The Structure of Symmetric Lie Algebras

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1. INTRODUCTION

In Winter [7], a certain class of Lie algebras, *symmetric Lie algebras*, and a corresponding class of combinatorial structures, *Lie rootsystems*, are introduced and studied independently. They are then interrelated, making it possible to study symmetric Lie algebras using Lie rootsystems. This is undertaken in Winter [4, 6] where further results on Lie rootsystems are obtained and used to study simple symmetric Lie algebras and two large classes of Lie algebras generalizing the Albert–Zassenhaus Lie algebras and the Kaplansky Lie algebras.

The Lie algebras and structures to which we refer are the symmetric Lie algebras and the Lie rootsystems which we define in Definitions 1.1 1.4 below. And the system of roots corresponding to a given symmetric Lie algebra to which we refer is the one we then describe in Theorem 1.5.

The purpose of this paper is to use the classification of Lie rootsystems of low rank to gain access to the structure of the automorphism group of a symmetric Lie algebra L; and then we use the theory of algebraic groups and the theory of Lie rootsystems to prove the following theorem, which expresses the structure of a symmetric Lie algebra L in terms of a classical Lie algebra L_S, a semisimple symmetric Lie algebra L^W whose root system is a Witt rootsystem (defined below) and solvable ideals. In the case of a ground field of characteristic 0, the theorem simply says that a symmetric Lie algebra of characteristic 0 is of the form $L = L_R \oplus \mathrm{Solv}\, L$ with L_R semi-simple, which follows from Levi's Theorem. So, we restrict ourselves in this paper to the much more difficult context of a ground field k of prime characteristic $p > 3$ (and sometimes 7).

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THEOREM 5.1. *Let (L, H) be a symmetric Lie algebra over a field of characteristic $p > 7$. Assume that (L, H) has no section of type T_2. Then (L, H) has a subalgebra L^W and an ideal I such that*

(1) $L = L^W + I$ *and* $L^W/\mathrm{Solv}\, L^W$ *is a symmetric Lie algebra with Witt rootsystem isomorphic to* R^W *and* $L^W \cap I \subseteq \mathrm{Solv}\, L^W$;

(2) I *has a "Levi decomposition"* $I = L_S \oplus \mathrm{Solv}\, I$ *where S is any classical complement of R^W in R, L_S is a classical Lie algebra of type S, and* $(\mathrm{Solv}\, I)^2$ *is nilpotent.*

1.1. DEFINITION. Given a Lie algebra $L = \sum_{a \in R} L_a$ (Cartan decomposition of L relative to a specified split Cartan subalgebra $H = L_0$), we let $L_a^1 = \{x \in L_a \,|\, [h, x] = a(h)\, x \text{ for all } h \in L_0\}$.

1.2. DEFINITION. A *symmetric Lie algebra* with *symmetric Cartan subalgebra* $H = L_0$ is a finite dimensional Lie algebra $L = \sum_{a \in R} L_a$ such that $a([L_{-a}^1, L_a^1]) \neq 0$ for all nonzero roots $a \in R$.

We will refer to a symmetric Lie algebra L with symmetric Cartan subalgebra H as the symmetric Lie algebra (L, H).

1.3. DEFINITION. A *rootsystem* is a pair (V, R) where V is a vectorspace and R is a finite subset of V containing 0 and spanning V which has a *symmetry* $r_a(v) = v - a^0(v)\, a$ $(v \in V)$ for each nonzero $a \in R$, the conditions defining "symmetry" being:

(1) $a^0 \in \mathrm{Hom}_k(V, k)$ and $a^0(a) = 2$;

(2) $r_a(R_b(a)) = R_b(a)$ for every *bounded a-orbit $R_b(a)$* $(b \in R)$.

In the above definition, $R_b(a)$ denotes the maximal string $\{b - ra, ..., b + qa\}$ of a-consecutive roots containing b, which is *unbounded* if it has p (the maximum possible number of) elements and bounded otherwise.

The *rank* of the root system R is the dimension of V, whereas the $\mathbb{Z}$-*rank* of R is the rank of its *groupoid dual*

$$\mathrm{Hom}(R, \mathbb{Z}) = \{f : R \to \mathbb{Z} \,|\, f(a + b) = f(a) + f(b)$$
$$\text{when all of } a, b, a + b \text{ are in } R\}.$$

The set $R \cap \mathbb{Z}a$ of elements of R that are integral multiples of a is denoted by Ra and is called the 1-*section* determined by a. Note that $\mathbb{Z}a$ has p elements if a is not zero, since the characteristic is $p > 0$.

For a nonzero root a, we introduce the following terminology:

(1) If $Ra = \mathbb{Z}a$, then a is called a *Witt root* and Ra is said to be of type W_1 (since it is isomorphic to the system of roots of a Witt algebra of rank 1).

(2) If $Ra = \{-a, 0, a\}$, then a is called a *classical root* and Ra is said to be of type A_1 (since it is isomorphic to the system of roots of a simple Lie algebra of type A_1).

This leads us to rootsystems that play a key role in studying symmetric Lie algebras.

1.4. DEFINITION. A *Lie rootsystem* is a rootsystem (V, R) such that

(1) R is the union of the subsets

$$R^W = \{a \in R \mid a = 0 \text{ or } a \text{ is a Witt root}\}$$

and

$$R^C = \{a \in R \mid a \text{ is a classical root}\};$$

(2) for every Witt root $a \in R^W$, the a-orbit $R_b(a)$ of any root b has 1, $p-1$, or p elements.

In particular, any nonzero rootsystem in the sense of 1.3 all of whose roots are classical or 0 is a Lie rootsystem called a *classical rootsystem*, condition (2) of 1.4 being satisfied vacuously in this case. Similarly, a Lie rootsystem all of whose roots are Witt roots is called a *Witt rootsystem*. Classical rootsystems play a key role in this paper, because of their frequent appearance in the structure theory and because they are the root-systems of the classical Lie algebras (cf. Theorem 2.2). Witt rootsystems also play a key role, as we see from the main structure theorem, Theorem 5.1.

Using the representation theory for the simple rank 1 symmetric Lie algebras (the classical and Witt simple Lie algebras of rank 1), the following is proved.

1.5. THEOREM (Winter [7]). *For any symmetric Lie algebra $L = \sum_{a \in R} L_a$, $(V, R) = (L_0^*, R)$ is a Lie root system, L_0^* being the dual space of L_0.*

In order to study the rootsystem locally, two roots at a time, as a tool to use in studying it globally, the rank two Lie rootsystems are determined by classifying the possibilities for pairs of roots. These possibilities for root pairs can be represented by the root pair types

$$A_1 \vee A_1, \ A_2, \ B_2, \ G_2, \ W_1 \vee A_1, \ W_2(m, n), \ W_1 \oplus A_1(m), \ S_2(m),$$

$$T_2 = S_2(A_1)(m).$$

Here, the parameters m and n are needed to take into account all possibilities for certain *Cartan integers* of the pairs of roots, as described in

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TABLE I

Possibilities for Pairs of Independent Roots a, b, Up to Change of Signs

No.	Diagram	Recognition conditions on a and b	Type of Rab	$a^0(b)$ $b^0(a)$
1.	$\bullet \quad \bullet$ $\;_a\;\;\;_b$	$a, b \in R^\bullet$, $R_b(a) = \{b\}$	$A \vee A$	0 0
2.	$\bullet\!\!-\!\!\bullet$ $\;_a\;\;\;_b$	$a, b \in R^\bullet$, $a^*(b)\, b^*(a) = 1$	A_2	-1 -1
3.	$\bullet\!\!\Leftarrow\!\!\bullet$ $\;_a\;\;\;_b$	$a, b \in R^\bullet$, $a^*(b)\, b^*(a) = a^*(b)/b^*(a) = 2$	B_2	-2 -1
4.	$\Leftleftarrows$ $\;_a\;\;\;_b$	$a, b \in R^\bullet$, $a^*(b)\, b^*(a) = a^*(b)/b^*(a) = 3$	G_2	-3 -1
5.	$\circ \quad \bullet$ $\;_a\;\;\;_b$	$a \in R^0$, $b \in R^\bullet$, $R_b(a) = \{b\}$	$W \vee A$	0 0
6.	$\circ \quad \circ$ $\;_a\;\;\;_b$	$a, b \in R^0$, $R_b(a) = \{b\}$	$W \vee W$	0 0
7.	$\circ\!\!\overset{mn}{-}\!\!\circ$ $\;_a\;\;\;_b$	$a, b \in R^0$, $R_b(a) = \mathbb{Z}a + b$	W_2	$-m$ $-n$
8.	$\circ\!\!\rightarrow\!\!\bullet$ $\;_a\;\;\;_b$	$a \in R^0$, $b \in R^\bullet$, $a + b \in R$	$W \oplus A$	$-m$ 0
9.	$\Rightarrow$ $\;_a\;\;\;_b$	$a, b \in R^\bullet$, $a^*(b)\, b^*(a) = 4$, $a^*(b)/b^*(a) = 1$	$W \oplus A$	-2 -2
10.	$\circ\!\!\Rrightarrow\!\!\circ$ $\;_a\;\;\;_b$	$a, b \in R^0$, $a^0(b)\, b^0(a) = 4$	S_2	$-m$ $-4/m$
11.	$\circ\!\!-\!\!\!\leftarrow\!\!\bullet$ $\;_a\;\;\;\cdot\;_b$	$a \in R^0$, $b \in R^\bullet$, $a + b \in R^0$	T_2	0 $-m$

Table I taken from Winter [7]. In the diagrams given in the table, black nodes represent classical roots and white nodes represent Witt roots.

This classification of root pairs induces a classification of Lie rootsystems of *rank* 2 (those generated by 2 but no fewer roots). Since the isomorphism type of a rank 2 Lie rootsystem does not change when only the parameters of the 2 roots generating it are changed, the isomorphism types of rank 2 rootsystems are obtained by simply dropping the parameters. So, the isomorphism types can be represented as

$$A_1 \vee A_1, \; A_2, \; B_2, \; G_2, \; W_1 \vee A_1, \; W_2, \; W_1 \oplus A_1, \; S_2, \; T_2 = S_2(A_1).$$

The last of these, the Lie rootsystem $T_2 = S_2(A_1)$, deserves special attention at this time because of the role it *does not* play in this paper:

Henceforth, we assume that there are no 2-sections of type T_2 in the Lie rootsystem of the symmetric Lie algebra under study.

The reasons for making this assumption are that no symmetric Lie algebra having T_2 as its Lie rootsystem is known to exist, and that we need it for our methods to work.

The Lie rootsystem $T_2 = S_2(A_1)$ that we have just banished is the Lie rootsystem $S_n(R) = S_n + R$ introduced in Winter [7] where $n = 2$ and $R = A_1$. Here S_n is the rootsystem

$$\{0\} \cup \{(r_1, ..., r_n) \mid r_1 + \cdots + r_n \neq 0\}$$

(where the $r_1, ..., r_n$ Lie in the prime field) and R is any Lie rootsystem contained in

$$\{(r_1, ..., r_n) \mid r_1 + \cdots + r_n = 0\}$$

(where the $r_1 + \cdots + r_n$ lie in the prime field). So,

$$T_2 = S_2 + \{(1, -1), (0, 0), (-1, 1)\} = S_2 \cup \{(1, -1), (-1, 1)\}.$$

It is a rootsystem having $p(p-1)$ Witt roots and two classical roots $\pm a$.

Given this classification, which gives the possibilities for the 2-*sections* of L (rootsystems of the symmetric subalgebras of toral rank 2 of L determined by two roots), general results about symmetric Lie algebras are proved in Winter [7] and then applied in Winter [4, 6] to get the following results:

 a. classification of all irreducible Witt rootsystems of ranks 1, 2, and 3;

 b. classification of all irreducible Witt rootsystems having no sections S_2, $W_1 \oplus (W_1 \vee W_1)$;

 c. the rootsystem of any simple nonclassical symmetric Lie algebra having no 2-section T_2 is a Witt rootsystem;

 d. classification of the rootsystems of the *generalized classical Albert–Zassenaus Lie Algebras* for $p > 3$;

 e. classification of the rootsystems of the *classical Albert–Zassenhaus–Kaplansky Lie algebras* for $p > 3$;

 f. construction of a *Weyl group* $W(R)$ of the rootsystem R of a symmetric Lie algebra which, when the Cartan matrix is nonsingular, acts transitively on certain *classical complements* S (certain maximal classical subrootsystems of R).

Now, in this paper, we pursue the structure of symmetric Lie algebras from another vantage point, that of its automorphism group Aut L. Although it is elementary that Aut L is an algebraic group for any finite dimensional Lie algebra L, it is rare that much more is known about Aut

L for general classes of Lie algebras of characteristic p. A notable exception is when L is a classical Lie algebra, in which case the existence of inner automorphisms is assured by the shortness of the lengths of root strings. This is explained in Seligman [2, Chap. III, pp. 50–72] which is devoted to determining the automorphism groups of the classical Lie algebras.

In this paper, we make use of the theory of Lie rootsystems and their classification for low ranks to prove, subject to the assumption that there be no 2-section of type T_2, that the lengths of root strings of *classical roots* are at most 4. For $p > 7$, this has rather strong implications for Aut L, namely that it is quite large and that much can be said about its structure in terms of certain special subgroups. The full power of the theory of algebraic groups now can be brought into play in studying L, since L is built from pieces having counterparts in Aut L. It is harnessing the power of the theory of algebraic groups in this way that lies at the heart of the proof of the structure theorem of Section 5.

2. PRELIMINARIES

Throughout the paper, k is a field of prime characteristic p, L is a Lie algebra, and $L = \sum_{a \in R} L_a$ is a Cartan decomposition where the Cartan subalgebra is $H = L_0$. We take the point of view throughout the paper that we are working with a fixed Lie algebra L and a fixed Cartan subalgebra H of it, never to be changed in the discussion. So, *we are really studying Lie algebras relative to given Cartan subalgebras.*

Assume next that L is a symmetric Lie algebra. We regard the rootsystem R of L as a subset of the k-span V of R in the vector space of functions from L_0 to k. By Theorem 1.5, (R, V) is a Lie rootsystem. The *type* of (L, H) is just the isomorphism class or type of its rootsystem R. For example, if R is of type W_2, then L is of type W_2.

If S is a subrootsystem of R, then L_S, defined below, is a symmetric Lie algebra with Cartan subalgebra H_S, also defined below, and rootsystem S.

2.1. DEFINITION. $L_S = \sum_{a \in S - \{0\}} ([L_a, L_{-a}] + L_a)$ and $H_S = \sum_{a \in S - \{0\}} [L_a, L_{-a}]$.

The intersection $R a_1 \cdots a_k$ of R with $\mathbb{Z} a_1 \oplus \cdots \oplus \mathbb{Z} a_k$ is a Lie rootsystem for any roots $a_1, ..., a_k$, called a *k-section*. We have seen in Section 1 that the types of possible 2-sections are

$$A_1 \vee A_1, A_2, B_2, G_2, W_1 \vee A_1, W_2, W_1 \oplus A_1, S_2, T_2.$$

If S is the k-section $S = R a_1 \cdots a_k$, the subalgebra L_S is a *k-section* of L.

Then L_S is a symmetric Lie algebra with rootsystem S and the type of L_S is just the isomorphism type of S.

Recall from Section 1 that R^W denotes the set of roots $a \in R$ such that

$$a, 2a, ..., (p-1)a$$

are also in R, the set of Witt roots of R; and $R^C = R - R^W$, the set of classical roots of R. The set R^W of Witt roots is a Witt rootsystem as defined in Section 1. So, by Winter [7], the possible types of 1-, 2-, and 3-sections of R^W are

$$W_1, W_2, S_2, W_3, W_1 \oplus (W_1 \vee W_1), W_1 \oplus S_2, S_3, S_3 \oplus (W_1 \vee W_1), S_3(S_2).$$

and *joins* (unions of pieces sharing exactly one element in common, namely 0) of two or more of these. Moreover, if there are no sections of type S_2 or $S_3(W_1 \vee W_1)$, the irreducible components of R^W are finite vector spaces.

We need the following results of Winter [4, 7].

2.2. THEOREM (Winter[7]). *A Lie rootsystem R is isomorphic to the rootsystem of a classical Lie algebra if and only if it is a classical rootsystem as defined in Section 1 (that is, a nonzero rootsystem all of whose nonzero roots are classical.)*

2.3. THEOREM (Winter[7]). *The rootsystem R of a symmetric Lie algebra L is classical if and only if*

$$L_R = \sum_{a \in R - \{0\}} ([L_a, L_{-a}] + L_a)$$

is classical with classical Cartan subalgebra $H_R = \sum_{a \in R - \{0\}} [L_a, L_{-a}]$.

2.4. THEOREM (Winter [4, 7]). *Let R be a Lie rootsystem which has no section of type T_2, and let $b \in R$ be any classical root of R. Then*

1. *R has a* splitting *f at b, that is, a mapping $f : R \to R$ such that $b \in f(R)$ and*

   (i) *$f(a) = 0$ if and only if $a \in R^W$;*

   (ii) *$f^2 = f$;*

   (iii) *$f(a)^0 (f(c)) = a^0(c)$ for all $a \in R^C$, $c \in R$;*

2. *for any splitting f of R, $S = f(R)$ is a classical rootsystem and $R \subseteq R^W + S$;*

3. *Any two splittings f, f' are* uniquely isomorphic; *that is, there exists a unique isomorphism $W : f(R) \to f'(R)$ of classical rootsystems such that $W(c) - c \in R^W$ for all $c \in f(R)$.*

If we let f be any splitting fr the rootsystem R of a symmetric Lie algebra L, we get a classical rootsystem $S = f(R)$. We'll refer to any S obtained in this fashion as a *classical complement* of R. By Theorem 2.3, the corresponding Lie subalgebra L_S defined below is a classical Lie algebra with classical Cartan subalgebra H_S.

2.5. DEFINITION.   For any classical complement S of R, the classical Lie algebra (L_S, H_S) where

$$L_S = \sum_{a \in S - \{0\}} ([L_a, L_{-a}] + L_a)$$

and $H_S = L_S \cap H$ is called a *classical complement* of L.

Although there is a classical complement L_S of L for any classical complement S of its rootsystem, we define only one *Witt core*, in the sense given below, since R^W is a subrootsystem of the rootsystem of L.

2.6. DEFINITION.   The *Witt core* of a symmetric Lie algebra (L, H) is the subalgebra $L_W = \sum_{a \in R^W} ([L_a, L_{-a}] + L_a)$.

3. THE GROUPS Aut L AND Inner Aut(L, H)

In this section, we study the group Aut L of automorphisms of a symmetric Lie algebra L. As a backdrop to this study, we fix a symmetric Cartan subalgebra H and corresponding Lie rootsystem R of L. Using R and its properties, we introduce the subgroup InnerAut(L, H) of inner automorphisms of L with respect to H. We also introduce some important subgroups of it. These are algebraic groups and their Lie algebras come into play during the discussion. By using their Lie algebras, we are able to get the structure of L by going to Aut L, using properties of algebraic groups and then transferring important information back.

In order to be sure that we can construct automorphisms from inner derivations by the method given below, *we assume for the remainder of the paper that R has no 2-section T_2 and that $p > 7$.*

Consider any roots $a \in R^C$, $b \in R$, and the section Rab determined by them. If Rab is reducible, then the a-orbit $R_b(a)$ of b is $\{b\}$. If Rab is irreducible, then Rab is isomorphic to one of

$$A_1, A_2, B_2, G_2, A_1 \oplus W_1,$$

since T_2 is excluded by hypthesis, and all other rank 2 Lie rootsystems contain no classical roots. In every case, the cardinality $|R_b(a)|$ of the

a-orbit $R_b(a)$ of b is at most 4. This is well known if R is classical, e.g., Jacobson [1, p. 117]. In the remaining case, R must be of type $A_1 \oplus W_1$, so that it is isomorphic to $\{a, 0, -a\} \oplus \{0, c, ..., (p-1) c\}$, by Winter [6]. In this case, it is also true since the only coefficients m of linear combinations $ma + nc \in A_1 \oplus W_1$ are -1, 0, 1. It follows that $(\text{ad } x)^4 = 0$ for all $x \in L_a$ $(a \in R^C)$.

Since we've assumed that $p > 7$, we have $4 < (p-1)/2$ and it follows that $\exp(\text{ad } x)$, for $a \in R^C$ and $x \in L_a$, is contained in the automorphism group $G = \text{Aut } L$ of L. So, for any classical root a and any $x \in L_a$, the algebraic group G contains the subgroup

$$U_x = \{\exp tx \,|\, t \in k\}.$$

As a homomorphic image (as a group and as an algebraic variety) of the closed connected additive group kx, U_x is a closed connected subgroup of G. It follows from the theory of algebraic groups that for any subset X of R, the subgroup G_X of G generated by the union of all U_x for which $x \in L_a$ for some $a \in X \cap R^C$ is a closed connected subgroup of L. We now single out certain of these groups of special interest to us here.

3.1. DEFINITION. The group G_R is called the *group of inner automorphisms* of L with respect to H and is denoted Inner Aut (L, H). If S is a classical complement of R, then G_S is called a *classical complement* of Aut L.

In Winter [4], we defined $\mathfrak{W}(X)$ for any $X \subseteq R$ to be the subgroup of the automorphism group Aut R of R generated by $\{r_b \,|\, b \in X \cap R^C\}$; and we defined $\hat{U}(X)$ to be the subgroup of Aut R generated by

$$\{r_b r_{a+b} \,|\, b \in X \cap R^C, a \in R^W, a + b \in R\}.$$

(Recall from Section 1 that $r_b(a)$ is defined as $a - b^0(a) b$.) We also considered the matrix $(a_j^0(a_i))$, $a_1, ..., a_n$ being a *base* (simple system of roots in the sense of Jacobson [1, pp. 119–121]) for a given classical complement of R. Since any two classical complements of R are isomorphic by some automorphism of R, by Theorem 2.4, the matrices $(a_j^0(a_i))$, $(b_j^0(b_i))$ relative to bases $a_1, ..., a_n$ and $b_1, ..., b_n$ of any two classical complements are similar. In this sense, we refer to any such matrix $(a_j^0(a_i))$ with respect to any base $a_1, ..., a_n$ for any classical complement as a *Cartan matrix* for R. Since $p > 7$, the Cartan matrix for R is nonsingular if and only if no irreducible component of a classical complement of R is of type A_n where $p \mid n + 1$. Given this, we state the following result on the rootsystem R of L.

3.2. THEOREM (Winter [4]). *Suppose that R has no section of type T_2 and let S be a classical complement of R with base of simple roots π. Then*

In any case, the module $\Pi_{[0]}(R)$ is the trivial submodule of R, and the module $\Pi_{[(p^r-1)\tilde{\rho}]}(R)$ is one dimensional. Thus by (3.2) we obtain

(3.7) PROPOSITION.   *Under the hypothesis of* (3.1), *we have a direct sum decomposition*

$$\mathrm{Res}^{TP_r''}_{(TP_r')^0 \cap TP_r''}(\mathrm{Ind}^{TP_r''}_{TP_r}(V))$$

$$\simeq \mathrm{Inf}^0(\mathrm{Ind}^{TL_r'}_{TP_r \cap TL_r'}(V)) \otimes \left(\bigoplus_{\gamma \in Z^+(\bar{\Delta}'' \setminus \bar{\Delta}')} \Pi_{[\gamma]}(R) \right). \qquad (3.8)$$

In particular, the modules of fixed points given in (3.5) *are* TL_r'-*module direct summands of* $\mathrm{Res}^{TP_r''}_{(TP_r')^0 \cap TP_r''}(\mathrm{Ind}^{TP_r''}_{TP_r}(V))$.

(3.9) *Remark.*   The ring R is a local ring of the form $k[\{x_i\}]$ where the $\{x_i\}$ are a suitable set of generators, and it is naturally a graded ring. The homogeneous components of R under this grading are invariant under the conjugation action of TL_r'. This gives rise to a second decomposition of R, $R = \oplus R_i$, where R_i is the homogeneous component of degree i relative to the above grading. Each R_i has a decomposition analogous to (3.6), and the decomposition of R obtained from the decompositions of the R_i's is a refinement of (3.6). One can use this to improve the decomposition (3.8).

4. LIFTING, EXTENSIONS, AND FILTRATIONS

(4.1) *Lifting.*   If TL_r is the Levi factor of a parabolic subgroup scheme TP_r, then it is also the Levi factor of TP_r^0. Thus we have two ways to inflate TL_r-modules to modules for a parabolic subgroup scheme of TG_r. Denote these inflation functors by Inf: $\mathcal{M}_{TL_r} \to \mathcal{M}_{TP_r}$ and Inf^0: $\mathcal{M}_{TL_r} \to \mathcal{M}_{TP_r^0}$, respectively.

The composition of inflation with induction is called *lifting*. If $TP_r \subset TP_r'$ are two parabolic subgroup schemes of TG_r, we obtain two lifting functors $\Lambda^{TP_r'}_{TL_r}: \mathcal{M}_{TL_r} \to \mathcal{M}_{TP_r'}$ and $(\Lambda^0)^{TP_r'}_{TL_r}: \mathcal{M}_{TM_r} \to \mathcal{M}_{TP_r'}$. When $TP_r' = TG_r$, we denote these functors by Λ_{TL_r} and $(\Lambda^0)_{TL_r}$, respectively.

(4.2) *Extensions.*   In this paragraph, we show that a large class of modules for TP_r^0 can be extended to rational modules for TG_r. The extensions are not unique in general, but in one important special case, they are unique up to isomorphism.

We let $Q^0 = \mathrm{Ind}^{TP_r^0}_{TL_r}(k)$ and recall that $\mathrm{Res}^{TP_r^0}_{U_{TP_r^0}}(Q^0)$ is the injective hull of the trivial $U_{TP_r^0}$-module.

(4.2.1) LEMMA. *Let M be a rational TL_r-module. Then the TP_r^0-module,*
$\operatorname{Ind}_{TL_r}^{TP_r^0}(M) \simeq \operatorname{Inf}^0(M) \otimes Q^0$ *can be extended to a rational TG_r-module. This extension is not unique in general.*

Proof. By (3.1), $\operatorname{Res}_{TP_r^0}^{TG_r}(A_{TL_r}(M)) \simeq \operatorname{Ind}_{TL_r}^{TP_r^0}(M)$, so an extension is simply $A_{TL_r}(M)$.

Suppose I is an injective indecomposable TG_r-module. The restriction of I to TP_r^0 is also injective since TG_r/TP_r^0 is affine [CPS 5, (2.1), (4.5)]. In general, it does not remain indecomposable. We write $I = I_1 \oplus I_2 \oplus, ..., I_n$ where the I_j's are injective indecomposable rational TP_r^0-modules.

Suppose S is an irreducible rational TL_r-module and $Q_L(S)$ is its injective hull. Then $\operatorname{Ind}_{TL_r}^{TP_r^0}(Q_L(S))$ is a rational injective TP_r^0-module. Since the socle of $\operatorname{Ind}_{TL_r}^{TP_r^0}(Q_L(S)) = \tilde{Q}$ is contained in the space of U_{TP^0}-fixed points of this module, it follows from (3.5) that $\operatorname{Soc}(\tilde{Q}) = \operatorname{Soc}(\tilde{Q}^{U_{TP_r^0}}) = \operatorname{Soc}(Q_L(S)) = \operatorname{Inf}^0(S)$, hence

$$\operatorname{Ind}_{TL_r}^{TP_r^0}(Q_L(S)) \text{ is the injective hull of } \operatorname{Inf}^0(S).$$

It follows that each of the modules I_j has an extension to a TG_r-module, hence $\operatorname{Res}_{TP_r^0}^{TG_r}(I)$ has at least two distinct extensions to a TG_r-module whenever $s > 1$. It is clear that examples where $s > 1$ occur in all types of rank at least two, hence the proof of the lemma is complete.

(4.2.2.) PROPOSITION. *Let M be a finite dimensional rational TL_r-module. Let $\lambda \in X$ and $Z = \lambda + \mathbf{Z}\Sigma_L$. If M is Z-generated, then the TP_r^0-module $\operatorname{Ind}_{TL_r}^{TP_r^0}(M)$ has an extension to a rational TG_r-module which is unique up to isomorphism.*

Proof. Suppose V is a rational TG_r-module with

$$\operatorname{Res}_{TP_r^0}^{TG_r}(V) \simeq \operatorname{Inf}^0(M) \otimes Q^0.$$

By (2.2.3) and (3.7), it follows that $\Pi_Z(V) \simeq \operatorname{Inf}^0(M) \otimes \operatorname{Soc}(Q^0)$ and that as an TL_r-module we have a direct sum decomposition

$$\operatorname{Res}_{TL_r}^{TG_r}(V) \simeq \Pi_Z(V) \oplus \Pi_{Z'}(V),$$

where Z' is the complement of Z in X. If θ is a weight of the module $\Pi_{Z'}(M)$, then the sum of θ and any non-negative integer linear combination of roots for root subgroups in U_{TP_r} is an element of Z'. Hence $\Pi_{Z'}(V)$ is in fact a TP_r-submodule of V, and the quotient map $V \to V/\Pi_{Z'}(V)$ is a homomorphism of TP_r-modules onto $\operatorname{Inf}^0(M) \otimes \operatorname{Soc}(Q^0)$. By universal mapping, we obtain a homomorphism of TG_r-modules, $V \to A_{TL_r}(M)$. This homomorphism is injective on the $U_{TP_r^0}$-socle of V, hence it is injective. Now a dimension count shows it is surjective.

On the other hand, applying our above equation for $r_b(a)$ to h gives

$$r_b(a)(h) = a(h) - 2(a(h_b)/b(h_b))\, b(h)$$
$$= a(h) - a(h_b)\, b(h)$$

since $b(h_b) = 2$. Comparing, we see that $w_b^*(a)(h) = r_b(a)(h)$ are equal for all h. ∎

For any subset X of R, we let $T(X)$ be the group of elements of G_R which map L_a into itself for $a = 0$ and all $a \in X \cap R^C$. Then $T(R)$ is a closed normal subgroup of $M(R)$. Moreover, it contains the products $w_b(s)\, w_b(t)$ and $w_b(s)^{-1}\, w_b(t)$ for any nonzero s and t, by a straightforward argument using Lemma 3.3 and Theorem 3.4. Taking $w_b = w_b(1)$, we can express

$$w_b(t) = w_b v(t),$$

where $v(t) = w_b^{-1}\, w_b(t) \in T(R)$. This having been said, for any subset X of R, we define $N(X)$ to be the subgroup of $M(R)$ generated by

$$\{ w_b \mid b \in X \cap R^C \} \cup T(R).$$

Then $N(X)$ contains $w_b(t)$ for all $b \in X \cap R^C$. We then let $W(X)$ be the factor group $N(X)/T(R)$.

The action of w_b^* on R is the same as the action of the reflection r_b of the Lie rootsystem R, by Theorem 3.4. So, we get the Weyl group $\mathfrak{W}(R)$ of the rootsystem R as a homomorphic image of the groups $N(R)$. Since $T(R)$ leaves each element of R^C fixed, any action of $N(R)$ factors to an action of $W(R)$ on R. So, we have a surjective homomorphism from $W(R)$ to $\mathfrak{W}(R)$. Since an element of the kernel of this mapping is represented by an element leaving invariant each L_a for $a \in R^C$, and since such elements are in $T(R)$, this kernel is trivial. So, our homomorphism is actually an isomorphism from $W(R)$ to $\mathfrak{W}(R)$.

Similarly, we get the same type of thing for each classical complement of S, where the discussion goes along the same lines as in the case of complex semisimple Lie algebras. The phenomena for R amounts to a *coherently patched together phenomena for the classical complements*. Specifically, for any classical complement S of R and any $b \in S$, S is mapped into itself by w_b^* and the action of w_b^* on S is the same as the action of the reflection r_b of the Lie rootsystem R on S, that is, $w_b^*|_S = r_b|_S$. So, we get the Weyl group of the classical rootsystem S as a homomorphic image of the groups $N(S)$. As in the case of R above, the action of $N(S)$ factors to an action of $W(S)$ on S, since $T(R)$ leaves each element of S fixed. So, as in the case of R, we have an isomorphism from $W(S)$ to $\mathfrak{W}(S)$.

We also define $V(X)$ as the group generated by the set of products

$$\{w_b w_{a+b} \mid b \in X \cap R^C, \, a \in R^W, \, a+b \in R\} \cup T(R).$$

Since $b \in X \cap R^C$, $a \in R^W$, $a+b \in R$ implies that $a+b \in R^C$, $V(X)$ is a subgroup of $W(X)$. In particular, it follows that $V(X)$ normalizes T, so that $U(X) = V(X)/T(R)$ is a group. Under the isomorphism from $W(R)$ to $\mathfrak{W}(R)$, $U(R)$ is mapped isomorphically to $\hat{U}(R)$. We now can lift Theorem 6.8 of Winter [4] from Aut R to Aut L as follows, using Theorem 3.4.

3.5. THEOREM. *Suppose that R has no section of type T_2 and let S be a classical complement of L with base of simple roots π. Then*

    1. *$W(R)$ and $\mathfrak{W}(R)$ are canonically isomorphic, and this isomorphism induces a canonical isomorphism from $U(R)$ to $\hat{U}(R)$;*

    2. *$W(\pi) = W(S)$;*

    3. *$U(\pi) = U(S)$, $U(S)$ is normalized by $W(S)$, $U(S) \cap W(S) = T(R)$, and $W(R) = W(S) U(S)$.*

Although the equation

$$g(L_a) = L_{g*(a)}$$

relates the actions of $\mathfrak{W}(R)$ on R and $W(R)$ on L, the relation between the action of $\mathfrak{W}(R)$ on R and the inner automorphism action of $W(R)$ on $G = \text{Aut } L$ is not so close. About the best that can be done easily is to exponentiate the relationship $g(L_a) = L_{g*(a)}$ for a classical complement, using the equation $g(\exp \text{ ad } x) g^{-1} = \exp \text{ ad } g(x)$. Of course, this means that *whenever the Weyl group $\mathfrak{W}(R)$ of R acts transitively on the classical complements S of R, the Weyl group $W(R)$ acts transitively by inner automorphisms on the classical complements G_S of* Aut L. We know that this is true when the Cartan matrix of R is nonsingular, by Theorem 3.2, which gives us the following version of Levi's Theorem for algebraic groups of characteristic 0.

3.6. THEOREM. *When the Cartan matrix of R is nonsingular, the classical complements G_S of the algebraic group* Aut L *are conjugate.*

The G_S are contained in a special part of Aut L whose Lie algebra has an ideal which we are able to study structurally, in the next section.

4. AN IDEAL IN THE LIE ALGEBRA OF THE GROUP Inner Aut(L, H)

Whereas the group $G_R = $ Inner Aut L is generated by the closed connected one parameter groups

$$U_a = \{\exp t \text{ ad } e_a \,|\, t \in k\} = \exp L_a \text{ (for } a \in R^C, e_a \in L_a - \{0\}),$$

we consider the subalgebra I generated by the one-dimensional Lie algebras L_a of these U_a (for $a \in R^C$, $e_a \in L_a - \{0\}$).

We show first that I is an ideal of L. For this, it suffices to prove that I is normalized by the Witt core L_W of L, since $L = L_W + I$. So, consider $a \in L^W - \{0\}$ and $b \in L^C$. If $a + b \in R$, then $a + b \in R^C$, as one sees from the classification of rootsystems of rank 2. It follows that $[L_a, L_b] \subseteq I$. On the other hand, $[L_a, L_b] = 0$ if $a + b$ is not in R. Using this, we see that ad L^W maps a generating set for the algebra I back into I, from which it follows that L_W normalizes I. So, I is an ideal of L.

The U_a (for $a \in R^C$) stabilize the ideal I, since they consist of exponentials of inner derivations by elements of I. So, the group G_R stabilizes I. The subalgebra Lie G_R of the Lie algebra Der L of derivations of L contains Lie $U_a = $ ad L_a for all $a \in R^C$. It follows that, Lie G_R contains the Lie subalgebra ad I of Der L generated by the ad L_a (for $a \in R^C$). Since the group G_R stabilizes the ideal I, its Lie algebra Lie G_R stabilizes I. It then follows that

$$[\text{Lie } G_R, \text{ ad } I] \subseteq \text{ad Lie } G_R(I) \subseteq \text{ad } I,$$

so that

1. ad I is an ideal of Lie G_R.

Since G_R is an algebraic group with a Cartan subgroup C_G containing a maximal torus of $T(R)$ containing the torus T generated by the elements $w_b(s)^{-1} w_b(t)$ (for nonzero scalars s, t and $b \in R^C$), its Lie algebra Lie G_R has Cartan subalgebra C containing Lie T and classical Lie algebra quotient Lie $G_R/(\text{Solv Lie } G_R)$ with classical Cartan subalgebra $\bar{C}$ containing $\overline{\text{Lie } T}$. Ideals and quotients of classical Lie algebras are classical, as one easily verifies from the definition of classical Lie algebra, Seligman [2, p. 28]. Since

$$(\text{ad } I + \text{Solv Lie } G_R)/\text{Solv Lie } G_R$$

is an ideal of the classical Lie algebra Lie $G_R/(\text{Solv Lie } G_R)$, it follows that

$$(\text{ad } I + \text{Solv Lie } G_R)/\text{Solv Lie } G_R$$

is a classical Lie algebra. By the fundamental homomorphism theorem, ad I/J is a classical Lie algebra where J is the intersection of ad I and Solv Lie G_R. Since J is solvable, it is contained in Solv ad I. So, since (Solv ad I)/J is a solvable ideal in a classical Lie algebra, it is 0 and Solv ad $I = J$. It follows that ad $I/J =$ ad I/Solv ad I, so that ad I/Solv ad I is a classical Lie algebra. Since Solv I contains the center of I, the algebras I/Solv I and ad I/ad(Solv I) are isomorphic. It follows that ad(Solv I) must be the solvable radical of ad I, so that ad(Solv I) = Solv ad I. But then I/Solv I and ad I/Solv ad I are isomorphic, from which we conclude that

   2. I/Solv I is a classical Lie algebra.

Since (Solv Lie G_R)^2 is nilpotent, by the Lie–Kolchin theorem for the connected solvable radical Solv G_R of the algebraic group G_R, it follows that

   3.   (Solv I)^2 is nilpotent.

5. THE STRUCTURE OF L

   We now use the ideal ad I of Lie Inner Aut(L, H) and its properties to prove

   5.1. THEOREM.   *Let* (L, H) *be a symmetric Lie algebra over a field of characteristic* $p > 7$. *Assume that* (L, H) *has no section of type* T_2. *Then* (L, H) *has a subalgebra* L^W *and an ideal* I *such that*

   (1) $L = L^W + I$ *and* L^W/Solv L^W *is a symmetric Lie algebra with Witt rootsystem isomorphic to* R^W *and* $L^W \cap I \subseteq$ Solv L^W;

   (2)   *I has a Levi decomposition* $I = L_S \oplus$ Solv I *where* S *is any classical complement of* R^W *in* R, L_S *is a classical Lie algebra of type* S, *and* (Solv I)^2 *is nilpotent.*

   Proof. We take L^W to be the Witt core of (L, H) introduced in Section 2, and we let I be the ideal introduced in Section 4. Since

$$R = R^W \cup R^C,$$

we have

$$L = L^W + I.$$

We now prove that L^W/Solv L^W is symmetric with Witt rootsystem R^W relative to $H +$ Solv L^W. For this, it suffices to show that for each $a \in R^W - \{0\}$, $e_a \in L_{-a}^1$, $e_a \in L_a^1$, $h_a = [e_{-a}, e_a]$ with $a(h_a) \neq 0$ that h_a is not

an element of Solv L^W. Suppose, to the contrary, that $h_a \in$ Solv L^W, under such circumstances. Then $L_a = [h_a, L_a]$ and $L_{-a} = [h_a, L_{-a}]$ are contained in Solv L^W, so that Solv L^W contains the semisimple algebra $ke_{-a} + kh_a + ke_a$. This is not possible, however, so we must conclude that h_a is not an element of Solv L^W, as we set out to show. Consequently, $L^W/$Solv L^W is a symmetric Lie algebra with Witt rootsystem R^W.

We next proceed to show, for any classical complement S of R^W in R, that $I = L_S +$ Solv I where $L_S = \sum_{a \in S}([L_{-a}, L_a] + L_a)$, that L_S is a classical Lie algebra, and that $L^W \cap I$ is solvable. Since the generators L_a (for $a \in R^C$) of the ideal I are one dimensional, by Theorems 2.2 and 2.3, $L^W \cap I$, L_S, I do not change when we replace L by $L_0 + \sum_{a \in R - \{0\}} L_a^1$. (Recall that $L_a^1 = \{x \in L_a | [h, x] = a(h) x$ for all $h \in L_0\}$.) Also, $L^W \cap I$, L_S, I do not change when we replace L by L^2. By the first of these two observations, we may assume, with no loss of generality, that ad L_0 acts diagonally on $\sum_{a \in R - \{0\}} L_a$. By the second, we may assume, with no loss of generality, that $L = L^2$. Since

$$L = L_0 + \sum_{a \in R - \{0\}} L_a$$

and $L = L^2$, we have

$$L_0 = [L_0, L_0] + \sum_{a \in R - \{0\}} [L_{-a}, L_a].$$

It follows that ad L_0 acts diagonally (as well as by nilpotent transformations) on the subset $\sum_{a \in R - \{0\}} [L_{-a}, L_a]$ of L_0 as well, so that ad $L_0 \sum_{a \in R - \{0\}} [L_{-a}, L_a] = 0$. From this, we see that

$$[L_0, L_0] = [L_0, [L_0, L_0]] + \left[L_0, \sum_{a \in R - \{0\}} [L_{-a}, L_a] \right] = [L_0, [L_0, L_0]]$$

and

$$[L_0, L_0] = [L_0, [L_0, L_0]].$$

It follows from this and the nilpotency of L_0 that $[L_0, L_0] = 0$. But then

$$L_0 = [L_0, L_0] + \sum_{a \in R - \{0\}} [L_{-a}, L_a] = \sum_{a \in R - \{0\}} [L_{-a}, L_a]$$

and ad L_0 acts diagonally on all of L.

It is easy to show that we may assume without loss of generality that L has center 0. This enables us to imbed L isomorphically via ad in the derivation algebra Der L of L as ad L. Consider the p-closure $\overline{\text{ad } L}$ of ad L

in Der L and take K to be any Cartan subalgebra of the centralizer of the torus ad L_0 in $\overline{\text{ad }L}$. One then easily verifies that K is a Cartan subalgebra of $\overline{\text{ad }L}$ which centralizes ad L_0.

Consequently, K stabilizes the rootspaces L_a (for $a \in R$) of L with respect to the Cartan subalgebra L_0. Since I is generated by the one dimensional rootspaces I_a (for $a \in R^C$), it follows, in particular, that K acts diagonally on I. Since $\overline{\text{ad }L}/\text{ad }L$ is abelian with Cartan subalgebra $(K + \text{ad }L)/\text{ad }L$, we have $\overline{\text{ad }L} = K + \text{ad }L$.

We now show that $K + \text{ad }I$ is a restricted subalgebra of $\overline{\text{ad }L}$. Taking the weight space decomposition

$$K + \text{ad }I = K + \sum_{a \in Q} \text{ad }I_a(K)$$

of ad K on $K + \text{ad }I$ and an element $x \in I_a(K)$ (for $a \in Q$), we have $[K, \text{ad }x] \subseteq k$ ad x (set of scalar multiples of ad x). The reason for this is that K acts diagonally on I, as we have seen. But then

$$0 = [\cdots [K, \text{ad }x], ..., \text{ad }x] = [K, (\text{ad }x)^p].$$

It follows from the self-normalizing property of Cartan subalgebras that $(\text{ad }x)^p$ is contained in the Cartan subalgebra K. Consequently, the algebra $K + I$ is spanned by elements whose pth powers are contained in $K + \text{ad }I$. By the theory of restricted Lie algebras of Jacobson [1, pp. 187–194], it follows that $K + \text{ad }I$ is a restricted subalgebra of Der L.

Since $K + \text{ad }I$ is a restricted subalgebra of Der L, the p-closure $\overline{\text{ad }I}$ of ad I is contained in $K + \text{ad }I$. Consequently, $\overline{\text{ad }I} = M + \text{ad }I$ where $M = K \cap \overline{\text{ad }I}$. Since $M \subseteq K$, M acts diagonally on I. We show now that, in fact, ad M acts diagonally on $\overline{\text{ad }I}$, so that ad M is a torus in the restricted Lie algebra $\overline{\text{ad }I}$. For this, let J be the centralizer of M in $\overline{\text{ad }I}$ and consider the subalgebra

$$J + \text{ad }I = J + \sum_{a \in Q} I_a(K)$$

of $\overline{\text{ad }I}$. Then for $x \in I_a(K)$, $a \in Q$ we have

$$[M, \text{ad }x] = k \text{ ad }x,$$

$$[\cdots [M, \text{ad }x], ..., \text{ad }x] = [M, (\text{ad }x)^p].$$

It follows, as in the preceding paragraph, that $J + \text{ad }I$ is a restricted subalgebra of $\overline{\text{ad }I}$. But then $J + \text{ad }I$ equals the p-closure $\overline{\text{ad }I}$. Since $\overline{\text{ad }I} = J + \text{ad }I$, $[M, J] = 0$, and M acts diagonally on I, it follows that ad M acts diagonally on $\overline{\text{ad }I}$. In particular, ad M acts diagonally on M, so

that ad $M(M) = 0$ and $M^2 = 0$, since M is nilpotent. This establishes that ad M acts as a torus on $\overline{\text{ad } I}$.

Since ad M acts a torus on $\overline{\text{ad } I}$, $N = \{D \in \overline{\text{ad } I} \mid [M, D] = 0\}$ is the Fitting nullspace of ad M acting on $\overline{\text{ad } I}$. Consequently, N is a self-normalizing subalgebra of $\overline{\text{ad } I}$. By the same argument, ad K acts diagonally on $\overline{\text{ad } I}$. Since K is nilpotent and contains M, it follows that ad K centralizes ad M. But then N is stable under the action of ad K, and $N = \sum_{a \in P} N_a(K)$ (weight space decomposition of N with respect to ad K). Let $x \in N_a(K)$. Then $[K, x] \subseteq kx$ implies that $[K, x^p] = 0$. Since $x^p \in N$ by the self-normalizing property of N, it follows that $x^p \in N_0(K) = K \cap N \subseteq K \cap \overline{\text{ad } I} = M$. But then $[x^p, N] = 0$, so that ad x is nilpotent on N. This establishes that the weakly closed set $\bigcup_{a \in P} N_a(K)$ of generators of N consists of nilpotent elements, so that N is nilpotent, by Jacobson [1, p. 33]. We now have shown that N is a self-normalizing subalgebra of $\overline{\text{ad } I}$ and N is nilpotent, so that N is a Cartan subalgebra of $\overline{\text{ad } I}$.

We now can show that M is a maximal torus of $\overline{\text{ad } I}$. For this, it suffices to show that M is a maximal torus of its centralizer N in $\overline{\text{ad } I}$. Since N is restricted with central toral ideal M, it suffices, by Winter [5], to show that the restricted Lie algebra N/M consists of nilpotent elements. As shown in the preceding paragraph, $N = \sum_{a \in P} N_a(K)$ and the pth powers of the elements of $\bigcup_{a \in P} \text{ad } N_a(K)$ are contained in ad M. We again invoke the results of Jacobson [1, p. 33] to conclude that N/M consists of nilpotent elements. Thus, M is a maximal torus of ad I.

We now know that the p-closure $\overline{\text{ad } I}$ of ad I in Der L has the form $\overline{\text{ad } L} = M + \text{ad } I$ where $M = K \cap \overline{\text{ad } I}$ and M is a maximal torus of $\overline{\text{ad } I}$. Moreover, $\overline{\text{ad } I}$ and M stabilize the ideal Solv I of I. We let $\overline{\text{Solv}}$ denote the p-closure of the radical Solv ad I of ad I and we consider the quotient $D = \overline{\text{ad } I}/\overline{\text{Solv}}$ and its ideal $C = (\text{ad } I + \overline{\text{Solv}})/\overline{\text{Solv}}$. Note that C is canonically isomorphic to the classical Lie algebras ad I/Solv ad I, I/Solv I. This follows from the fundamental homomorphism theorems by the following argument. We know from Section 4 that I/Solv I is classical. Since it is isomorphic to

$$\text{ad } I/\text{Solv ad } I,$$

the latter is also classical. The algebra C is isomorphic to ad I/ad J where ad J is the intersection of ad I and $\overline{\text{Solv}}$. Since ad J is contained in Solv ad I and conversely, we have ad J = Solv ad I. It follows that C is isomorphic to ad I/Solv ad I, as we had asserted. By Seligman [2, p. 48], the classical Lie algebra C is restricted. Since C has center 0, this restricted structure for C is unique. Consequently, C is a restricted ideal of the restricted Lie algebra D.

Since $\overline{\text{ad } I} = M + \text{ad } I$ and M is a maximal torus of $\overline{\text{ad } I}$ whose centralizer

N is a Cartan subalgebra of $\overline{\operatorname{ad} I}$, $M^* = (M + \overline{\operatorname{Solv}})/\overline{\operatorname{Solv}}$ is a maximal torus of D and $N^* = (N + \overline{\operatorname{Solv}})/\overline{\operatorname{Solv}}$ is a Cartan subalgebra of D, by Winter [5], and $D = M^* + C$. Since C is a restricted ideal in D and N^* is a Cartan subalgebra of D, we see as in the fourth preceding paragraph, that the Fitting null space C_0 of ad $(N^* \cap C)$ in C is a Cartan subalgebra of C. Since C is classical, it follows that C_0 is a torus. For if it were not, we could extend the ground field to the algebraic closure and get a classical Lie algebra having a Cartan subalgebra which is not a torus. But this is impossible by the conjugacy of Cartan subalgebras in this case, as established in Seligman [2, p. 116]. Since the maximal torus M^* of D centralizes C_0, it follows that $C_0 \subseteq M^*$. But then we have

$$M^* \cap C = C_0 = N^* \cap C.$$

Since $D = M^* + C$ with $M^* \subseteq N^* \subseteq D$, we have

$$N^* = M^* + N^* \cap C = M^* + C_0 = M^*.$$

This establishes that $M^* = N^*$ and M^* is a toral Cartan subalgebra of D such that $C_0 = M^* \cap C$ is a Cartan subalgebra of C.

Since we reduced to the case where ad L_0 acts diagonally on L, ad L_0 is contained in the Cartan subalgebra K of $\overline{\operatorname{ad} I}$, by an easy argument. In fact, we have $K \cap \operatorname{ad} L = \operatorname{ad} L_0$, since L_0 is a Cartan subalgebra of L. Now $C_0 = M^* \cap C$ where $M = K \cap \overline{\operatorname{ad} L}$. Since

$$C = (\operatorname{ad} I + \overline{\operatorname{Solv}})/\overline{\operatorname{Solv}},$$

it follows that

$$C_0 = ((K \cap \operatorname{ad} I) + \overline{\operatorname{Solv}})/\overline{\operatorname{Solv}}.$$

Since

$$K \cap \operatorname{ad} I = (K \cap \operatorname{ad} L) \cap \operatorname{ad} I = \operatorname{ad} L_0 \cap \operatorname{ad} I = \operatorname{ad} I_0,$$

where $I_0 = L_0 \cap I$, it follows that $C_0 = (\operatorname{ad} I_0 + \overline{\operatorname{Solv}})/\overline{\operatorname{Solv}}$.

We now take $a \in R - \{0\}$ and consider the corresponding

$$C_a = (\operatorname{ad} L_a + \overline{\operatorname{Solv}})/\overline{\operatorname{Solv}}.$$

If $a \in R^C$ (the case where a is a classical root), then we have $L_a = I_a$ (intersection of L_a and I) and then

$$C^{(a)} = C_{-a} + [C_{-a}, C_a] + C_a$$

is isomorphic to

$$L^{(a)} = L_{-a} + [L_{-a}, L_a] + L_a$$

(which in turn is isomorphic to $\overline{Sl_2}$), since

$$C^{(a)} = (\text{ad } L^{(a)} + \overline{\text{Solv}})/\overline{\text{Solv}}$$

and

$$\text{ad } L^{(a)} \cap \overline{\text{Solv}} = \{0\}$$

(by the semi-simplicity of $L^{(a)}$). Suppose next that $a \in R^W$ (the case where a is a Witt root). When we pass from a to $a^*: C_0 \to k$ defined by $a^*(\text{ad } h + \overline{\text{Solv}}) = a(h)$ (for $h \in I_0$), we consider the two possibilities $a^* = 0$ and $a^* \neq 0$. If $a^* = 0$, then $[I_0, I_a] = 0$ and $[C_0, C_a] = 0$ implies that C_a is contained in the Cartan subalgebra

$$C_0 = (\text{ad } I_0 + \overline{\text{Solv}})/\overline{\text{Solv}}.$$

Since $I_a \cap I_0 = \{0\}$ and $\overline{\text{Solv}}$ is a sum of weight spaces for M, it follows that $I_a \subseteq \overline{\text{Solv}}$. Suppose next that $a^* \neq 0$ and choose $h \in I_0$ such that $a(h) \neq 0$. Then the ideal I contains $[L_{ia}, h] = L_{ia}$ for $i = 1, ..., p-1$. But then it follows that the rootsystem $R^* = \{a^* | a \in R\}$ of C with respect to C_0 contains the roots $0^*, a^*, ..., (p-1) a^*$. But then $2a^*$ is a root in R^* and the rootsystem R^* is not a reduced classical rootsystem. We claim that this cannot be. For if it did, then by extending the ground field to the algebraic closure, it could happen also over an algebraically closed field. Then C would be a classical Lie algebra over an algebraically closed field. In this setting, by the conjugacy of Cartan subalgebras of Seligman [2, p. 116], the rootsystem of C with respect to *every* Cartan subalgebra is reduced classical. This is a contradiction which forces us to conclude that the case $a^* \neq 0$, $a \in R^W$ never occurs. We conclude that the rootsystem $R^* = \{a^* | a \in R\}$ of C with respect to C_0 coincides with

$$R = \{a^* | a \in R^C\} \cup \{0\},$$

and that $a^* = 0^*$ for all $a \in R^W$. It follows that the algebra

$$\sum_{a \in R^W} ([I_{-a}, I_a] + I_a)$$

is contained in Solv I. From this, it follows that $L^W \cap I$ is solvable and, therefore, is contained in Solv L^W.

We now let S be a classical complement of R^W in R and consider the subalgebra $L_S = \sum_{a \in S}([L_{-a}, L_a] + L_a)$ of I. Then L_S is a classical Lie algebra, by Theorem 2.2. Consequently, $L_S \cap \text{Solv } I = \{0\}$. We now proceed to show that $I = L_S + \text{Solv } I$. We know that $I/\text{Sol } I$ is a classical Lie algebra with Cartan subalgebra $I_0 = \sum_{a \in R^C} [L_{-a}, L_a]$, since $C = (\text{ad } I + \overline{\text{Solv}})/\overline{\text{Solv}}$ is a classical Lie algebra with Cartan subalgebra

$C_0 = (\text{ad } I_0 + \overline{\text{Solv}})/\overline{\text{Solv}}$. Also, by identification of C and $I/\text{Solv } I$, we have seen that the Cartan decomposition of

$$I^* = I/\text{Solv } I$$

is

$$I^* = \sum_{a^* \in R^*} I^*_{a^*},$$

where

   (1) a^* is the function on $I^*_0 = (I_0 + \text{Solv } I)/\text{Solv } I$ defined by $a^*(h + \text{Solv } I) = a(h)$ (for $a \in R$);
   (2) $a^* = 0$ if $a \in R^W$ and $a^* \neq 0$ if $a \in R^C$;
   (3) $R^* = \{a^* \mid a \in R\}$.

Since $R \subseteq R^W + S$ and $a^* = 0$ for $a \in R^W$, it follows that R^* coincides with the set $S^* = \{a^* \mid a \in S\}$. Finally, since $a^* \neq 0$ for all $a \in S - \{0\}$ (since $a^* \neq 0^*$ for all $a^* \in R^C$), the Cartan decomposition for I^* is $I^* = \sum_{a^* \in S^*} I^*_{a^*}$. Taking any $a \in S - \{0\}$, we have

   (1) $(I_a + \text{Solv } I)/\text{Solv } I \subseteq I^*_{a^*}$ and dim $I^*_{a^*} = 1$;
   (2) $I_a \cap \text{Solv } I = \{0\}$ (since $I_{-a} + [I_{-a}, I_a] + I_a$ is semisimple).

It follows that $((I_a + \text{Solv } I)/\text{Solv } I = I^*_{a^*})$ for all $a \in S$ and, consequently, that

$$I/\text{Solv } I = \sum_{a \in S} I^*_{a^*} = \sum_{a \in S} (I_a + \text{Solv } I)/\text{Solv } I = (I_S + \text{Solv } I)/\text{Solv } I.$$

Thus, $I = I_S + \text{Solv } I$ as asserted. ∎

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