Robustness margin need not be a continuous function of the problem data

B.R. Barmish  
Department of Electrical and Computer Engineering, University of Wisconsin-Madison, Madison, WI 53706, U.S.A.

P.P. Khargonekar  
Department of Electrical Engineering and Computer Science, University of Michigan, Ann Arbor, MI 48109, USA

Z.C. Shi  
Department of Electrical and Computer Engineering, University of Wisconsin-Madison, Madison, WI 53706, USA

R. Tempo  
CENS-CNR, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

Received 20 January 1990  
Revised 13 April 1990

Abstract. For systems with structured real perturbations, it is shown that the robustness margin for stability can be a discontinuous function of the problem data.

Keywords. Robustness, stability; uncertainty; ill-conditioning.

1. Introduction

Consider a linear control system with a transfer function or state space description parameterized in terms of a vector of uncertain parameters \( q \in \mathbb{R}^r \). A fundamental problem addressed in a large number of papers is: Determine the maximum uncertainty bound, call it \( r_{\text{max}} \), such that the system is stable for all \( q \in \mathbb{R}^r \) with \( \| q \| \leq r_{\text{max}} \). Note that the chosen norm for \( q \) is almost always \( \ell^2 \) or \( \ell^\infty \) and \( r_{\text{max}} \) is generally called the robustness margin; see Section 2 for a precise definition.

In many cases, a slightly different formulation of the problem above is considered; i.e., given an uncertainty bound \( r > 0 \), determine if the system is stable for all \( q \in \mathbb{R}^r \) with \( \| q \| \leq r \). In this case, only a 'yes' or 'no' answer is required. In the sequel, all analysis is carried out in the robustness margin framework but it should be noted that the consequences apply equally well to this alternative yes/no formulation; e.g., see the discussion associated with the example in Section 3.

The simple paradigms above are at the heart of many robust stability analysis techniques; e.g., see the literature ranging from real \( \mu \) as in Doyle and Packard to the post-Kharitonov literature (see Barmish and Jury for reviews of the continuous-time and discrete-time cases respectively), to polytope stability problems as in Bartlett, Hollot and Huang and to the theory dealing with frequency sweeping methods; e.g., see de Gaston and Safonov, Biernacki, Hwang and Bhattacharyya, Hinrichsen and Pritchard and Barmish.

Our main technical objective in this paper is to demonstrate that the robustness margin \( r_{\text{max}} \) is not necessarily continuous with respect to the problem data; the notion of problem data will be fully explained in the sequel. This discontinuity phenomenon is seen to be independent of the computational algorithm used to actually calculate \( r_{\text{max}} \). Matters are further complicated by the fact that at the point of discontinuity in the space of problem data, the robustness margin may be much smaller than at neighboring points. This may lead to potentially deceptive conclusions.

We feel that the most important implication of the present paper is that there is a serious issue pertaining to conditioning properties of the robustness margin. Despite the possibility that discontinuity of this margin may be nongeneric, in regions close to the discontinuity set, ill-conditioning of \( r_{\text{max}} \) must nevertheless be addressed. Therefore, our conclusion is that a thorough analysis of conditioning properties of the robust stability problem is an important area for future research.

The case which we make for discontinuity of the robustness margin is based on a simple example of a unity feedback system – the plant has
uncertain parameters entering linearly into numerator and denominator coefficients. We call this a linear uncertainty structure. Using \( d \) to represent the data describing the system, the robustness margin is written explicitly as \( r_{\text{max}}(d) \) and we prove the following: There exists a sequence of data \( (d(n))_{n=1}^{\infty} \) converging to some \( d^* \) such that

\[
\lim_{n \to \infty} r_{\text{max}}(d(n)) > r_{\text{max}}(d^*)
\]

That is, if one solves the sequence of robustness margins problems corresponding to \( d(n) \), the margins \( r_{\text{max}}(d(n)) \) may differ considerably from \( r_{\text{max}}(d^*) \). This happens even as the data \( d(n) \) gets arbitrarily close to \( d^* \).

For the simple case of linear uncertainty structures as in Section 3, it is felt that it should be possible to perform some sort of apron check for discontinuity. However, the fact that the discontinuity phenomenon occurs at the level of linear uncertainty structures serves as a ‘warning’ that care must be exercised when dealing with more complicated nonlinear problems. This is consistent with the example involving a nonlinear uncertainty structure in the paper by Ackermann, Hu and Kaesbauer [11] – severe computational problems arise as a certain data parameter is changed.

2. Notation and definition of the robustness margin

We consider polynomials with real coefficients \( a_i \) which depend continuously on a vector of uncertain parameters \( q \in \mathbb{R}^l \) whose \( i \)-th component is \( q_i \). To denote the dependence of \( a_i \) on \( q \), we write \( a_i(q) \). Hence, we take an uncertain polynomial to be of the form

\[
p(s, q) = s^m + \sum_{i=0}^{m-1} a_i(q) s^i.
\]

In Section 3, \( a_i(q) \) is affine linear and in Section 4, \( a_i(q) \) is multilinear. When \( q = 0 \), we obtain the so-called nominal polynomial \( p(s, 0) \), which is assumed to be strictly stable; i.e., its roots lie in the open left half plane.

A bounding set for the vector of uncertain parameters \( q \) will be a box parameterized by its radius \( r \); this box is denoted as \( Q_r \), and is described by

\[
Q_r = \{ q \in \mathbb{R}^l : \| q_i \| \leq r ; i = 1, 2, \ldots, l \}.
\]

Note that the discussion to follow can easily be adapted to handle the case when \( Q_r \) is a sphere; i.e., the discontinuity phenomenon is not particular to the \( L_{\infty} \) norm on uncertain parameters. In addition, discontinuities can occur when working with many other stability regions besides the open left half plane – the unit disk being a prime example.

Robustness margin In accordance with the discussion in Section 1, the robustness margin (for stability) is given by

\[
r_{\text{max}} = \sup \{ r : p(s, q) \text{ is strictly stable for all } q \in Q_r \}
\]

Dependence on problem data. In each of the examples to follow, the integers \( l = \dim q \) and \( m = \deg p(s, q) \) are held fixed and problem data consists of the coefficient functions \( a_0( \cdot ) , a_1( \cdot ) , \ldots , a_{m-1}( \cdot ) \) To illustrate the discontinuity phenomenon, we use a finite-dimensional space for this problem data. That is, each \( a_i(\cdot) \) is viewed as a mapping on data vectors \( d \in \mathbb{R}^p \) to continuous functions of \( q \). For example, a family of problems might be described by \( p = 6, l = 2, m = 2 \) and

\[
p(s, q) = s^2 + (d_1 + d_2 q_1 + d_3 q_2) s + (d_4 + d_5 q_1 + d_6 q_2)
\]

A specific robustness margin problem is obtained with \( d_1 = 2, d_2 = 1, d_3 = 4, d_4 = 3, d_5 = 6 \) and \( d_6 = 12 \). This leads to

\[
p(s, q) = s^2 + (2 + q_2 + 4 q_2) s + (3 + 6 q_1 + 12 q_2)
\]

Within this data space framework, two problems are deemed to be ‘close together’ if their associated data vectors (call them \( d^1 \) and \( d^2 \)) are close together in some arbitrary but fixed norm on \( \mathbb{R}^p \); i.e., \( \| d^1 - d^2 \| \) is small.

To denote dependence on \( d \), we henceforth write \( p_d(s, q) \) and \( r_{\text{max}}(d) \) in lieu of \( p(s, q) \) and \( r_{\text{max}} \) respectively. We are now prepared to present our main example.

3. Example establishing discontinuity of the robustness margin

Before formally proceeding, it is important to note that it is easy to construct relatively trivial
examples for which discontinuity of \( r_{\text{max}} \) can easily be demonstrated. Such examples involve cases when there is only one uncertain parameter, cases when the uncertainty structure is highly nonlinear, cases when the limiting polynomial \( p_{d^*}(s, q) \) is only marginally stable and cases when \( p_{d^*}(s, q) \) has lower degree or a smaller number of uncertainties than \( p_d(s, q) \). In contrast, the example below is simple yet nontrivial.

Indeed, consider a unity feedback system with open loop transfer function denoted by

\[
P_d(s, q) = K_d \frac{N_d(s, q)}{D_d(s, q)}
\]

where \( N_d(s, q) \) and \( D_d(s, q) \) are uncertain polynomials and \( K_d \) is the loop gain. The subscript 'd' is used to emphasize dependence on the data. In this example, \( l = 2 \), \( m = 4 \) and with \( d = d^* \), consider

\[
K_{d^*} = a,
\]

\[
N_{d^*}(s, q) = 4a + 10aq_1,
\]

and

\[
D_{d^*}(s, q) = s^4 + (20 - 20q_2)s^3
\]

\[
+ (44 + 2a + 10q_1 - 40q_2)s^2
\]

\[
+ (20 + 8a + 20aq_1 - 20q_2)s + a^2,
\]

where

\[
a = 3 + 2\sqrt{2}.
\]

Using our data notation, we write

\[
K_d = d_0,
\]

\[
N_d(s, q) = d_1 + d_2q_1,
\]

and

\[
D_d(s, q) = s^4 + (d_3 + d_4q_2)s^3
\]

\[
+ (d_5 + d_6q_1 + d_7q_2)s^2
\]

\[
+ (d_8 + d_9q_1 + d_{10}q_2)s + d_{11}.
\]

By comparing the expressions for \( K_{d^*} \), \( N_{d^*}(s, q) \) and \( D_{d^*}(s, q) \) with \( K_d \), \( N_d(s, q) \) and \( D_d(s, q) \), respectively, it is clear that the \( d_i^* \) are readily available, e.g., \( d_0^* = a \), \( d_1^* = 4a \), \( d_2^* = 10a \), \( d_3^* = 20 \), etc.

Now, we consider the data sequence \( \langle d(n) \rangle_{n=1}^{\infty} \) described by

\[
d_i(n) = \begin{cases} 
  d_i^* & \text{for } i \neq 0, \\
  a_n & \text{for } i = 0,
\end{cases}
\]

where

\[
a_n = a - 1/n.
\]

This sequence corresponds to the case where the plant data is fixed and the gain \( a_n \) is converging to \( a \).

**Robustness margin.** In order to obtain the robustness margin along the \( d(n) \) sequence for the feedback system above, we study the closed loop polynomial

\[
p_{d(n)}(s, q) = K_{d(n)}N_{d(n)}(s, q) + D_{d(n)}(s, q)
\]

\[
= s^4 + (20 - 20q_2)s^3
\]

\[
+ (44 + 2a + 10q_1 - 40q_2)s^2
\]

\[
+ (20 + 8a + 20aq_1 - 20q_2)s + a^2,
\]

and for the limiting case, we study the closed loop polynomial

\[
p_{d^*}(s, q) = aN_{d^*}(s, q) + D_{d^*}(s, q)
\]

\[
= s^4 + (20 - 20q_2)s^3
\]

\[
+ (44 + 2a + 10q_1 - 40q_2)s^2
\]

\[
+ (20 + 8a + 20aq_1 - 20q_2)s
\]

\[
+ (5a^2 + 10a^2q_1).
\]

**Discontinuity claim** (see next subsection for proof). We claim that

\[
0.417 = \lim_{n \to \infty} r_{\text{max}}(d(n)) > r_{\text{max}}(d^*) = 0.234.
\]

That is, we claim that the robustness margin is **discontinuous** at the data point \( d^* \). In fact, for this example, along the data sequence \( \langle d(n) \rangle_{n=1}^{\infty} \), the robustness margin is given by

\[
\lim_{n \to \infty} r_{\text{max}}(d(n)) = 1 - \frac{1}{10} a = 0.417
\]
However, precisely at \( d^* \), the robustness margin becomes

\[
\rho_{\text{max}}(d^*) = \frac{7 - a}{5} = 0.234.
\]

This example illustrates the ‘false sense of security’ associated with the robustness margin. To further elaborate, if \( q_1^* = q_2^* = 0.234 \), two of the roots of the closed loop polynomial \( p_{d(n)}(s, q^*) \) approach the imaginary axis as \( n \to \infty \). That is, \( p_{d(n)}(s, q^*) \) is ‘nearly’ destabilized by an uncertainty vector \( q^* \) whose norm is 0.234 despite the fact that the predicted margin is approximately 0.417.

**Proof of claim.** Along the data sequence, we examine the closed loop polynomial \( p_{d(n)}(s, q) \) given by (1). Then, to obtain the robustness margin, we use the fact that the leading minors of the Hurwitz testing matrix must be positive. This leads to the following four conditions:

**Condition 1:**

\[
(20 - 20q_2) > 0
\]

**Condition 2:**

\[
(20 - 20q_2)(44 + 2a + 10q_1 - 40q_2) - (20 + 8a + 20aq_1 - 20q_2) > 0,
\]

**Condition 3:**

\[
(20 - 20q_2)(44 + 2a + 10q_1 - 40q_2)
\]

\[
(20 + 8a + 20aq_1 - 20q_2)
\]

\[
- (20 - 20q_2)\left[5a^2 - \frac{4}{n}a + 10a\left(a - \frac{1}{n}\right)q_1\right]
\]

\[
- (20 + 8a + 20aq_1 - 20q_2)^2 > 0.
\]

**Condition 4:**

\[
5a^2 - \frac{4}{n}a + 10a\left(a - \frac{1}{n}\right)q_1 > 0.
\]

Note that \( r_{\text{max}}(d(n)) \) is the supremal value of \( r \) such that the four inequalities above hold for all \( q \in Q_r \). That is, letting

\[
r_r(d(n)) = \sup\{ r : \text{Condition } i \text{ holds for all } q \in Q_r \},
\]

it follows that

\[
r_{\text{max}}(d(n)) = \min_{i=1, \ldots, 4} \{ r_i(d(n)) \}.
\]

The remainder of the proof will proceed via a number of steps.

**Step 1:** We claim that

\[
\lim_{n \to \infty} r_3(d(n)) = 1 - \frac{a}{10} \approx 0.417
\]

To prove this claim, it is first verified that Condition 3 is equivalent to

\[
2\left[4\sqrt{a} - 20 + 10\sqrt{a}q_1 + 20q_2\right]^2
\]

\[
\cdot \left[(20 - 2a - 20q_2)^2 + \frac{1}{n}a(4 + 10q_1)(20 - 20q_2)^2\right] > 0.
\]

Now, to obtain the quantity \( \lim_{n \to \infty} r_3(d(n)) \), we rewrite the inequality above as

\[
2F_1(q)F_2(q) + G_n(q) > 0
\]

where

\[
F_1(q) = \left[4\sqrt{a} - 20 + 10\sqrt{a}q_1 + 20q_2\right]^2.
\]

\[
F_2(q) = (20 - 2a - 20q_2),
\]

and

\[
G_n(q) = \frac{1}{n}a(4 + 10q_1)(20 - 20q_2)^2.
\]

To establish the desired limit for \( r_3(d(n)) \), we observe that if \( q_2 = 1 - \frac{1}{10}a \) and \( q_1 = -0.4 \), then

\[
2F_1(q)F_2(q) + G_n(q) = 0.
\]

That is, for all \( n \),

\[
r_3(d(n)) \leq 1 - \frac{1}{10}a.
\]

Furthermore, it is also easy to verify that for arbitrarily small \( \varepsilon > 0 \), there exists an integer \( N_\varepsilon \) having the following property: For any fixed \( n > N_\varepsilon \) and any uncertainty \( q \in Q_1-a/10-\varepsilon \),

\[
2F_1(q)F_2(q) + G_n(q) > 0.
\]

Hence, for \( n > N_\varepsilon \),

\[
r_3(d(n)) \geq 1 - \frac{1}{10}a - \varepsilon
\]

From the two inequalities involving \( r_3(d(n)) \) above, we conclude that

\[
\lim_{n \to \infty} r_3(d(n)) = 1 - \frac{1}{10}a.
\]

Hence, the claim is established
Step 2: We claim that for $n$ sufficiently large,

$$r_i(d(n)) > r_3(d(n))$$

(3)

for $i = 1, 2, 4$. Indeed from Condition 1, it is trivial to see that

$$r_1(d(n)) = 1.$$

To verify (3) for $r_2(d(n))$, we view the left hand side of Condition 2 as a function of $(q_1, q_2)$. It suffices to show that this function is positive on a box of radius $r = 1 - \frac{1}{10}a$. To this end, notice that for arbitrary $|q_1| < r < 0.5$ and $q_2 < r < 0.5$, it is easy to verify that we have crude bounds

$$(20 - 20q_2)(44 + 2a + 10q_1 - 40q_2) > 300$$

and

$$20 + 8a + 20aq_1 - 20q_2 < 150.$$

Therefore, the left hand side in Condition 2 remains positive as required.

Finally, setting the left hand side of Condition 4 to zero, it is straightforward to obtain the formula

$$r_4(d(n)) = \frac{5na - 4}{10(na - 1)}.$$

Hence, it is easy to see that

$$\lim_{n \to \infty} r_4(d(n)) = \frac{1}{2},$$

which implies that $r_4(d(n)) > r_3(d(n))$ for $n$ sufficiently large.

Step 3: We claim that

$$\lim_{n \to \infty} r_{\max}(d(n)) = 1 - \frac{1}{10}a \approx 0.417.$$

This claim follows easily from Steps 1 and 2. That is, we have

$$\lim_{n \to \infty} r_{\max}(d(n)) = \lim_{n \to \infty} r_3(d(n)) = 1 - \frac{1}{10}a \approx 0.417.$$

Step 4: We claim that

$$r_{\max}(d^*) = \frac{7-a}{5} \approx 0.234.$$

Indeed, as in the $d(n)$ analysis, we use the formula

$$r_{\max}(d^*) = \min_{i=1, \ldots, 4} \{r_i(d^*)\}$$

(4)

where $r_i(d^*)$ is obtained from the $i$-th Hurwitz inequality at $d^*$. Analogous to Steps 1–3, we first analyze Condition 3 with $n \to \infty$. By a straightforward computation, it is easy to verify that Condition 3 is equivalent to

$$2(4\sqrt{a} - 20 + 10\sqrt{q_1 + 20q_2})^2 \cdot (20 - 2a - 20q_2) > 0.$$

Now, we examine each factor separately and obtain the margin

$$r_3(d^*) = \frac{7-a}{5} \approx 0.234.$$

Next, reasoning exactly as in Steps 2–3, it is easy to verify that

$$r_1(d^*) = 1 > r_3(d^*),$$

$$r_2(d^*) > r_3(d^*),$$

$$r_4(d^*) = \frac{1}{2} > r_3(d^*).$$

Hence, from (4), we obtain

$$r_{\max}(d^*) = r_3(d^*) = \frac{7-a}{5} \approx 0.234.$$

The proof of the claim is now complete.

Yes–No Problem. The discontinuity claim above can also be interpreted in terms of the yes/no problem formulation discussed in the Introduction. To illustrate, consider a robust stability problem with given uncertainty bound $r = 0.3$. Now, the following problem arises: When using $d^*$, the answer to the robust stability question is “no” but taking $d(n)$ the answer is “yes”.

Remarks. In practice, the robustness margin can be computed via a number of methods. For example, instead of using a Hurwitz matrix as in the proof of the claim above, one can use the well-known frequency sweep method. That is, letting

$$r_{\max}(d, \omega) = \sup \{r: p_d(j\omega, q) \neq 0 \text{ for all } q \in Q, \},$$

it then follows that

$$r_{\max}(d) = \inf_{\omega} r_{\max}(d, \omega).$$
In our specific example, solution by frequency sweep method for \( n \) finite leads to

\[
\begin{align*}
\max \left\{ C_n(\omega), D_n(\omega) \right\} & \quad \text{if } \omega \neq 0, \\
\frac{5na - 4}{10(na - 1)} & \quad \text{if } \omega = 0,
\end{align*}
\]

where

\[
\begin{align*}
C_n(\omega) &= \left( \frac{\omega^2 - a}{10n} \right)^2 - 0.4, \\
D_n(\omega) &= \left( \frac{\omega^2 - a}{10n} \right)^2 - 1,
\end{align*}
\]

and

\[\Delta_n(\omega) = \omega^4 - a \left( 2 - \frac{1}{n} \right) \omega^2 + a \left( a - \frac{1}{n} \right).\]

In the limiting case, we obtain

\[
\max \left\{ \frac{\omega^2 - 5}{10}, \left| 1 - \frac{a}{10} \right| \right\}
\]

Similarly, we take

\[
g_d(\omega) = p_d(\omega) f(\omega)
\]

where \( p_d(\omega) \) is given by (2). Note that the uncertainty structures above are multilinear and the first factors in \( g_d(n)(\omega) \) and \( g_d^*(s, q) \) are the same polynomials which were used in Section 3. Moreover, since the robustness margin of \( f(s, q) \) is unity, it follows that the margins for \( g_d(n)(\omega) \) and \( g_d^*(s, q) \) are exactly the same as those found for \( p_d(n)(\omega) \) and \( p_d^*(s, q) \) in Section 3. Notice that if \( g_d(n)(\omega) \) and \( g_d^*(s, q) \) are given in expanded form rather than factored as above, the detection of the discontinuity becomes much more difficult. That is, if one has a theory to flag discontinuity at the level of affine linear uncertainties, then one is faced with a complicated factorization problem. As seen below, matters can be even worse because it is easy to construct examples for which a factorization does not exist.

**Modification which does not permit factorization.**

Take \( f(s, q) \), \( g_d(n)(\omega) \), \( g_d^*(s, q) \), \( p_d(n)(\omega) \), \( p_d^*(s, q) \), \( d(n) \) and \( d^* \) as above and let \( \langle b_n \rangle \) be any sequence of positive real numbers converging to zero. Now, using the polynomials \( g_d(n)(\omega) + b_n \) and \( g_d^*(s, q) \), it can be shown (see Appendix) that the same discontinuity phenomenon occurs along appropriately constructed subsequences \( \langle d(n) \rangle \) of \( \langle d(n) \rangle \) and \( \langle b_n \rangle \). Moreover, by examination of the coefficients of \( s^0 \), it is easy to see that a factorization of the modified polynomials into a product of polynomials each having affine uncertainty structure is impossible. That is, it is straightforward to verify that the quantity

\[
\text{coef}(s^0) = a \left( 5a - \frac{4}{n} + 10 \left( a - \frac{1}{n} \right) q_1 \right) (1 + q_3)
\]

cannot be factored as a product of two affine linear functions of \( q \).

**5. Conclusion**

In view of the arguments and examples given in this paper, it is felt that investigation into conditioning properties of the robust stability problem is an important area for future research.
Appendix

We consider the same setup as in Section 3; i.e., \( p_d(s, q) \) is a monic polynomial with robustness margin \( r_{\text{max}}(d) \) and \( \langle d(n) \rangle_{n=1}^{\infty} \) is a sequence converging to \( d^* \). Given any \( \epsilon > 0 \), define

\[
\tilde{p}_d(s, q, \epsilon) = p_d(s, q) + \epsilon
\]

whose robustness margin is denoted by \( \tilde{r}_{\text{max}}(d, \epsilon) \).

Now, we establish a basic lemma.

**Lemma.** Suppose that for each \( n \),

\[
r_{\text{max}}(d(n)) > \beta.
\]

Then there exists an integer \( N \) and a sequence \( \langle \epsilon_n \rangle_{n=1}^{\infty} \) of positive numbers converging to zero such that

\[
\tilde{r}_{\text{max}}(d(n), \epsilon_n) > \beta
\]

for all \( n \geq N \).

**Proof.** Since \( d(n) \) converges to \( d^* \), pick \( N \) such that

\[
|d(n) - d^*| \leq 1
\]

for all \( n \geq N \). Now, letting \( n \geq N \) be fixed, the proof continues with a sequence of claims.

Claim 1. There exists an \( \omega_0 > 0 \) having the following property: For arbitrary \( |\omega| > \omega_0 \), \( q \in Q_\beta \) and \( \epsilon \in [0, 1] \),

\[
|\tilde{p}_{d(n)}(j\omega, q, \epsilon)| \neq 0.
\]

This claim is easily established after noting that for \( \omega \) sufficiently large,

\[
\omega^m > \max \{|\tilde{p}_{d(n)}(j\omega, q) - (j\omega)^m| : q \in Q_\beta; \epsilon \in [0, 1]; |d - d^*| \leq 1\}.
\]

Claim 2. With \( \omega_0 \) as in Claim 1 and \( \epsilon \in [0, 1] \), let

\[
F_n(\epsilon) = \min \{|\tilde{p}_{d(n)}(j\omega, q, \epsilon)| : q \in Q_\beta; |\omega| \leq \omega_0\}.
\]

Then

\[
F_n(0) > 0.
\]

This claim is established by contradiction. Indeed, if \( F_n(0) = 0 \), it follows that \( |\tilde{p}_{d(n)}(j\omega^*, q^*, 0)| = 0 \) for some \( |\omega^*| \leq \omega_0 \) and \( q^* \in Q_\beta \). Hence,

\[
r_{\text{max}}(d(n)) = \tilde{r}_{\text{max}}(d(n), 0) \leq \beta,
\]

which is a contradiction.

**Claim 3.** \( F_n(\epsilon) \) is continuous with respect to \( \epsilon \in [0, 1] \). This claim follows from continuous dependence of \( \tilde{p}_{d(n)}(j\omega, q, \epsilon) \) on \( (\omega, q, \epsilon) \) and compactness of \( [-\omega_0, \omega_0] \) and \( Q_\beta \). That is, \( F_n(\epsilon) \) is an infimal value function; e.g., see Berge [12].

**Claim 4.** There exists some \( \epsilon^*_n > 0 \) such that

\[
F_n(\epsilon) > 0
\]

for all \( \epsilon \in [0, \epsilon^*_n] \). This claim is immediate from the fact that \( F_n(0) > 0 \) and \( F_n(\epsilon) \) is continuous.

To complete the proof of the lemma, take

\[
\epsilon_n = \min \left\{ \frac{1}{n}, \epsilon^*_n \right\}
\]

Now, by construction, it follows that the sequence \( \langle \epsilon_n \rangle_{n=1}^{\infty} \) converges to zero and

\[
F_n(\epsilon_n) > 0
\]

for all \( n \geq N \). Consequently,

\[
\tilde{p}_{d(n)}(j\omega, q, \epsilon_n) \neq 0
\]

for all \( \omega \in \mathbb{R} \) and \( q \in Q_\beta \). This implies that

\[
\tilde{r}_{\text{max}}(d(n), \epsilon_n) > \beta
\]

for all \( n \geq N \). The proof of the lemma is now complete.

Acknowledgements

This work was performed while B.R. Barmish was a Senior NATO Guest Fellow at CENS-CNR, Torino (Italy). The support of Professor A.R. Meo is gratefully acknowledged. Partial support of this project was also provided by the U.S. National Science Foundation under Grants ECS-8612948 and ECS-9096109, by the U.S. Air Force Office of Scientific Research under Grant AFOSR-88-0020 and by Honeywell and Boeing.

The authors are especially grateful to Dr. F. Kraus of ETH, Zurich. In his detailed comments on an earlier draft of this paper which appeared in the Proceedings of the 1989 CDC (see Barmish, Khargonekar, Shi and Tempo [13]), he exposed an error motivating this corrected version of the work.

References


