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SMOOTH MASSLESS LIMIT OF QED *

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Massless QED is studied by introducing different regularization schemes for the resulting mass singularities. It is demonstrated, for the one-loop corrections to electron scattering off an external potential, that cancellation of regulator-dependent *finite* parts within a Lee and Nauenberg (LN) set renders the observable process unique, thus establishing existence of its smooth massless limit.

1. Introduction

It is well known that gauge theories containing massless particles develop infinities at the S-matrix level, associated with mass singularities. These disappear from physical quantities if one includes all processes belonging to the same set of degenerate (physically indistinguishable) initial and final states [1, 2]. It becomes necessary, nevertheless, to introduce a set of regulators giving mathematical meaning to the singular individual processes within the set and to take the limit of these regulators to zero at the end of the calculation. However, the different regularization schemes applicable to the mass singularities of gauge theories have somewhat different physical content. Specifically, if one chooses to regulate the mass singularities of a gauge theory by using massive regulators (i.e. by assigning masses to all massless particles existing in the theory and taking the limit of these masses to zero at the end of the calculation), one has to consider the extra degrees of freedom introduced by the inclusion of these masses (for massless vector bosons for example) or the non-vanishing helicity-flip amplitudes (for massless fermions) in certain collinear processes. Similarly, if one chooses dimensional regularization of the mass singularities [3–6], one has to consider the extra degrees of freedom (for massless vector bosons) coming from the extra $n-2$ transverse directions existing in an n-dimensional space-time (the continuation is $n > 4$).

While these contributions will not affect the leading singularities, they appear in the lower-order singularities and, of course, in the finite parts. Hence the question

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of whether the observable quantities are uniquely determined, independently of the regularization scheme one uses for treating the mass singularities of a theory is, to the best of our knowledge an open one, which has drawn little attention [7, 8]. Furthermore, in processes that are highly selective as far as which singularities remain after taking the regulator limit, such as helicity-flip processes, uniqueness of a gauge theory is profoundly troublesome since such processes actually appear to give *finite* cross sections [1, 9], if one uses massive regularization, but trivially give zero if one uses dimensional regularization. Such processes may be handled by including degenerate processes corresponding to disconnected Feynman graphs [1], but whether inclusion of the latter is physically relevant for studying the high-energy limit of massive gauge theories and/or restores uniqueness is an open question.

The above considerations lead naturally to the suspicion that, if uniqueness of massless gauge theories is a fact, then the important idea of cancellation of mass singularities within a set of degenerate processes, as was stated by Lee and Nauenberg (LN) and independently by Kinoshita, may be extended to cancellation of *finite* regulator-dependent quantities within that set, thus restoring not only the finiteness, but also the uniqueness of the massless theory.

In this paper we consider a specific but fundamental process in QED, namely the one-loop corrections to electron scattering off an external potential. In sect. 2 we treat what we call one-mass singularities, i.e. the singularities arising from considering only one massless particle and specifically the ones appearing from the masslessness of the photon. We show uniqueness of the process comparing the massive (subsect. 2.1)* and the dimensional (subsect. 2.2) regularization of the infrared singularity. In subsect. 2.3 we summarize our results. This part of the paper serves for establishing notation as well as methodology. In addition, the formulas for the finite parts obtained, will be used in the remainder of the paper. In sect. 3 we consider the same process in massless QED (two-mass singularities). As is evident, there is a choice of regulator combinations to be made, hinting at the nontriviality of establishing the uniqueness of the process. We choose to compare what we consider an interesting combination of mixed regulators (subsect. 3.1) with a completely dimensional one (subsect. 3.2). Uniqueness is again recovered. As a by-product we discover, at the analytical level, that the generally accepted correspondence between the infinite parts $\ln m \leftrightarrow 1/\epsilon$, $\ln^2 m \leftrightarrow 2/\epsilon^2$ where $\epsilon = n - 4$ and n is the dimensionality of space-time, can be extended to the finite parts as the same analytic continuation of hypergeometric functions of different arguments. This is shown in detail in subsect. 3.2.2 and appendix B. Finally, we demonstrate the LN cancellation of finite regulator-dependent pieces, and thus are confident that the process works in general, even in the more problematic cases of helicity-flip processes. The results of the section are summa-

* The results in subsect. 2.1 are well known and can be found in many places in the literature [10].

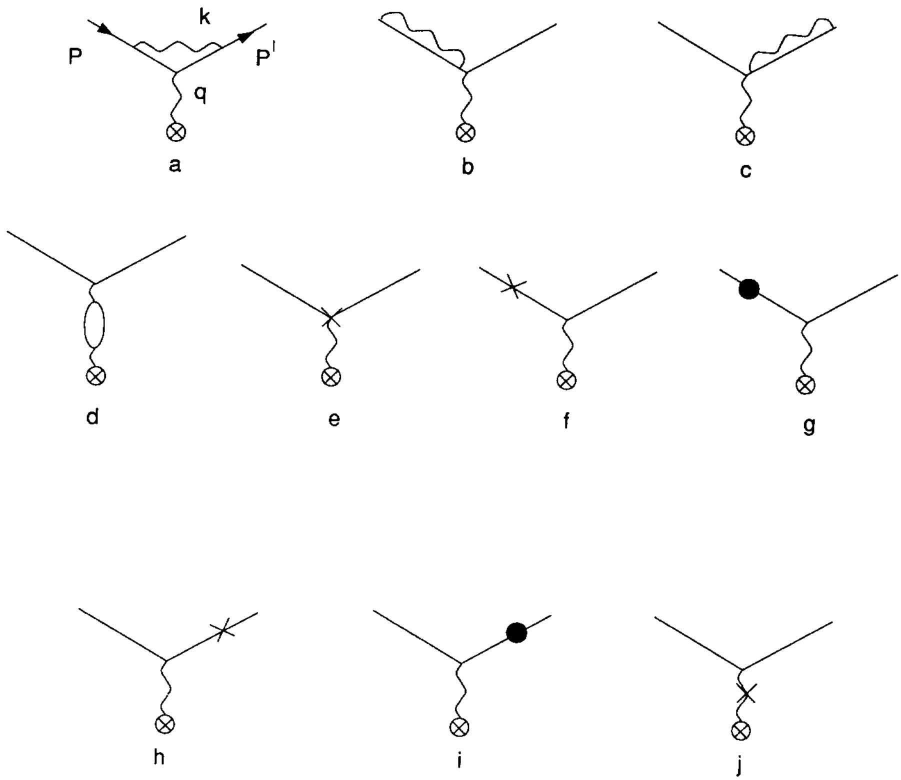

Fig. 1. Radiative corrections to electron scattering off an external potential. The crosses and the dots denote vertex or wave function and mass counterterms.

rized in subsect. 3.3. In the end (sect. 4) we summarize our conclusions and formulate some speculations of a more general nature.

2. One-mass singularities

We shall consider as a typical QED process the one-loop radiative corrections to scattering from an external potential. We will be working in the Feynman gauge. The corresponding diagrams are shown in fig. 1.

Our quantities will in general have two sources of infinities. Correspondingly, we should reserve two regulator indices for those quantities which contain at least one kind of infinity. The first regulator index corresponds to the UV singularities but since we shall be using dimensional regularization, minimal subtraction and intermediate renormalization throughout this paper, this index will never appear explicitly. The corresponding regulator will always be $\epsilon_{UV} = 4 - n$. As stated in sect. 1, uniqueness of QED in the perturbative sense would amount to equalities of the form

$$\lim_{\mu_1\to 0}\sum_{D(E_b)D(E_a)}|S_{ba}^{(\mu_1\equiv m_\gamma)}|^2-\lim_{\mu_1\to 0}\sum_{D(E_b)D(E_a)}|S_{ba}^{(\mu_1\equiv\epsilon)}|^2=0. \tag{2.1}$$

In the above μ_1 is the mass regulator scheme, $\epsilon\equiv\epsilon_{IR}=n-4$, $S_{ba}^{(\mu_1)}$ is the regularized S-matrix element for the transition between states a and b (of energies E_a, E_b respectively) and the sum extends over degenerate LN sets $D(E_b)$, $D(E_a)$ for usual QED (massive electron, massless photon) processes*. Similarly,

$$\lim_{\mu_1\to 0}\sum_{D(E_b)D(E_a)}|S_{ba}^{(\mu_1\equiv m_e)}|^2-\lim_{\mu_1\to 0}\sum_{D(E_b)D(E_a)}|S_{ba}^{(\mu_1\equiv\epsilon)}|^2=0 \tag{2.2}$$

for the massive photon and massless electron.

It is well known, as noted above, that to regulate UV singularities dimensionally, one has to consider initially a number of space-time dimensions $n=4-\epsilon_{UV}<4$, while to regulate the mass singularities in the same scheme, $n=4+\epsilon>4$. This will be described by the correspondence $\epsilon_{UV}\to-\epsilon$, whose meaning is that the theory is not *simultaneously* UV-regular and mass-singularity-free in any number of dimensions. Thus, the generally accepted *prescription* is to first renormalize the theory dimensionally and, once the counterterms are included, to continue $n=4+\epsilon$. We shall show that eq. (2.1) holds for the particular process shown in fig.1.

2.1. $\mu_1=m_\gamma$ REGULARIZATION SCHEME

The radiative corrections will arise from virtual and real processes. The former will come from the vertex and self-energy graphs, while the latter from bremsstrahlung graphs.

2.1.1. Virtual corrections. From fig. 1a we obtain for the vertex correction

$$\begin{aligned}\Lambda_\mu^{(m_\gamma)}(p,p') &= -ie^2\int\frac{d^{4-\epsilon_{UV}}k}{(2\pi)^{4-\epsilon_{UV}}}\gamma_\nu(\not p'-\not k+m_e)\gamma_\mu(\not p-\not k+m_e)\gamma^\nu\\ &\quad\times\frac{1}{(k^2-m_\gamma^2)(k^2-2(p'k))(k^2-2(pk))}\\ &=\frac{\alpha}{4\pi}\gamma_\mu\int_0^1 dx\int_0^1 y\,dy\Bigg[\frac{4}{\epsilon_{UV}}+2(\ln 4\pi-\gamma-2\ln m_e-2)-2\ln L^2\\ &\quad-\frac{1}{L^2}\left(4-4y-2y^2+r^2(2-2y+2y^2x(1-x))\right)\Bigg]+R(\mu_1),\end{aligned} \tag{2.3}$$

*Throughout the paper we shall follow the notation of refs. [1,5,6] as closely as possible.

where γ is Euler's constant and $R(\mu_1)$ denotes terms regular at the limit $\mu_1 = 0$. These terms will not be shown explicitly since they cancel (by definition) from equalities like (2.2). In eq. (2.3) we have defined

$$L^2 \equiv y^2\left[1 + r^2x(1-x)\right] + \lambda^2(1-y), \quad \lambda^2 \equiv \frac{m_\gamma^2}{m_e^2}, \quad r^2 \equiv \frac{-q^2}{m_e^2}.$$

Performing the y-integration we have

$$\Lambda_\mu^{(\lambda)} = \frac{\alpha}{4\pi}\gamma_\mu \int_0^1 dx \left[\frac{2}{\epsilon_{UV}} + (\ln 4\pi - \gamma - 2\ln m_e - 2) - 2I_0 \right.$$
$$\left. - \frac{\rho^2}{1-\xi^2}\left(4I_1 - 4I_2 - 2I_3 + r^2(2I_1 - 2I_2 + 2x(1-x)I_3)\right)\right], \quad (2.4)$$

where

$$\rho^2 = \frac{m_e^2}{E^2}, \qquad \xi^2 = 1 - \rho^2\left(1 + r^2x(1-x)\right),$$

E being the initial energy and

$$I_0 = \int_0^1 dy\, y \ln L^2, \qquad I_n = \int_0^1 dy\, y^n \frac{1}{y^2 - y\beta + \beta}, \quad \beta = \frac{\lambda^2\rho^2}{1-\xi^2}.$$

We can easily calculate the following limits:

$$\lim_{\lambda\to 0} I_0 = \tfrac{1}{2}\ln\left(\frac{1-\xi^2}{\rho^2}\right) - \tfrac{1}{2}, \qquad \lim_{\lambda\to 0} I_1 = -\ln\lambda + \tfrac{1}{2}\ln\left(\frac{1-\xi^2}{\rho^2}\right),$$

$$\lim_{\lambda\to 0} I_2 = 1, \lim_{\lambda\to 0} I_3 = \tfrac{1}{2}.$$

After inclusion of the UV counterterms as shown in fig. 1e, we can write

$$\lim_{\lambda\to 0} \Lambda_\mu^{(\lambda)} = \frac{\alpha}{4\pi}\gamma_\mu \int_0^1 dx \left[\ln 4\pi - \gamma - 2\ln m_e - 2 - \ln C^2 \right.$$
$$\left. - \frac{2+r^2}{C^2}\ln C^2 + \frac{2(3+r^2)}{C^2} + \frac{2(2+r^2)}{C^2}\ln\lambda\right] \quad (2.5)$$

with

$$C^2 \equiv \frac{1-\xi^2}{\rho^2} = 1 + r^2x(1-x).$$

From fig. 1b we obtain for the self-energy correction

$$-i\Sigma^{(m_\gamma)}(p) = -e^2\int\frac{\mathrm{d}^{4-\epsilon_{\mathrm{UV}}}k}{(2\pi)^{4-\epsilon_{\mathrm{UV}}}}\frac{(\epsilon_{\mathrm{UV}}-2)(\not{p}-\not{k})+(4-\epsilon_{\mathrm{UV}})m_{\mathrm{e}}}{\left(k^2-m_\gamma^2\right)\left((p-k)^2-m_{\mathrm{e}}^2\right)}. \tag{2.6}$$

Performing a Taylor expansion

$$-i\Sigma^{(m_\gamma)}(p) = A^{(m_\gamma)} + B^{(m_\gamma)}(\not{p}-m_{\mathrm{e}}) + \Sigma_f^{(m_\gamma)}(p)(\not{p}-m_{\mathrm{e}})^2, \tag{2.7}$$

we observe that $A^{(m_\gamma)}$ just renormalizes the electronic mass. The wave function renormalization constant $B^{(m_\gamma)}$, nevertheless, contributes $\frac{1}{2}B^{(m_\gamma)}$ per diagram as an observable radiative correction. From eqs. (2.6) and (2.7) we deduce

$$\begin{aligned}
B^{(m_\gamma)} = &-e^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^{4-\epsilon_{\mathrm{UV}}}k}{(2\pi)^{4-\epsilon_{\mathrm{UV}}}}\frac{(\epsilon_{\mathrm{UV}}-2)(1-x)}{\left[k^2-\left(m_{\mathrm{e}}^2x^2+m_\gamma^2(1-x)\right)\right]^2}\\
&+e^2\int_0^1\mathrm{d}x\int\frac{\mathrm{d}^{4-\epsilon_{\mathrm{UV}}}k}{(2\pi)^{4-\epsilon_{\mathrm{UV}}}}4x(1-x)m_{\mathrm{e}}^2\frac{(\epsilon_{\mathrm{UV}}-2)(1-x)+4-\epsilon_{\mathrm{UV}}}{\left[k^2-\left(m_{\mathrm{e}}^2x^2+m_\gamma^2(1-x)\right)\right]^3}\\
= &\frac{i\alpha}{2\pi\epsilon_{\mathrm{UV}}}-\frac{i\alpha}{4\pi}\left[\gamma-\ln 4\pi+1+2\ln m_{\mathrm{e}}+2\int_0^1\mathrm{d}x(1-x)\ln D^2\right]\\
&-\frac{i\alpha}{\pi}\int_0^1\mathrm{d}x\frac{x(1-x^2)}{D^2},
\end{aligned} \tag{2.8}$$

where $D^2 = x^2 + \lambda^2(1-x)$. One can easily show that

$$\lim_{\lambda\to 0}\int_0^1\mathrm{d}x(1-x)\ln D^2 = -\tfrac{3}{2}, \qquad \lim_{\lambda\to 0}\int_0^1\mathrm{d}x\frac{x(1-x^2)}{D^2} = -\ln\lambda-\tfrac{1}{2}.$$

Diagram 1d is of the $\mathrm{R}(\mu_1)$ type, i.e. non-singular, hence we do not consider it. Once the UV counterterms are included, the total virtual correction $2\delta_{\mathrm{V}}^{(\lambda)}$ will be defined as

$$\lim_{\lambda\to 0} 2\delta_{\mathrm{V}}^{(\lambda)}\gamma_\mu \equiv 2\lim_{\lambda\to 0}\left[\Lambda_\mu^{(\lambda)}+\tfrac{1}{2}i\gamma_\mu B^{(\lambda)}\times 2\right].$$

Hence we find for the virtual contribution

$$\lim_{\lambda \to 0} 2\delta_{\mathrm{V}}^{(\lambda)} = \frac{\alpha}{\pi}\left(\int_0^1 \mathrm{d}x \frac{2+r^2}{1+r^2x(1-x)} - 2\right)\ln\lambda$$
$$+ \frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\left[-6 - \ln\left(1+r^2x(1-x)\right)\right.$$
$$\left. - \frac{2+r^2}{1+r^2x(1-x)}\ln\left(1+r^2x(1-x)\right) + \frac{2(3+r^2)}{1+r^2x(1-x)}\right]. \quad (2.9)$$

2.1.2. Real corrections. As is well known, the physically observable cross sections must include bremsstrahlung contributions. In this case (one-mass singularities) only soft bremsstrahlung contributes. Its contribution is

$$\delta_{\mathrm{BS}}^{(m_\gamma)} = e^2 \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{1}{2\omega} \sum_{l=1}^{3} \left| \frac{(\epsilon^{(l)}p')}{(p'k) - \frac{1}{2}m_\gamma^2} - \frac{(\epsilon^{(l)}p)}{(pk) - \frac{1}{2}m_\gamma^2} \right|^2 . \quad (2.10)$$

In the above, the momenta (p, p', k) label the initial, final electron and final photon respectively. Calculating the momentum integral we find

$$\delta_{\mathrm{BS}}^{(m_\gamma)} = \frac{\alpha\rho^2}{\pi}\left[\left(1+\tfrac{1}{2}r^2\right)\int_0^1 \mathrm{d}x I_x(\lambda) - I(\lambda)\right], \quad (2.11)$$

where

$$I_x(\lambda) \equiv \int_0^{\Delta E/m_e} \mathrm{d}k\, k^2 \int_{-1}^{1} \mathrm{d}y \frac{1}{(k^2+\lambda^2)^{1/2}} \frac{1}{\left[(k^2+\lambda^2)^{1/2} - \frac{1}{2}\rho\lambda^2 - k\xi y\right]^2},$$

$$I(\lambda) \equiv \int_0^{\Delta E/m_e} \mathrm{d}k\, k^2 \int_{-1}^{1} \mathrm{d}y \frac{1}{(k^2+\lambda^2)^{1/2}} \frac{1}{\left[(k^2+\lambda^2)^{1/2} - \frac{1}{2}\rho\lambda^2 - k\xi_0 y\right]^2},$$

where ξ has been defined earlier, ΔE is the energy resolution of the detector and

$\xi_0 = (1-\rho^2)^{1/2}$. These integrals have been calculated in appendix A. The result is

$$\lim_{\lambda\to 0} \delta_{\mathrm{BS}}^{(\lambda)} = -\frac{\alpha}{\pi}\left(\int_0^1 \mathrm{d}x \frac{2+r^2}{1+r^2x(1-x)} - 2\right)\ln\lambda$$
$$+\frac{\alpha}{2\pi}\int_0^1 \mathrm{d}x\left[\frac{2+r^2}{1+r^2x(1-x)}\frac{1}{\xi}\ln\left(\frac{1-\xi}{1+\xi}\right)\right.$$
$$\left.+\left(\frac{2+r^2}{1+r^2x(1-x)}-2\right)\ln\left(\frac{2\Delta E}{m_{\mathrm{e}}}\right)-\frac{2}{\xi_0}\ln\left(\frac{1-\xi_0}{1+\xi_0}\right)\right]. \quad (2.12)$$

2.2. $\mu_1 = \epsilon$ REGULARIZATION SCHEME

In this scheme extra care is needed in order not to confuse the renormalization process with the one describing the structure of the mass singularity, since both of them are regulated dimensionally.

2.2.1. Virtual corrections. For the vertex condition we have*

$$\Lambda_\mu^{(\epsilon_{\mathrm{UV}})} = \frac{e^2}{(4\pi)^{2-\epsilon_{\mathrm{UV}}/2}}\int_0^1 \mathrm{d}x\int_0^1 y\,\mathrm{d}y\left\{\gamma_\mu\frac{(\epsilon_{\mathrm{UV}}-2)^2}{2}\Gamma\left(\tfrac{1}{2}\epsilon_{\mathrm{UV}}\right)y^{-\epsilon_{\mathrm{UV}}}\left(C^2\right)^{-\epsilon_{\mathrm{UV}}/2}m_{\mathrm{e}}^{-\epsilon_{\mathrm{UV}}}\right.$$
$$-\gamma_\mu\Gamma\left(1+\tfrac{1}{2}\epsilon_{\mathrm{UV}}\right)y^{-(2+\epsilon_{\mathrm{UV}})}\left(C^2\right)^{-(1+\epsilon_{\mathrm{UV}}/2)}m_{\mathrm{e}}^{-(2+\epsilon_{\mathrm{UV}})}$$
$$\times\left[m_{\mathrm{e}}^2\left(4-4y+(\epsilon_{\mathrm{UV}}-2)y^2\right)\right.$$
$$\left.\left.+(-q^2)\left(2-2y+(2-\epsilon_{\mathrm{UV}})y^2x(1-x)\right)\right]\right\}. \quad (2.13)$$

At this stage we are able to recognize the UV singularity as the pole part of the Γ-function. This will be: $(\alpha/4\pi\epsilon_{\mathrm{UV}})\gamma_\mu$. Including, then, the UV counterterms, we can use eq. (2.13) by discarding the above piece and by substituting: $\epsilon_{\mathrm{UV}} \to -\epsilon$ in the rest. The result is

$$\lim_{\epsilon\to 0}\Lambda_\mu^{(\epsilon)} = -\frac{\alpha}{2\pi\epsilon}\gamma_\mu\int_0^1 \mathrm{d}x\frac{2+r^2}{1+r^2x(1-x)}$$
$$+\frac{\alpha}{4\pi}\gamma_\mu\int_0^1 \mathrm{d}x\left[\left(\ln 4\pi-\gamma-2\ln m_{\mathrm{e}}\right)\left(1+\frac{2+r^2}{1+r^2x(1-x)}\right)\right.$$
$$-2-\ln\left(1+r^2x(1-x)\right)+\frac{2(3+r^2)}{1+r^2x(1-x)}$$
$$\left.-\frac{2+r^2}{1+r^2x(1-x)}\ln\left(1+r^2x(1-x)\right)\right]. \quad (2.14)$$

* The integrand of this expression is different from the one given in ref. [5]. However, the two expressions are identical, once the integrations are performed.

Similarly for self-energy contribution we get

$$B^{(\epsilon_{UV})} = -\frac{i\alpha}{4\pi}\left(4\pi m_e^{-2}\right)^{\epsilon_{UV}/2}\Gamma\left(\tfrac{1}{2}\epsilon_{UV}\right)\left[\int_0^1 dx(\epsilon_{UV}-2)x^{-\epsilon_{UV}}(1-x)\right.$$
$$\left.+\epsilon_{UV}\int_0^1 dx\, x^{-(1+\epsilon_{UV})}(1-x)\left((\epsilon_{UV}-2)(1-x)+4-\epsilon_{UV}\right)\right]. \quad (2.15)$$

Proceeding as above, we obtain

$$\lim_{\epsilon\to 0} B^{(\epsilon)} = -\frac{i\alpha}{\pi\epsilon} + \frac{i3\alpha}{4\pi}\left(\ln 4\pi - \gamma - 2\ln m_e + \tfrac{4}{3}\right) \quad (2.16)$$

Therefore

$$\lim_{\epsilon\to 0} 2\delta_V^{(\epsilon)} = -\frac{\alpha}{\pi\epsilon}\left(\int_0^1 dx\frac{2+r^2}{1+r^2x(1-x)} - 2\right) + \frac{\alpha}{2\pi}\int_0^1 dx\left[\left(\frac{2+r^2}{1+r^2x(1-x)} - 2\right)\right.$$
$$\times\left(\ln 4\pi - \gamma - 2\ln m_e\right) - 6 - \ln\left(1+r^2x(1-x)\right)$$
$$\left.+\frac{2(3+r^2)}{1+r^2x(1-x)} - \frac{2+r^2}{1+r^2x(1-x)}\ln\left(1+r^2x(1-x)\right)\right]. \quad (2.17)$$

2.2.2. Real corrections. The soft bremsstrahlung contribution will be

$$\delta_{BS}^{(\epsilon)} = \frac{\alpha}{\pi}\left(\frac{\Delta E^2}{4\pi}\right)^{\epsilon/2}\frac{1}{\epsilon\Gamma\left(1+\frac{1}{2}\epsilon\right)}\rho^2\left[\left(1+\tfrac{1}{2}r^2\right)\int_0^1 dx\, I_x(\epsilon) - I(\epsilon)\right], \quad (2.18)$$

where

$$I_x(\epsilon) \equiv \int_{-1}^1 dy\frac{\left(1-y^2\right)^{\epsilon/2}}{(1-\xi y)^2},$$

$$I(\epsilon) \equiv \int_{-1}^1 dy\frac{\left(1-y^2\right)^{\epsilon/2}}{(1-\xi_0 y)^2}$$

Hence, expanding around $\epsilon = 0$, we have

$$\lim_{\epsilon\to 0}\delta_{BS}^{(\epsilon)} = \frac{\alpha}{\pi\epsilon}\left(\int_0^1 dx\frac{2+r^2}{1+r^2x(1-x)} - 2\right)$$
$$+\frac{\alpha}{2\pi}\int_0^1 dx\left[\left(\frac{2+r^2}{1+r^2x(1-x)} - 2\right)\ln 4 + \frac{2+r^2}{1+r^2x(1-x)}\frac{1}{\xi}\ln\left(\frac{1-\xi}{1+\xi}\right)\right.$$
$$\left.+\left(\frac{2+r^2}{1+r^2x(1-x)} - 2\right)\left(2\ln\Delta E + \gamma - \ln 4\pi\right) - \frac{2}{\xi_0}\ln\left(\frac{1-\xi_0}{1+\xi_0}\right)\right].$$
$$(2.19)$$

2.3. RESULTS

Our process will be

$$\lim_{\mu_1\to 0} \sum_{D(E_a)D(E_b)} |S_{ba}^{(\mu_1)}|^2 = \lim_{\mu_1\to 0} \left[2\delta_V^{(\mu_1)} + \delta_{BS}^{(\mu_1)}\right].$$

From eqs. (2.9), (2.12), (2.17) and (2.19) we see that
(i) Every sum in eq. (2.1) is finite, as expected by the simple Bloch–Nordsieck mechanism.

$$\text{(ii)}\quad \lim_{\lambda\to 0}\left[2\delta_V^{(\lambda)} + \delta_{BS}^{(\lambda)}\right] = \lim_{\epsilon\to 0}\left[2\delta_V^{(\epsilon)} + \delta_{BS}^{(\epsilon)}\right]$$

$$= \frac{\alpha}{2\pi}\left\{-6 - J_0 + 2\left[(2+r^2)\ln\left(\frac{2\Delta E}{m_e}\right) + 3 + r^2\right]J_1\right.$$

$$\left. + (2+r^2)(J_3 - J_2) - 4\ln\left(\frac{2\Delta E}{m_e}\right) - \frac{2}{\xi_0}\ln\left(\frac{1-\xi_0}{1+\xi_0}\right)\right\},$$

where

$$J_0 \equiv \int_0^1 dx\,\ln\left(1 + r^2x(1-x)\right), \qquad J_1 \equiv \int_0^1 dx\frac{1}{1+r^2x(1-x)},$$

$$J_2 \equiv \int_0^1 dx\frac{1}{1+r^2x(1-x)}\ln\left(1+r^2x(1-x)\right),$$

$$J_3 \equiv \int_0^1 dx\frac{1}{1+r^2x(1-x)}\frac{1}{\xi}\ln\left(\frac{1-\xi}{1+\xi}\right),$$

Hence equality (2.1) is proved and uniqueness is recovered. The above integrals have been calculated and are given, for completeness, below.

$$J_0 = -\ln\eta^2 - 2 - 2(z-1)\ln(z-1) + 2z\ln z,$$

$$J_1 = \frac{2\eta^2}{2z-1}\ln\left(\frac{z^2}{\eta^2}\right),$$

$$J_2 = \frac{2\eta^2}{2z-1}\left[\ln^2\left(\frac{\eta^2}{z}\right) - \tfrac{1}{2}\ln^2\eta^2 + \ln(2z-1)\ln\left(\frac{z^2}{2z-1}\right) - 2\,\mathrm{Li}_2\left(\frac{z}{2z-1}\right) + \zeta(2)\right].$$

In the above we have defined $\eta^2 \equiv r^{-2} = -m_e^2/q^2$, $z \equiv \frac{1}{2}(1+(1+4\eta^2)^{1/2})$, $\mathrm{Li}_2(x) \equiv \int_0^x -(\mathrm{d}y/y)\ln(1-y)$ is the dilogarithm and $\zeta(2) = \pi^2/6$.

Finally,

$$J_3 = \frac{\rho^2}{\tau(1-\xi_1^2)^{1/2}} \Bigg\{ 2\ln(\xi_+)\ln\left[\frac{1}{4}\left(\frac{\xi_+ + \xi'}{\xi_+ - \xi'}\right)\left(\frac{\xi_- + \xi'}{\xi_- - \xi'}\right)\right] + \tfrac{1}{2}\ln^2(\xi_- + \xi')$$

$$- \tfrac{1}{2}\ln^2(\xi_- - \xi') + \tfrac{1}{2}\ln^2(\xi_+ + \xi') - \tfrac{1}{2}\ln^2(\xi_+ - \xi') - \ln(\xi_+^2 - \xi'^2)\ln\left(\frac{\xi_- + \xi'}{\xi_- - \xi'}\right)$$

$$+ \ln(\xi_- - \xi')\ln\left(\frac{\xi_- + \xi'}{2\xi_-}\right) - \ln(\xi_+ - \xi')\ln\left(\frac{\xi_+ + \xi'}{2\xi_+}\right) + 2\,\mathrm{Li}_2\left(\frac{\xi_- - \xi'}{2\xi_-}\right)$$

$$- 2\,\mathrm{Li}_2\left(\frac{\xi_+ - \xi'}{2\xi_+}\right) + 2\,\mathrm{Li}_2\left(\frac{\xi_+ - \xi_-}{\xi_+ + \xi'}\right) - 2\,\mathrm{Li}_2\left(\frac{\xi_+ - \xi_-}{\xi_+ - \xi'}\right)\Bigg\},$$

where

$$\xi_1^2 \equiv 1 - \rho^2 - \tfrac{1}{4}\tau^2 \equiv \xi_0^2 - \tfrac{1}{4}\tau^2, \qquad \xi_+^2 \equiv \frac{1+\xi_1}{1-\xi_1} \equiv \frac{1}{\xi_-^2},$$

$$\xi'^2 \equiv \frac{\xi_0 - \xi_1}{\xi_0 + \xi_1}, \qquad \tau^2 \equiv \frac{-q^2}{E^2} = \rho^2 r^2.$$

3. Two-mass singularities

This is the case of massless QED. In general we shall have three sources of infinities, one UV and two mass singularities. Correspondingly our mass regulators will be denoted by (μ_1, μ_2) and will regulate the (m_γ, m_e) singularities, respectively. To check uniqueness, we can form three linearly independent equalities of the form (2.1) and (2.2) (we do not consider momentum cutoffs among the regulators).

We consider, by far the most interesting of these equalities, the ones corresponding to the smooth electronic mass limit of massless QED. These will exhibit mixed mass regulators, namely combinations of massive regulator for the electron and dimensional for the photon, as opposed to totally massive or totally dimensional. Hence, two independent equalities can be formed. Both involve, at some point, treatment of the regulator-induced degrees of freedom and the resulting finite contributions. We choose, in the remainder of this paper, to study the equality:

$$\lim_{m_e\to 0}\lim_{\epsilon\to 0}\sum_{(DE_a)(DE_b)}|S_{ba}^{(\epsilon,m_e)}|^2-\lim_{\epsilon\to 0}\sum_{(DE_a)(DE_b)}|S_{ba}^{(\epsilon,\epsilon)}|^2=0. \tag{3.1}$$

The other choice, necessarily involving a totally massive combination, can be treated similarly and in fact part of it has already been calculated in subsect. 2.1.

3.1. $(\mu_1,\mu_2)=(\epsilon,m_\epsilon)$ REGULARIZATION SCHEME (MIXED MASS REGULATORS)

As in the one-mass singularity case, we shall have to consider virtual and real processes. The bremsstrahlung contributions will come, though, from both soft and hard collinear radiation.

3.1.1. Virtual corrections. After renormalization, the virtual corrections have already been calculated in formula (2.17). To be exact, we should also consider possible contributions from vacuum polarization fig. 1d. This diagram was of the $R(\mu_1)$ type and hence we did not consider it so far.

For an off-mass-shell photon of momentum q, the vacuum polarization graph gives

$$\Pi_{\mu\nu}(q)=-(-ie)^2\int\frac{d^{4-\epsilon_{UV}}k}{(2\pi)^{4-\epsilon_{UV}}}\mathrm{Tr}\left[\gamma_\mu\frac{i(\not k+m_e)}{k^2-m_e^2}\gamma_\nu\frac{i(\not k-\not q+m_e)}{(k-q)^2-m_e^2}\right]. \tag{3.2}$$

The corresponding contribution, after renormalization, will be $2\delta_{VP}^{(\epsilon,m_e)}$, where

$$\lim_{\epsilon\to 0}\delta_{VP}^{(\epsilon,m_e)}\equiv\partial_{VP}^{(0,m_e)}=\frac{2\alpha}{\pi}\left[\frac{1}{6}(\gamma-\ln 4\pi+2\ln m_e)+\int_0^1 dx\,x(1-x)\ln\left(1+r^2x(1-x)\right)\right]. \tag{3.3}$$

Therefore, in the limit

$$\lim_{m_e\to 0}\lim_{\epsilon\to 0}\delta_{VP}^{(\epsilon,m_e)}=\frac{2\alpha}{\pi}\left[\frac{1}{6}(\gamma-\ln 4\pi+2\ln m_e)-1-\frac{1}{6}\left(2\ln m_e-\ln(-q^2)\right)\right]$$
$$=\delta_{VP}^{(0,0)}! \tag{3.4}$$

Hence that contribution is of the $R(\mu_1,\mu_2)$ type and therefore will not be considered. We already saw that the $1/\epsilon$ terms disappear from the sum $2\delta_V^{(\epsilon,m_e)}+\delta_{BS}^{(\epsilon,m_e)}$ (eqs. (2.17) and (2.19)). Hence we can use the above equations, discarding the infrared poles, for our present purposes.

In evaluating the limit of the massive regulator, we shall need the limits of certain integrals. Derivation of some of these limits is not entirely trivial. For convenience, we present a table of all the limits necessary for the computation of the radiative corrections that follow, in appendix B. Hence, putting everything together, we obtain from eq. (2.17)

$$\lim_{m_e\to 0}\lim_{\epsilon\to 0} 2\delta_{\mathrm{V}}^{(\epsilon,m_e)} = \frac{\alpha}{2\pi}\left\{\ln^2\eta^2 + 2\ln m_e\left[-1+2\left(\gamma-\ln 4\pi+\ln(-q^2)\right)\right]\right\} + \frac{\alpha}{2\pi}$$
$$\times\left\{2(\gamma-\ln 4\pi) - 4 + 2\zeta(2) - \ln(-q^2)\left[-3+2\left(\gamma-\ln 4\pi+\ln(-q^2)\right)\right]\right\}. \tag{3.5}$$

3.1.2. Real corrections: soft bremsstrahlung. Similarly, from eq. (2.19) we obtain

$$\lim_{m_e\to 0}\lim_{\epsilon\to 0} \delta_{\mathrm{BS}}^{(\epsilon,m_e)} = \frac{\alpha}{2\pi}\left\{-\ln^2\eta^2 - 2\ln m_e\right.$$
$$\left.\times\left[4\ln\left(\frac{\Delta E}{E}\right) + 2 + 2\left(\gamma-\ln 4\pi+\ln(-q^2)\right)\right]\right\}$$
$$+\frac{\alpha}{2\pi}\left\{-2\zeta(2) - 2(\gamma-\ln 4\pi) - 4\ln\left(\frac{\Delta E}{E}\right)\right.$$
$$\left.+\ln(-q^2)\left[4\ln\left(\frac{\Delta E}{E}\right) + 2\left(\gamma-\ln 4\pi+\ln(-q^2)\right)\right] + f(\tau^2)\right\}, \tag{3.6}$$

where, in appendix B, we defined

$$f(\tau^2) \equiv \ln(\tau^2)\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)_k\frac{1}{k!}B(k,k)(\tau^2)^k$$
$$-\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)_k\frac{1}{k!}\left[\Psi(k+1)-\Psi(k+\tfrac{1}{2})-2\Psi(k)+2\Psi(2k)\right]B(k,k)(\tau^2)^k. \tag{3.7}$$

All of the symbols appearing in eqs. (3.5)–(3.7) are defined in appendix B.

3.1.3. Real corrections: hard collinear bremsstrahlung. Finally, as is well known, we shall have to consider the extra degeneracy of initial and final states due to collinear hard radiation as shown in fig. 2.

This contribution will be the regulator-dependent part of the incoherent sum of the above pairs of diagrams (final and initial state degeneracy respectively). The contributions of the two pairs are equal, so we shall just double the final-state degeneracy contribution. This will contain terms depending on $1/(p_1 k)^2, 1/(p_1 k)$

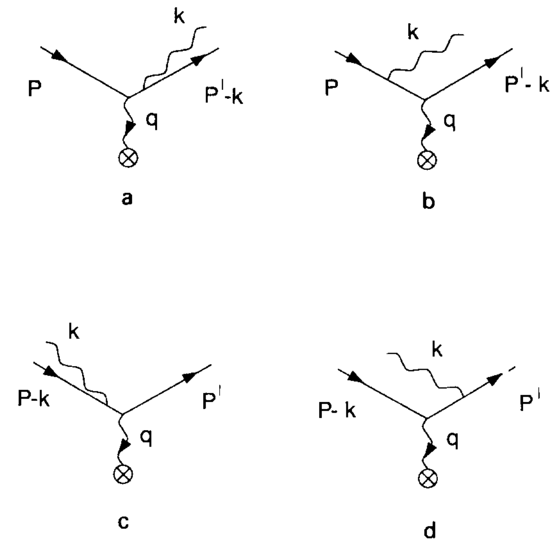

Fig. 2. Hard collinear bremsstrahlung. (a, b): Final state degeneracy (c, d): Initial state degeneracy.

and independent of $1/(p_1 k)$ where $p_1 \equiv p' - k$. After summing over the photon polarizations, and discarding terms independent of $1/(p_1 k)$ since these are of the $R(\mu_1, \mu_2)$ type, the hard collinear bremsstrahlung contribution will be $\lim_{m_e \to 0} \lim_{\epsilon \to 0} 2\delta_{BH}^{(\epsilon, m_e)} \equiv \lim_{m_e \to 0} 2\delta_{BH}^{(0, m_e)}$, where

$$\lim_{m_e \to 0} \delta_{BH}^{(m_e)} \frac{d\sigma_0}{d\Omega} = \frac{1}{16\pi(2\pi)^3} \frac{1}{E\left(1 - (m_e/E)^2\right)^{1/2}} \times \int_{m_e}^{E-\Delta E} dE_1\, E_1(E - E_1)\left(1 - \left(\frac{m_e}{E_1}\right)^2\right)^{1/2} \int_0^{\delta} d\theta\, \theta |S_{BH}^{(m_e)}|^2 . \quad (3.8)$$

In the above expression $d\sigma_0/d\Omega$ is the non-radiative process, θ the collinear direction angle, $|S_{BH}^{(m_e)}|^2$ the regulator-dependent part of the Bethe–Heitler process, and δ the photon jet opening angle. One can find

$$|S_{BH}^{(m_e)}|^2 = 64\pi^3\alpha \left[\frac{1}{(p_1 k)^2}(-m_e^2) + \frac{1}{(p_1 k)}\left(\frac{2E}{E - E_1} - 1 - \frac{E_1}{E}\right)\right] \frac{d\sigma_0}{d\Omega} . \quad (3.9)$$

Notice that it is wrong to set the mass $m_e^2 = 0$ in the numerator of the $1/(p_1 k)^2$ term from the beginning, since this term becomes a *finite* observable contribution once integrated over the collinear direction [1, 9]:

$$\lim_{m_e \to 0} m_e^2 \int_0^{\delta} \mathrm{d}\theta\, \theta \frac{1}{(p_1 k)^2}$$
$$= \lim_{m_e \to 0} \frac{m_e^2}{(E_1(E-E_1))^2}$$
$$\times \int_0^{\delta} \frac{\mathrm{d}\theta^2}{2} \frac{1}{\left[1-\left(1-(m_e/E_1)^2\right)^{1/2}+\left(1-(m_e/E_1)^2\right)^{1/2}\theta^2/2\right]^2}$$
$$= \frac{2}{(E-E_1)^2}. \tag{3.10}$$

This term is the notorious helicity-flip finite contribution which exists in polarized and non-polarized processes, but is singled out in the former, since the rest of the radiative corrections, having been multiplied by positive powers of the regulator mass, go to zero. In the present case, nevertheless, this term is essential for the restoration of uniqueness since it combines with the soft-bremsstrahlung m_e^2-dependent term that gave the finite $(1/\xi_0)\ln[(1-\xi_0)/(1+\xi_0)]$-term in eq. (2.19). This is fortunate since both of these terms will be identically zero in the purely dimensional regularization scheme. We find

$$\lim_{m_e \to 0} 2\delta_{\mathrm{BH}}^{(m_e)} = \frac{\alpha}{2\pi} \ln m_e \left[8 \ln\left(\frac{\Delta E}{E}\right)+6\right] + \frac{\alpha}{2\pi}\left[-\ln\delta\left(8\ln\left(\frac{\Delta E}{E}\right)+6\right)\right.$$
$$\left. -8\ln E \ln(\Delta E/E) - 8\zeta(2) + 9 - 6\ln E + 4\ln\left(\frac{\Delta E}{E}\right)\right]. \tag{3.11}$$

3.2. $(\mu_1, \mu_2) = (\epsilon, \epsilon)$ REGULARIZATION SCHEME

We must be especially careful here since, in the virtual corrections, we encounter the phenomenon of an UV pole turning into an infrared one [6] while, in the real corrections, the extra polarizations of the photons must be consistently considered.

3.2.1. Virtual corrections. Using previous expressions found in the one-mass singularity case, we have

$$\lim_{\epsilon \to 0} 2\delta_{\mathrm{V}}^{(\epsilon,\epsilon)} = \frac{\alpha}{2\pi}\left[-\frac{8}{\epsilon^2}+\frac{4}{\epsilon}\left(\ln 4\pi + \tfrac{3}{2} - \gamma - \ln(-q^2)\right)\right] + \frac{\alpha}{2\pi}$$
$$\times\left[-8+\gamma^2+\zeta(2)+3(\gamma - \ln 4\pi + \ln(-q^2)) + 2\gamma \ln\left(\frac{4\pi}{-q^2}\right) - \ln^2\left(\frac{4\pi}{-q^2}\right)\right]. \tag{3.12}$$

In the above we observe that the wave function renormalization constant contribution will be as follows:

$$B^{(\epsilon_{\rm UV},\epsilon_{\rm UV})} = -\frac{i\alpha}{4\pi}\left(\frac{4\pi}{-p^2}\right)^{\epsilon_{\rm UV}/2}(\epsilon_{\rm UV}-2)B\left(2-\tfrac{1}{2}\epsilon_{\rm UV}, 1-\tfrac{1}{2}\epsilon_{\rm UV}\right)\Gamma\left(\tfrac{1}{2}\epsilon_{\rm UV}\right), \tag{3.13}$$

from where the UV singularity will be: $i\alpha/2\pi\epsilon_{\rm UV}$. Once the UV counterterm is found, we continue to $n > 4$, i.e. $\epsilon_{\rm UV} \to -\epsilon$. But then

$$B^{(\epsilon_{\rm UV},\epsilon_{\rm UV})} - \frac{i\alpha}{2\pi\epsilon_{\rm UV}} \to -\frac{i\alpha}{4\pi}\left(\frac{-p^2}{4\pi}\right)^{\epsilon/2}(-\epsilon-2) \times B\left(2+\tfrac{1}{2}\epsilon, 1+\tfrac{1}{2}\epsilon\right)\Gamma\left(-\tfrac{1}{2}\epsilon\right) + \frac{i\alpha}{2\pi\epsilon} = \frac{i\alpha}{2\pi\epsilon}, \tag{3.14}$$

since the first term becomes identically zero, p^2 being on mass shell.

We remark that this process of an UV pole "turning into" an infrared one is idiosyncratic to the completely massless theory. Stated differently, there is no space-time region where the theory is free of both kinds of divergences. Masses separate the two space-time regions in which the theory is free of, alternatively, UV and mass singularities. Once this "barrier" is set to zero, the two regions merge, resulting in the above effect.

3.2.2. Real corrections: soft bremsstrahlung. This time the second term in eq. (2.18) is absent. We have

$$\delta_{\rm BS}^{(\epsilon,\epsilon)} = \frac{\alpha}{\pi}\left(\frac{\Delta E^2}{4\pi}\right)^{\epsilon/2}\frac{1}{\epsilon}\frac{1}{\Gamma\left(1+\frac{1}{2}\epsilon\right)}\frac{\tau^2}{2}\int_0^1 {\rm d}x\, I_x(\epsilon)\,. \tag{3.15}$$

The integral $I_x(\epsilon)$ was encountered in eq. (2.18) but, in the completely massless case, it develops a pole. It is both useful and instructive to compute it using the following technique which suggests a relation between the finite parts obtainable by the different regularization methods. Writing

$$I_x(\epsilon) = \frac{1}{\xi^2}\int_0^2 {\rm d}y \frac{y^{\epsilon/2}(2-y)^{\epsilon/2}}{\left(\dfrac{1}{\xi}-1+y\right)^2},$$

and making use of the formula [11]*

$$\int_0^u {\rm d}x x^{\nu-1}(u-x)^{\mu-1}(x^m+\beta^m)^\lambda$$
$$= \beta^{m\lambda}u^{\mu+\nu-1}B(\mu,\nu)\,{}_{m+1}F_m\left(-\lambda, \frac{\nu}{m}\ldots\frac{\nu+m-1}{m}; \frac{\mu+\nu}{m}\ldots\frac{\mu+\nu+m-1}{m}; -\frac{u^m}{\beta^m}\right), \tag{3.16}$$

* Note that the formula in the above reference is wrong, as a simple scale transformation can show.

we have

$$I_x(\epsilon) = \frac{1}{(1-\xi)^2} 2^{1+\epsilon} B\left(1+\tfrac{1}{2}\epsilon, 1+\tfrac{1}{2}\epsilon\right) F\left(2, 1+\tfrac{1}{2}\epsilon; 2\left(1+\tfrac{1}{2}\epsilon\right); \frac{2\xi}{\xi-1}\right).$$

In the above, $F = {}_2F_1$ is the (simple) hypergeometric function.

The idea is to transform the argument of the above expression, using the analytic continuation formulas relevant to hypergeometric functions, to an x-integrable argument involving τ^2 alone, since that was the dependence of the finite part obtained in subsect. 3.1.2.

If $\pm(1-c)$, $\pm(a-b)$, $\pm(a+b-c)$ are such that two of them are equal, or one of them is equal to $\frac{1}{2}$, then there exists a quadratic transformation [12]

$$F(a,b;2b;z) = \left(1-\tfrac{1}{2}z\right)^{-a} F\left(\tfrac{1}{2}a, \tfrac{1}{2}+\tfrac{1}{2}a; \tfrac{1}{2}+b; \left(\frac{z}{2-z}\right)^2\right). \tag{3.17}$$

Here the former alternative between the last two combinations holds. Hence

$$I_x(\epsilon) = 2^{1+\epsilon} B\left(1+\tfrac{1}{2}\epsilon, 1+\tfrac{1}{2}\epsilon\right) F\left(1, \tfrac{3}{2}; \tfrac{3}{2}+\tfrac{1}{2}\epsilon; \xi^2\right).$$

Applying now the linear transformation [12]

$$\begin{aligned} F(a,b;c;z) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a,b;a+b-c+1;1-z) \\ &\quad + (1-z)^{c-a-b} \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} F(c-a,c-b;c-a-b+1;1-z), \end{aligned} \tag{3.18}$$

that is only possible if the arguments of the right-hand side functions give meaningful analytic continuations (which is the case here due to the fact that we are in $4+\epsilon$ dimensions), we obtain

$$\begin{aligned} \int_0^1 \mathrm{d}x\, I_x(\epsilon) &= 2^{1+\epsilon} B\left(1+\tfrac{1}{2}\epsilon, 1+\tfrac{1}{2}\epsilon\right) \left[\frac{1+\epsilon}{\epsilon-2} \int_0^1 \mathrm{d}x\, F\left(1, \tfrac{3}{2}; 2-\tfrac{1}{2}\epsilon; \tau^2 x(1-x)\right) \right. \\ &\quad \left. + \frac{\Gamma\left(\tfrac{3}{2}+\tfrac{1}{2}\epsilon\right)\Gamma\left(1-\tfrac{1}{2}\epsilon\right)}{\Gamma\left(\tfrac{3}{2}\right)} \int_0^1 \mathrm{d}x \left(\tau^2 x(1-x)\right)^{\epsilon/2-1} F\left(\tfrac{1}{2}+\tfrac{1}{2}\epsilon, \tfrac{1}{2}\epsilon; \tfrac{1}{2}\epsilon; \tau^2 x(1-x)\right) \right]. \end{aligned} \tag{3.19}$$

Hence the soft bremsstrahlung contribution will be

$$\delta_{\mathrm{BS}}^{(\epsilon,\epsilon)} = \frac{\alpha}{2\pi} \left[\frac{1}{\epsilon} \left(\frac{\Delta E^2}{4\pi} \right)^{\epsilon/2} \frac{1}{\Gamma\left(1+\tfrac{1}{2}\epsilon\right)} f_1(\epsilon) + \frac{2}{\epsilon} B\left(\tfrac{1}{2}\epsilon, \tfrac{1}{2}\epsilon\right) \frac{1}{\Gamma\left(\tfrac{3}{2}\right)} f_2(\epsilon) \right], \tag{3.20}$$

where

$$f_1(\epsilon) \equiv 2^{1+\epsilon} B\left(1+\tfrac{1}{2}\epsilon, 1+\tfrac{1}{2}\epsilon\right)\left\{\frac{1+\epsilon}{\epsilon-2}\sum_{k=0}^{\infty}\left(\tfrac{3}{2}\right)_k \frac{1}{\left(2-\frac{\epsilon}{2}\right)_k} B(k+1,k+1)\left(\tau^2\right)^{k+1}\right.$$

$$\left.+\frac{\Gamma\left(\frac{3}{2}+\frac{1}{2}\epsilon\right)\Gamma\left(1-\frac{1}{2}\epsilon\right)}{\Gamma\left(\frac{3}{2}\right)}\left(\tau^2\right)^{\epsilon/2}\sum_{k=1}^{\infty}\left(\tfrac{1}{2}+\tfrac{1}{2}\epsilon\right)_k \frac{1}{k!} B\left(k+\tfrac{1}{2}\epsilon, k+\tfrac{1}{2}\epsilon\right)\left(\tau^2\right)^k\right\}, \tag{3.21}$$

$$f_2(\epsilon) \equiv \left(\frac{\tau^2 \Delta E^2}{\pi}\right)^{\epsilon/2} \Gamma\left(1+\tfrac{1}{2}\epsilon\right)\Gamma\left(\tfrac{3}{2}+\tfrac{1}{2}\epsilon\right)\Gamma\left(1-\tfrac{1}{2}\epsilon\right)\Gamma^{-1}(2+\epsilon). \tag{3.22}$$

Making the appropriate Taylor expansions we obtain (see appendix C)

$$f_1(0) = 0, \tag{3.23}$$

$$f_1'(0) = \left[-3+\Psi(2)+\Psi\left(\tfrac{3}{2}\right)-\Psi(1)-\Psi\left(\tfrac{1}{2}\right)+\ln(\tau^2)\right]\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)k\frac{1}{k!}B(k,k)\left(\tau^2\right)^k$$

$$-\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)_k\frac{1}{k!}\left[\Psi(k+1)-\Psi\left(\tfrac{1}{2}+k\right)-2\Psi(k)+2\Psi(2k)\right]B(k,k)\left(\tau^2\right)^k = f(\tau^2), \tag{3.24}$$

with $f(\tau^2)$ defined in eq. (3.7), and

$$\frac{f_2(0)}{\Gamma\left(\frac{3}{2}\right)} = 1, \qquad \frac{2f_2'(0)}{\Gamma\left(\frac{3}{2}\right)} = \ln\left(\frac{\tau^2\Delta E^2}{4\pi}\right)+\gamma, \qquad (3.25), (3.26)$$

$$\frac{4f_2''(0)}{\Gamma\left(\frac{3}{2}\right)} = \left[\ln\left(\frac{\tau^2\Delta E^2}{4\pi}\right)+\gamma\right]^2+\zeta(2). \tag{3.27}$$

Hence substituting we get

$$\lim_{\epsilon\to 0}\delta_{\mathrm{BS}}^{(\epsilon,\epsilon)} = \frac{\alpha}{2\pi}\left[\frac{8}{\epsilon^2}+\frac{4}{\epsilon}\left(\gamma-\ln 4\pi+\ln(-q^2)+2\ln\left(\frac{\Delta E}{E}\right)\right)\right]+\frac{\alpha}{2\pi}$$

$$\times\left[\ln^2\left(\left(\frac{-q^2}{4\pi}\right)\left(\frac{\Delta E}{E}\right)^2\right)+2\gamma\ln\left(\left(\frac{-q^2}{4\pi}\right)\left(\frac{\Delta E}{E}\right)^2\right)-\left(\gamma^2+\zeta(2)\right)+f(\tau^2)\right]. \tag{3.28}$$

3.2.3. Real corrections: hard collinear bremsstrahlung. The cross section is obtained by the formula

$$\mathrm{d}\sigma_{\mathrm{BH}}^{(\epsilon,\epsilon)} = \frac{\pi}{E}\int \frac{\mathrm{d}^{3+\epsilon}k}{2\omega(2\pi)^{3+\epsilon}} \frac{\mathrm{d}^{3+\epsilon}p_1}{2E_1(2\pi)^{3+\epsilon}} \delta(E-\omega-E_1)|S_{\mathrm{BH}}^{(\epsilon)}|^2 . \tag{3.29}$$

We can write the regulator-dependent quantity $|S_{\mathrm{BH}}^{(\epsilon)}|^2$ as

$$|S_{\mathrm{BH}}^{(\epsilon)}|^2 = |S_{\mathrm{BH}}^{(\epsilon;2)}|^2 + |S_{\mathrm{BH}}^{(\epsilon;\epsilon)}|^2 , \tag{3.30}$$

where the first term on the right-hand side corresponds to the two transverse degrees of polarization of the photon and the second to the remaining ϵ transverse degrees of polarization, introduced by the dimensional regulator. We find

$$|S_{\mathrm{BH}}^{(\epsilon;2)}|^2 = 64\pi^3\alpha\frac{1}{(p_1k)}\left(\frac{2E}{E-E_1} - 1 - \frac{E_1}{E}\right)\left(\frac{2\pi}{E}\right)^{\epsilon}\frac{\mathrm{d}\sigma_0}{\mathrm{d}\Omega} , \tag{3.31}$$

$$|S_{\mathrm{BH}}^{(\epsilon;\epsilon)}|^2 = 64\pi^3\alpha\frac{\epsilon}{(p_1k)}\frac{\omega}{2E}\left(\frac{2\pi}{E}\right)^{\epsilon}\frac{\mathrm{d}\sigma_0}{\mathrm{d}\Omega} . \tag{3.32}$$

Notice the important factor $(2\pi/E)^{\epsilon}$, necessary in order to express the left-hand side in factorizable form in terms of the non-radiative cross section in n dimensions*. We obtain

$$\lim_{\epsilon\to 0} 2\delta_{\mathrm{BH}}^{(\epsilon,\epsilon)} = -\frac{\alpha}{2\pi}\frac{1}{\epsilon}\left[8\ln\left(\frac{\Delta E}{E}\right)+6\right] + \frac{\alpha}{2\pi}\left[-\ln\delta\left(8\ln\left(\frac{\Delta E}{E}\right)+6\right) + 4\ln^2 E - 4\ln^2\Delta E - 8\zeta(2) + 13 - 6\ln E - (\gamma - \ln 4\pi)\left(4\ln\left(\frac{\Delta E}{E}\right)+3\right)\right] . \tag{3.33}$$

3.3. RESULTS

Our process will be

$$\lim_{\mu_2\to 0}\lim_{\mu_1\to 0}\sum_{(\mathrm{D}E_a)(\mathrm{D}E_b)} |S_{ba}^{(\mu_1,\mu_2)}|^2 = \lim_{\mu_2\to 0}\lim_{\mu_1\to 0}\left[2\delta_{\mathrm{V}}^{(\mu_1,\mu_2)} + \delta_{\mathrm{BS}}^{(\mu_1,\mu_2)} + 2\delta_{\mathrm{BH}}^{(\mu_1,\mu_2)}\right] .$$

*The quantity in eq. (3.32) contributes to the cross section an amount $2\delta_{\mathrm{BH}}^{(\epsilon,\epsilon;\epsilon)} = \alpha/2\pi$. This is exactly opposite to the contribution coming from the vertex correction, due to the n-dimensional γ-algebra. In other words the extra ϵ photon degrees of freedom give *finite* contributions that cancel the ones coming from the n-dimensionality of the fermionic space-time index. This constitutes another example of finite LN cancellations.

Summing up the results of subsects. 3.1 and independently 3.2 we obtain:
(i) The well-known cancellation of the singular pieces [1, 6]. Hence finiteness is recovered.

(ii)
$$\lim_{m_e \to 0} \lim_{\epsilon \to 0} \left[2\delta_{\mathrm{V}}^{(\epsilon, m_e)} + \delta_{\mathrm{BS}}^{(\epsilon, m_e)} + 2\delta_{\mathrm{BH}}^{(\epsilon, m_e)}\right]$$
$$= \lim_{\epsilon \to 0} \left[2\delta_{\mathrm{V}}^{(\epsilon, \epsilon)} + \delta_{\mathrm{BS}}^{(\epsilon, \epsilon)} + 2\delta_{\mathrm{BH}}^{(\epsilon, \epsilon)}\right]$$
$$= \frac{\alpha}{2\pi}\left[5 - 8\zeta(2) + 3\ln\left(\frac{-q^2}{E^2}\right) + 4\ln\left(\frac{-q^2}{E^2}\right)\ln\left(\frac{\Delta E}{E}\right) - \ln\delta\left(8\ln\left(\frac{\Delta E}{E}\right) + 6\right) + f(\tau^2)\right],$$

with $f(\tau^2)$ given in eq. (3.7). Therefore, equality (3.1) is proved and uniqueness is recovered.

4. Conclusions

We demonstrated that, in this particular QED process, LN probabilities are not only finite, but also unique, even in the completely massless limit of the theory. We believe that the details involved are generic, indicating that the proposition should work in general in QED. We also demonstrated that special attention should be paid to the idiosyncracies of each regularization scheme, especially when more than one particle is massless.

In particular, the extra physical (but regulator-induced) degrees of freedom for the photon, and the helicity-flip terms proportional to m_e^2 should be kept, if multiplied by $1/\epsilon$ poles and collinear singularities respectively, because they give finite contributions. These spurious finite contributions cancel when we add up all processes belonging to the same degenerate LN set. The uniqueness of the finite parts becomes obvious when, as it was shown at the analytical level, one expresses the seemingly incomparable integrals in terms of hypergeometric functions. These lead to the same finite parts if continued analytically (in the dimensional case) or expanded in a Taylor series (in the massive case) at exactly the point where analytic continuation, possible in the dimensional case because of the extra space-time dimensions, breaks down.

We believe that uniqueness of LN probabilities is no mere accident but rather *follows* from the structure of the singularities and their subsequent cancellation within a LN set. The different members of such a set are actually pieces of different Green functions that become physically degenerate once put on mass-shell. Therefore, we believe in the possibility of unifying these different pieces, possibly at the Green functions level, by an inclusive formalism which would show that, if one obtains finiteness, one will obtain uniqueness as a matter of course. This formalism would necessarily address the question of collinear helicity-flip processes and the corresponding finite contributions that, at this stage, seem quite non-unique. These are of no mere academic interest, since their contribution may be a radiative background making dubious the interpretation of many experiments [9, 13, 14]. We shall return to these more general questions in a future publication.

We would like to thank M.B. Einhorn for suggesting this problem and his many valuable insights and discussions, for proofreading the manuscript and for his collaboration on the problem of the massless limit.

Appendix A

Consider the integral

$$I_x(\lambda) = \int_0^{\Delta E/m_e} \mathrm{d}k\,k^2 \int_{-1}^{1} \mathrm{d}y \frac{1}{(k^2+\lambda^2)^{1/2}} \frac{1}{\left[(k^2+\lambda^2)^{1/2} - \frac{1}{2}\rho\lambda^2 - k\xi y\right]^2}.$$

Since we are interested in the limit $\lambda \to 0$, the singular and regular structure of the integral remains the same if we drop the $\rho\lambda^2$ term. Hence we shall calculate

$$\lim_{\lambda\to 0} I_x(\lambda) = \lim_{s\to\infty} \int_{-1}^{1} \mathrm{d}y J_x(s,\xi,y),$$

where

$$J_x(s,\xi,y) = \int_0^{s\equiv\Delta E/m_e\lambda} \mathrm{d}t \frac{t^2}{(1+t^2)^{1/2}} \frac{1}{\left[(1+t^2)^{1/2} - t\xi y\right]^2}.$$

Making the transformation: $t = \tan\theta$, we have

$$J_x(s,\xi,y) = \int_0^{\arctan s} \mathrm{d}\theta \frac{1}{\cos\theta(1-\xi y\sin\theta)^2} - \frac{1}{\xi y}\frac{1}{1 - s\xi y/(1+s^2)^{1/2}} + \frac{1}{\xi y}.$$

The first integral can be calculated easily if we expand $1/(1-\xi y\sin\theta)$ in a Taylor series (notice that $0 < \xi, y, \sin\theta < 1$). We obtain

$$J_x(s,\xi,y) = \sum_{k=0}^{\infty} (\xi y)^{2k} \int_0^{\arctan s} \mathrm{d}\theta \frac{\sin^{2k}\theta}{\cos\theta} + 2\sum_{k=0}^{\infty} (k+1)(\xi y)^{2k+1}$$

$$\times \int_0^{\arctan s} \mathrm{d}\theta \frac{\sin^{2k+1}\theta}{\cos\theta} - \frac{1}{\xi y}\frac{1}{1 - s\xi y/(1+s^2)^{1/2}} + \frac{1}{\xi y}.$$

Using also

$$\int \mathrm{d}\theta \frac{\sin 2k\theta}{\cos\theta} = \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\theta\right) - \sum_{l=1}^{k} \frac{\sin^{2l-1}\theta}{2l-1},$$

$$I_x(\lambda) = \int_0^1 \mathrm{d}y \left[J_x(s,\xi,y) + J_x(s,\xi,-y) \right],$$

$$\lim_{s\to\infty} \ln\tan\left(\tfrac{1}{4}\pi + \tfrac{1}{2}\arctan s\right) = \ln 2s,$$

and the identity

$$\sum_{k=1}^{\infty} \sum_{l=1}^{k} \frac{\xi^{2k}}{2l-1} = \frac{1}{2\xi}\ln\left(\frac{1-\xi}{1+\xi}\right) - \frac{1}{1-\xi^2}\frac{1}{2\xi}\ln\left(\frac{1-\xi}{1+\xi}\right),$$

we obtain

$$\lim_{\lambda\to 0} I_x(\lambda) = 2\left\{ \left[\ln\left(\frac{2\Delta E}{m_{\mathrm{e}}}\right) - \ln\lambda \right] \frac{1}{1-\xi^2} + \frac{1}{1-\xi^2}\frac{1}{2\xi}\ln\left(\frac{1-\xi}{1+\xi}\right) \right\}.$$

In order to perform the x-integration on the second term of this integral, we introduce two constants α and β such that

$$1 - \rho^2 - \rho^2 r^2 \alpha(1-\alpha) = \beta^2.$$

Then we perform the change of variables:

$$x = \alpha - \frac{(2\alpha-1)\rho^2 r^2 \omega^2 - 2\beta\omega}{\rho^2 r^2 \omega^2 - 1}.$$

Thus we complete the square of ξ:

$$\xi = \beta - \frac{(2\alpha-1)\rho^2 r^2 \omega - 2\beta}{\rho^2 r^2 \omega^2 - 1}.$$

Hence the logarithmic integral splits into standard dilogarithms. The integration appearing in subsect. 2.3 was performed at the conveniently symmetric point $\alpha = \frac{1}{2}$.

Appendix B

For convenience, we supply a list of limits we used in the text. The corresponding parameters appearing are: $r^2 = -q^2/m_{\mathrm{e}}^2$, $\eta^2 = 1/r^2$, $\tau^2 = -q^2/E^2$, $\xi^2 =$

$1-\tau^2[\eta^2+x(1-x)]$, $\xi_0^2=1-\tau^2\eta^2$. The following table of limits was used:

$$\lim_{m_e\to 0}\int_0^1 dx\ln\left(1+r^2x(1-x)\right) = -2-\ln\eta^2,$$

$$\lim_{m_e\to 0}\int_0^1 dx\frac{1}{1+r^2x(1-x)}=0,$$

$$\lim_{m_e\to 0} r^2\int_0^1 dx\frac{1}{1+r^2x(1-x)} = -2\ln\eta^2,$$

$$\lim_{m_e\to 0}\int_0^1 dx\frac{\ln\left(1+r^2x(1-x)\right)}{1+r^2x(1-x)}=0,$$

$$\lim_{m_e\to 0} r^2\int_0^1 dx\frac{\ln\left(1+r^2x(1-x)\right)}{1+r^2x(1-x)}=\ln^2\eta^2-2\zeta(2),$$

$$\lim_{m_e\to 0}\ln m_e\int_0^1 dx\frac{1}{1+r^2x(1-x)}=0,$$

$$\lim_{m_e\to 0} r^2\ln m_e\int_0^1 dx\frac{1}{1+r^2x(1-x)} = -\ln^2\eta^2-\ln(-q^2)\ln\eta^2,$$

$$\lim_{m_e\to 0}\int_0^1 dx\frac{1}{1+r^2x(1-x)}\frac{1}{\xi}\ln\left(\frac{1-\xi}{1+\xi}\right)=0,$$

$$\lim_{m_e\to 0} r^2\int_0^1 dx\frac{1}{1+r^2x(1-x)}\frac{1}{\xi}\ln\left(\frac{1-\xi}{1+\xi}\right)$$

$$= -\ln^2\eta^2-2\left[\Psi\left(\tfrac{1}{2}\right)-\Psi(1)+\ln\tau^2\right]\ln\eta^2-2\zeta(2)+\ln\tau^2\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)_k\frac{1}{k!}B(k,k)(\tau^2)^k$$

$$-\sum_{k=1}^{\infty}\left(\tfrac{1}{2}\right)_k\frac{1}{k!}\left[\Psi(k+1)-\Psi\left(k+\tfrac{1}{2}\right)-2\Psi(k)+2\Psi(2k)\right]B(k,k)(\tau^2)^k,$$

$$\lim_{m_e\to 0}\frac{1}{\xi_0}\ln\left(\frac{1-\xi_0}{1+\xi_0}\right)=\ln\eta^2+\ln\tau^2-\ln 4.$$

In the above, $\zeta(x)$ is the Riemann zeta function ($\zeta(2)=\pi^2/6$), $\Psi(x)\equiv d\ln\Gamma(x)/dx$, Γ, B the gamma and beta functions, respectively, and the Pochhammer symbol is defined as

$$(a)_k\equiv\frac{\Gamma(a+k)}{\Gamma(a)}.$$

A derivation of the next to last limit goes as follows: Denoting the integral by $I(\eta^2)$ and noticing that in general $\tau^2[\eta^2 + x(1-x)] < 1$, and that, $0 < \xi < 1$, we can write [11]

$$\frac{1}{\xi}\ln\left(\frac{1-\xi}{1+\xi}\right) = -2F\left(\tfrac{1}{2}, 1; \tfrac{3}{2}; \xi^2\right),$$

where

$$F(a,b;c;z) \equiv \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!}$$

is the (simple) hypergeometric function. Also noticing that for $|1-z| < 1$ we can write the expansion [12]

$$F(a,b;a+b;z) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(k!)^2} \times [2\Psi(k+1) - \Psi(a+k) - \Psi(b+k) - \ln(1-z)](1-z)^k,$$

to obtain

$$\begin{aligned}\lim_{\eta^2\to 0} I(\eta^2) = &-\left[\Psi(1) - \Psi\left(\tfrac{1}{2}\right) - \ln\tau^2\right] \lim_{\eta^2\to 0}\int_0^1 \mathrm{d}x \frac{1}{\eta^2 + x(1-x)} \\ &+ \lim_{\eta^2\to 0}\int_0^1 \mathrm{d}x \frac{\ln(\eta^2 + x(1-x))}{\eta^2 + x(1-x)} \\ &- \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)_k \frac{1}{k!}\left\{\left(\Psi(k+1) - \Psi\left(k+\tfrac{1}{2}\right)\right)(\tau^2)^k \lim_{\eta^2\to 0}\int_0^1 \mathrm{d}x\left(\eta^2 + x(1-x)\right)^{k-1}\right. \\ &\left. -(\tau^2)^k \lim_{\eta^2\to 0}\int_0^1 \mathrm{d}x \ln\left[\tau^2\left(\eta^2 + x(1-x)\right)\right]\left[\eta^2 + x(1-x)\right]^{k-1}\right\}.\end{aligned}$$

We can set $\eta^2 = 0$ in the limits existing in the sum, since the corresponding integrals are convergent and the series is convergent. Also, substituting the rest of the limits from the table and using the formula

$$\int_0^1 \mathrm{d}x \ln(x(1-x))[x(1-x)]^{k-1} = 2B(k,k)[\Psi(k) - \Psi(2k)],$$

we obtain

$$\lim_{\eta^2\to 0} I(\eta^2) = -\ln^2\eta^2 - 2\left[\Psi\left(\tfrac{1}{2}\right) - \Psi(1) + \ln\tau^2\right]\ln\eta^2 - 2\zeta(2) + f(\tau^2),$$

with

$$f(\tau^2) \equiv \ln(\tau^2) \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)_k \frac{1}{k!} B(k,k)(\tau^2)^k$$
$$- \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)_k \frac{1}{k!} \left[\Psi(k+1) - \Psi\left(k+\tfrac{1}{2}\right) - 2\Psi(k) + 2\Psi(2k)\right] B(k,k)(\tau^2)^k .$$

Appendix C

We shall prove eqs. (3.23) and (3.24). Looking at the definition of $f_1(\epsilon)$, we obtain

$$f_1(0) = 2\left[-\tfrac{1}{2} \sum_{k=0}^{\infty} \left(\tfrac{3}{2}\right)_k \frac{1}{(2)_k} B(k+1,k+1)(\tau^2)^{k+1} + \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)_k \frac{1}{k!} B(k,k)(\tau^2)^k\right].$$

But $\left(\tfrac{3}{2}\right)_k = 2\left(\tfrac{1}{2}\right)_{k+1}$, $(2)_k = (k+1)!$. Hence $f_1(0) = 0$. Similarly,

$$f_1'(0) = \left[\left.\frac{\mathrm{d}2^{\epsilon}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} + \left.\frac{\mathrm{d}B\left(1+\frac{1}{2}\epsilon, 1+\frac{1}{2}\epsilon\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\right] f_1(0)$$
$$+2\left\{\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\frac{1+\epsilon}{\epsilon-2}\right)\right|_{\epsilon=0} 2\sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)_k \frac{1}{k!} B(k,k)(\tau^2)^k - \tfrac{1}{2}\sum_{k=0}^{\infty} \left(\tfrac{3}{2}\right)_k \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\left(2-\tfrac{1}{2}\epsilon\right)_k\right]^{-1}\Big|_{\epsilon=0}\right.$$
$$\times B(k+1,k+1)(\tau^2)^{k+1} + \frac{1}{\Gamma\left(\frac{3}{2}\right)} \frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[\Gamma\left(\tfrac{3}{2}+\tfrac{1}{2}\epsilon\right)\Gamma\left(1-\tfrac{1}{2}\epsilon\right)(\tau^2)^{\epsilon/2}\right]\Big|_{\epsilon=0}$$
$$\times \sum_{k=1}^{\infty} \left(\tfrac{1}{2}\right)_k \frac{1}{k!} B(k,k)(\tau^2)^k + \sum_{k=1}^{\infty} \frac{1}{k!}\left[\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\tfrac{1}{2}+\tfrac{1}{2}\epsilon\right)_k\Big|_{\epsilon=0} B(k,k)\right.$$
$$\left.\left. + \left(\tfrac{1}{2}\right)_k \left.\frac{\mathrm{d}B\left(k+\frac{1}{2}\epsilon, k+\frac{1}{2}\epsilon\right)}{\mathrm{d}\epsilon}\right|_{\epsilon=0}\right](\tau^2)^k\right\}.$$

Using the relations

$$\left.\frac{\mathrm{d}(C)^{a\epsilon}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = a \ln C,$$
$$\left.\frac{\mathrm{d}\Gamma(a\epsilon+b)}{\mathrm{d}\epsilon}\right|_{\epsilon=0} = a\Gamma(b)\Psi(b),$$
$$\left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left[(a\epsilon+b)_k\right]^{\pm 1}\right|_{\epsilon=0} = \pm a\left[(b)_k\right]^{\pm 1}\left[\Psi(b+k) - \Psi(b)\right],$$

as well as the special values, whenever appropriate,

$$\Psi(1) = -\gamma, \qquad \Psi(2) = -\gamma + 1, \qquad \Psi(\tfrac{1}{2}) = -\gamma - \ln 4,$$

$$\Psi(\tfrac{3}{2}) = 2 - \gamma - \ln 4, \qquad \Psi'(1) = \zeta(2),$$

$$\Psi'(2) = \zeta(2) - 1, \qquad \Psi'(\tfrac{3}{2}) = 3\zeta(2) - 4,$$

we obtain eq. (3.24).

Expanding similarly $f_2(\epsilon)$ and using the above formulas, as well as

$$\left.\frac{\mathrm{d}^2(C)^{a\epsilon}}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} = a^2 \ln^2 C,$$

$$\left.\frac{\mathrm{d}^2\Gamma(a\epsilon + b)}{\mathrm{d}\epsilon^2}\right|_{\epsilon=0} = a^2\Gamma(b)\left[\Psi^2(b) + \Psi'(b)\right],$$

we can derive the rest of the expansions.

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