_a]$ of $L_0$ as well, so that $\text{ad} L_0 \sum_{a \in R - \{0\}} [L_{-a}, L_a] = 0$. From this, we see that

$$[L_0, L_0] = [L_0, [L_0, L_0]] + \left[ L_0, \sum_{a \in R - \{0\}} [L_{-a}, L_a] \right] = [L_0, [L_0, L_0]]$$

and

$$[L_0, L_0] = [L_0, [L_0, L_0]].$$

It follows from this and the nilpotency of $L_0$ that $[L_0, L_0] = 0$. But then

$$L_0 = [L_0, L_0] + \sum_{a \in R - \{0\}} [L_{-a}, L_a] = \sum_{a \in R - \{0\}} [L_{-a}, L_a]$$

and $\text{ad} L_0$ acts diagonally on all of $L$.

It is easy to show that we may assume without loss of generality that $L$ has center 0. This enables us to imbed $L$ isomorphically via $\text{ad}$ in the derivation algebra Der $L$ of $L$ as $\text{ad} L$. Consider the $p$-closure $\text{ad} L$ of $\text{ad} L$.
in $\text{Der} \ L$ and take $K$ to be any Cartan subalgebra of the centralizer of the torus $\text{ad} \ L_0$ in $\text{ad} \ L$. One then easily verifies that $K$ is a Cartan subalgebra of $\text{ad} \ L$ which centralizes $\text{ad} \ L_0$.

Consequently, $K$ stabilizes the rootspaces $L_a$ (for $a \in R$) of $L$ with respect to the Cartan subalgebra $L_0$. Since $I$ is generated by the one dimensional rootspaces $I_a$ (for $a \in R^C$), it follows, in particular, that $K$ acts diagonally on $I$. Since $\text{ad} \ L / \text{ad} \ L$ is abelian with Cartan subalgebra $(K + \text{ad} \ L) / \text{ad} \ L$, we have $\text{ad} \ L = K + \text{ad} \ L$.

We now show that $K + \text{ad} \ I$ is a restricted subalgebra of $\text{ad} \ L$. Taking the weight space decomposition

$$K + \text{ad} \ I = K + \sum_{a \in Q} \text{ad} I_a(K)$$

of $\text{ad} \ K$ on $K + \text{ad} \ I$ and an element $x \in I_a(K)$ (for $a \in Q$), we have $[K, \text{ad} x] \subseteq k \text{ ad} x$ (set of scalar multiples of $\text{ad} x$). The reason for this is that $K$ acts diagonally on $I$, as we have seen. But then

$$0 = [\cdots [K, \text{ad} x], ..., \text{ad} x] = [K, (\text{ad} x)^p].$$

It follows from the self-normalizing property of Cartan subalgebras that $(\text{ad} x)^p$ is contained in the Cartan subalgebra $K$. Consequently, the algebra $K + I$ is spanned by elements whose $p$th powers are contained in $K + \text{ad} \ I$. By the theory of restricted Lie algebras of Jacobson [1, pp. 187–194], it follows that $K + \text{ad} \ I$ is a restricted subalgebra of $\text{Der} \ L$.

Since $K + \text{ad} \ I$ is a restricted subalgebra of $\text{Der} \ L$, the $p$-closure $\text{ad} I$ of $\text{ad} \ I$ is contained in $K + \text{ad} \ I$. Consequently, $\text{ad} I = M + \text{ad} \ I$ where $M = K \cap \text{ad} I$. Since $M \subseteq K$, $M$ acts diagonally on $I$. We show now that, in fact, $\text{ad} M$ acts diagonally on $\text{ad} I$, so that $\text{ad} M$ is a torus in the restricted Lie algebra $\text{ad} I$. For this, let $J$ be the centralizer of $M$ in $\text{ad} I$ and consider the subalgebra

$$J + \text{ad} \ I = J + \sum_{a \in Q} I_a(K)$$

of $\text{ad} I$. Then for $x \in I_a(K)$, $a \in Q$ we have

$$[M, \text{ad} x] = k \text{ ad} x,$$

$$[\cdots [M, \text{ad} x], ..., \text{ad} x] = [M, (\text{ad} x)^p].$$

It follows, as in the preceding paragraph, that $J + \text{ad} I$ is a restricted subalgebra of $\text{ad} I$. But then $J + \text{ad} I$ equals the $p$-closure $\text{ad} I$. Since $\text{ad} I = J + \text{ad} I$, $[M, J] = 0$, and $M$ acts diagonally on $I$, it follows that $\text{ad} M$ acts diagonally on $\text{ad} I$. In particular, $\text{ad} M$ acts diagonally on $M$, so
that $\text{ad } M(M) = 0$ and $M^2 = 0$, since $M$ is nilpotent. This establishes that $\text{ad } M$ acts as a torus on $\overline{\text{ad } I}$.

Since $\text{ad } M$ acts a torus on $\overline{\text{ad } I}$, $N = \{ D \in \overline{\text{ad } I} | [M, D] = 0 \}$ is the Fitting nullspace of $\text{ad } M$ acting on $\overline{\text{ad } I}$. Consequently, $N$ is a self-normalizing subalgebra of $\overline{\text{ad } I}$. By the same argument, $\text{ad } K$ acts diagonally on $\overline{\text{ad } I}$. Since $K$ is nilpotent and contains $M$, it follows that $\text{ad } K$ centralizes $\text{ad } M$. But then $N$ is stable under the action of $\text{ad } K$, and $N = \sum_{\alpha \in \rho} N_\alpha(K)$ (weight space decomposition of $N$ with respect to $\text{ad } K$). Let $x \in N_\alpha(K)$. Then $[K, x] \subseteq kx$ implies that $[K, x^p] = 0$. Since $x^p \in N$ by the self-normalizing property of $N$, it follows that $x^p \in N_0(K) = K \cap N \subseteq K \cap \overline{\text{ad } I} = M$. But then $[x^p, N] = 0$, so that $\text{ad } x$ is nilpotent on $N$. This establishes that the weakly closed set $\bigcup_{\alpha \in \rho} N_\alpha(K)$ of generators of $N$ consists of nilpotent elements, so that $N$ is nilpotent, by Jacobson [1, p. 33].

We now have shown that $N$ is a self-normalizing subalgebra of $\overline{\text{ad } I}$ and $N$ is nilpotent, so that $N$ is a Cartan subalgebra of $\overline{\text{ad } I}$.

We now can show that $M$ is a maximal torus of $\overline{\text{ad } I}$. For this, it suffices to show that $M$ is a maximal torus of its centralizer $N$ in $\overline{\text{ad } I}$. Since $N$ is restricted with central toral ideal $M$, it suffices, by Winter [5], to show that the restricted Lie algebra $N/M$ consists of nilpotent elements. As shown in the preceding paragraph, $N = \sum_{\alpha \in \rho} N_\alpha(K)$ and the $p$th powers of the elements of $\bigcup_{\alpha \in \rho} \text{ad } N_\alpha(K)$ are contained in $\text{ad } M$. We again invoke the results of Jacobson [1, p. 33] to conclude that $N/M$ consists of nilpotent elements. Thus, $M$ is a maximal torus of $\overline{\text{ad } I}$.

We now know that the $p$-closure $\overline{\text{ad } I}$ of $\text{ad } I$ in $\text{Der } L$ has the form $\overline{\text{ad } I} = M + \overline{\text{ad } I}$ where $M = K \cap \overline{\text{ad } I}$ and $M$ is a maximal torus of $\overline{\text{ad } I}$. Moreover, $\overline{\text{ad } I}$ and $M$ stabilize the ideal $\text{Solv } I$ of $I$. We let $\text{Solv }$ denote the $p$-closure of the radical $\text{Solv ad } I$ of $\text{ad } I$ and we consider the quotient $D = \text{ad } I/\text{Solv}$ and its ideal $C = (\text{ad } I + \text{Solv})/\text{Solv}$. Note that $C$ is canonically isomorphic to the classical Lie algebras $\text{ad } I/\text{Solv ad } I$, $I/\text{Solv } I$. This follows from the fundamental homomorphism theorems by the following argument. We know from Section 4 that $I/\text{Solv } I$ is classical. Since it is isomorphic to $\text{ad } I/\text{Solv ad } I$,

the latter is also classical. The algebra $C$ is isomorphic to $\text{ad } I/\text{ad } J$ where $\text{ad } J$ is the intersection of $\text{ad } I$ and $\text{Solv }$. Since $\text{ad } J$ is contained in $\text{Solv ad } I$ and conversely, we have $\text{ad } J = \text{Solv ad } I$. It follows that $C$ is isomorphic to $\text{ad } I/\text{Solv ad } I$, as we had asserted. By Seligman [2, p. 48], the classical Lie algebra $C$ is restricted. Since $C$ has center 0, this restricted structure for $C$ is unique. Consequently, $C$ is a restricted ideal of the restricted Lie algebra $D$.

Since $\overline{\text{ad } I} = M + \text{ad } I$ and $M$ is a maximal torus of $\overline{\text{ad } I}$ whose centralizer
$N$ is a Cartan subalgebra of $\text{ad } I$, $M^* = (M + \text{Solv})/\text{Solv}$ is a maximal torus of $D$ and $N^* = (N + \text{Solv})/\text{Solv}$ is a Cartan subalgebra of $D$, by Winter [5], and $D = M^* + C$. Since $C$ is a restricted ideal in $D$ and $N^*$ is a Cartan subalgebra of $D$, we see as in the fourth preceding paragraph, that the Fitting null space $C_0$ of $\text{ad } (N^* \cap C)$ in $C$ is a Cartan subalgebra of $C$. Since $C$ is classical, it follows that $C_0$ is a torus. For if it were not, we could extend the ground field to the algebraic closure and get a classical Lie algebra having a Cartan subalgebra which is not a torus. But this is impossible by the conjugacy of Cartan subalgebras in this case, as established in Seligman [2, p. 116]. Since the maximal torus $M^*$ of $D$ centralizes $C_0$, it follows that $C_0 \subseteq M^*$. But then we have

$$M^* \cap C = C_0 = N^* \cap C.$$ 

Since $D = M^* + C$ with $M^* \subseteq N^* \subseteq D$, we have

$$N^* = M^* + N^* \cap C = M^* + C_0 = M^*.$$ 

This establishes that $M^* = N^*$ and $M^*$ is a toral Cartan subalgebra of $D$ such that $C_0 = M^* \cap C$ is a Cartan subalgebra of $C$.

Since we reduced to the case where $\text{ad } L_0$ acts diagonally on $L$, $\text{ad } L_0$ is contained in the Cartan subalgebra $K$ of $\text{ad } I$, by an easy argument. In fact, we have $K \cap \text{ad } L = \text{ad } L_0$, since $L_0$ is a Cartan subalgebra of $L$. Now $C_0 = M^* \cap C$ where $M = K \cap \text{ad } L$. Since

$$C = (\text{ad } I + \text{Solv})/\text{Solv},$$

it follows that

$$C_0 = ((K \cap \text{ad } I) + \text{Solv})/\text{Solv}.$$ 

Since

$$K \cap \text{ad } I = (K \cap \text{ad } L) \cap \text{ad } I = \text{ad } L_0 \cap \text{ad } I = \text{ad } I_0,$$

where $I_0 = L_0 \cap I$, it follows that $C_0 = (\text{ad } I_0 + \text{Solv})/\text{Solv}$.

We now take $a \in R \setminus \{0\}$ and consider the corresponding

$$C_a = (\text{ad } L_a + \text{Solv})/\text{Solv}.$$ 

If $a \in R^C$ (the case where $a$ is a classical root), then we have $L_a = I_a$ (intersection of $L_a$ and $I$) and then

$$C^{(a)} = C_a + [C_{-a}, C_a] + C_a$$

is isomorphic to

$$L^{(a)} = L_{-a} + [L_{-a}, L_a] + L_a$$
(which in turn is isomorphic to $S_{12}$), since

$$C^{(a)} = (\text{ad } L^{(a)} + \text{Solv})/\text{Solv}$$

and

$$\text{ad } L^{(a)} \cap \text{Solv} = \{0\}$$

(by the semi-simplicity of $L^{(a)}$). Suppose next that $a \in R^W$ (the case where $a$ is a Witt root). When we pass from $a$ to $a^* : C_0 \to k$ defined by $a^*(\text{ad } h + \text{Solv}) = a(h)$ (for $h \in I_0$), we consider the two possibilities $a^* = 0$ and $a^* \neq 0$. If $a^* = 0$, then $[I_0, I_a] = 0$ and $[C_0, C_a] = 0$ implies that $C_a$ is contained in the Cartan subalgebra

$$C_0 = (\text{ad } I_0 + \text{Solv})/\text{Solv}.$$ 

Since $I_0 \cap I_0 = \{0\}$ and $\text{Solv}$ is a sum of weight spaces for $M$, it follows that $I_a \subseteq \text{Solv}$. Suppose next that $a^* \neq 0$ and choose $h \in I_0$ such that $a(h) \neq 0$. Then the ideal $I$ contains $[L_{ia}, h] = L_{ia}$ for $i = 1, \ldots, p - 1$. But then it follows that the root system $R^* = \{a^* | a \in R\}$ of $C$ with respect to $C_0$ contains the roots $0^*, a^*, \ldots, (p - 1)a^*$. But then $2a^*$ is a root in $R^*$ and the root system $R^*$ is not a reduced classical root system. We claim that this cannot be. For if it did, then by extending the ground field to the algebraic closure, it could happen also over an algebraically closed field. Then $C$ would be a classical Lie algebra over an algebraically closed field. In this setting, by the conjugacy of Cartan subalgebras of Seligman [2, p. 116], the root system of $C$ with respect to every Cartan subalgebra is reduced classical. This is a contradiction which forces us to conclude that the case $a^* = 0$, $a \in R^W$ never occurs. We conclude that the root system $R^* = \{a^* | a \in R\}$ of $C$ with respect to $C_0$ coincides with

$$R = \{a^* | a \in R^C\} \cup \{0\},$$

and that $a^* = 0^*$ for all $a \in R^W$. It follows that the algebra

$$\sum_{a \in R^W} ([I_{-a}, I_a] + I_a)$$

is contained in $\text{Solv } I$. From this, it follows that $L^W \cap I$ is solvable and, therefore, is contained in $\text{Solv } L^W$.

We now let $S$ be a classical complement of $R^W$ in $R$ and consider the subalgebra $L_S = \sum_{a \in S} ([L_{-a}, L_a] + I_a)$ of $I$. Then $L_S$ is a classical Lie algebra, by Theorem 2.2. Consequently, $L_S \cap \text{Solv } I = \{0\}$. We now proceed to show that $I = L_S + \text{Solv } I$. We know that $I/\text{Solv } I$ is a classical Lie algebra with Cartan subalgebra $I_0 = \sum_{a \in R^C} [L_{-a}, L_a]$, since $C = (\text{ad } I + \text{Solv})/\text{Solv}$ is a classical Lie algebra with Cartan subalgebra
$C_0 = (\text{ad } I_0 + \text{Solv})/\text{Solv}$. Also, by identification of $C$ and $I/\text{Solv } I$, we have seen that the Cartan decomposition of

$$I^* = I/\text{Solv } I$$

is

$$I^* = \sum_{a^* \in R^*} I_{a^*},$$

where

1. $a^*$ is the function on $I^*_5 = (I_0 + \text{Solv } I)/\text{Solv } I$ defined by $a^*(h + \text{Solv } I) = a(h)$ (for $a \in R$);
2. $a^* = 0$ if $a \in R^W$ and $a^* \neq 0$ if $a \in R^C$;
3. $R^* = \{a^* \mid a \in R\}$.

Since $R \subseteq R^W + S$ and $a^* = 0$ for $a \in R^W$, it follows that $R^*$ coincides with the set $S^* = \{a^* \mid a \in S\}$. Finally, since $a^* \neq 0$ for all $a \in S - \{0\}$ (since $a^* \neq 0$ for all $a^* \in R^C$), the Cartan decomposition for $I^*$ is $I^* = \sum_{a^* \in S^*} I_{a^*}$. Taking any $a \in S - \{0\}$, we have

1. $(I_a + \text{Solv } I)/\text{Solv } I \subseteq I_{a^*}$ and $\dim I_{a^*} = 1$;
2. $I_a \cap \text{Solv } I = \{0\}$ (since $I_{-a} + [I_{-a}, I_a] + I_a$ is semisimple).

It follows that $((I_a + \text{Solv } I)/\text{Solv } I = I_{a^*})$ for all $a \in S$ and, consequently, that

$$I/\text{Solv } I = \sum_{a \in S} I_{a^*} = \sum_{a \in S} (I_a + \text{Solv } I)/\text{Solv } I = (I_S + \text{Solv } I)/\text{Solv } I.$$

Thus, $I = I_S + \text{Solv } I$ as asserted.

REFERENCES