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INITIAL POSTBUCKLING BEHAVIOR OF BEAMS ON NON-LINEAR ELASTIC FOUNDATIONS

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Introduction

Several researchers have studied the problem of buckling of beams attached to elastic foundations. Many of these studies exploited the simplicity of the beam-elastic foundation model in order to characterize the different types of postbuckling behavior as well as to demonstrate the variety of techniques that are available to study similar problems. As a result several simplifications were incorporated into the equations that bring out a particular type of response. For example in [1], non linear terms entering into the equations as derivatives of the lateral displacement $W(x)$, were dropped from the equation while still retaining terms of the same order in $W(x)$ in the restoring force of the foundation. Several other studies dealt with the problem in the context of civil engineering applications. A recent paper [2], which includes an extensive bibliography, addresses this problem within the framework of the improved Koiter[3] postbuckling theory. This same author addresses the issue of mode interaction in [4], and discusses its importance under certain circumstances. Basically, the approach adopted in [2,3 and 4] amounts to developing asymptotic expressions for the potential energy in the neighborhood of the critical load in terms of a small parameter, which is identified as the amplitude of the buckling mode. Koiters original work was extended in [5] where a formalism based on the principle of virtual work was used. The approach adopted in the present paper is to use asymptotic expansion considerations to differential equations obtained via the principle of stationary potential energy.

In the present study the linear part of the foundation modulus is taken to be dependent linearly on the axial compression on the beam. The beam is thought of as modeling a single fiber of a unidirectional one fiber composite, while the foundation represents the supporting matrix. Thus, the assumed dependency of the foundation modulus on the axial compression on the beam is based on the premise that the lateral support offered by the foundation is a function of the overall compression on the composite.

Problem Formulation

Perfect beams

We consider a rectangular, uniform beam of length L, unit breadth , and thickness h, subjected to an axial compressive load P applied at the ends. We assume that strains are small compared to unity and that planes normal to the centroidal line in the undeformed state remain normal to the deformed centroidal line. Further, we assume that the centroidal line of the beam remains inextensible. We choose a Lagrangian description with a fixed right handed rectangular Cartesian frame of reference to be used, with the X axis coinciding with the centroidal line of the beam in the initial unloaded state and the Z axis normal to it. Let the components of the displacement of a particle positioned initially at (X, Z) be denoted by U, W. In the Lagrangian description, the Greens strain tensor, referred to the initial configuration is used, whose components are

$$\begin{aligned} E_X &= \frac{dU}{dX} + \frac{1}{2}[(\frac{dU}{dX})^2 + (\frac{dW}{dX})^2] \\ E_Z &= \frac{dW}{dZ} + \frac{1}{2}[(\frac{dW}{dZ})^2 + (\frac{dU}{dX})^2] \\ E_{XZ} &= \frac{1}{2}[\frac{dU}{dZ} + \frac{dW}{dX} + \frac{dU}{dX}\frac{dU}{dZ} + \frac{dW}{dX}\frac{dW}{dZ}]. \end{aligned} \tag{1}$$

Consistent with the assumptions stated, we have

$$\begin{aligned} U(X,Z) &= U_0(X) - Z\frac{dW_0}{dX} \\ W(X,Z) &= W_0(X) + Z\frac{dU_0}{dX}. \end{aligned} \tag{2}$$

Here, a subscript '0' is used to denote quantities associated with the centroidal line. The condition of inextensibility of the centroidal line renders a relation between $U_0(X)$ and $W_0(X)$ in the form

$$2E_{X_0} = (\frac{dU_0}{dX} + 1)^2 + (\frac{dW}{dX})^2 - 1 = 0. \tag{3}$$

Substituting (2) in (1) and using (3), we find that the only non zero strain component is E_X. Thus, the displacement field (2), along with (3), simplifies the non-linear strain components of the Greens strain tensor to the extent that the shearing strain E_{XZ} and the transverse normal strain E_Z are both zero. Corresponding to the Greens strain tensor, we have Kirchhof's stress tensor S_{ij}, which will be used in the analysis. Assuming a state of plane stress in the XZ plane and with the assumption on the smallness of E_X, we invoke Hooke's law in the form

$$S_X = \mathrm{E}_f E_X, \tag{4}$$

where, E_f is the Youngs modulus of the beam material. Next, we write the potential energy (Π) of the beam,

$$\begin{aligned} \Pi =& \frac{1}{2}\int_0^L \int_{-\frac{h}{2}}^{\frac{h}{2}} S_X E_X dZ dX + P\int_0^L \frac{dU_0}{dX} dX \\ &+ \int_0^L (\frac{1}{2}K_1 {W_0}^2 + \frac{1}{4}K_3 {W_0}^4) dX. \end{aligned} \tag{5}$$

In the above expression, K_1 and K_3 are the linear and cubic spring constants of the elastic foundation. Π can be expressed in terms of, $W_0(X)$, the lateral displacement of the centroidal line, and its derivatives by using (4), (1) and (3) in (5). Then, carrying out the integration in Z and defining

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} Z^2 dZ = I, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{Z^4}{4} dZ = J, \tag{6}$$

we obtain,

$$\begin{aligned}\Pi =& \frac{E_f I}{2}\int_0^L W_0''^2(1+W_0'^2)dX + \frac{E_f J}{2}\int_0^L W_0''^4 dX - \frac{P}{2}\int_0^L (W_0'^2 + \frac{1}{4}W_0'^4)dX \\ &+ \int_0^L (\frac{1}{2}K_1 W_0^2 + \frac{1}{4}K_3 W_0^4)dX,\end{aligned} \tag{7}$$

where, we have retained all the quartic terms entering in (5). A prime indicates differentiation with respect to X (or x in the normalized expressions to appear later). The second term in Π is the result of retaining quadratic terms (in Z) in the expression for E_X. It is seen that this term contributes a quartic (in W_0) term to Π. However, it will be shown later that this term can be omitted on account of its negligible contribution to Π. Noting that $K_1 = K_0 + \hat{\alpha}P$, the following non-dimensional quantities are introduced next.

$$\begin{aligned}&w = \frac{W}{\lambda},\ x = \frac{X}{\lambda},\ l = \frac{L}{\lambda},\ \sigma = \frac{P\lambda^2}{4EI}, \\ &k_3 = \frac{K_3\lambda^6}{EI},\ \alpha = \hat{\alpha}\lambda^2,\ \beta = \frac{J}{I\lambda^2}, \\ &\lambda = (\frac{4EI}{K_0})^{\frac{1}{4}},\ \Pi^* = \frac{2\Pi\lambda}{EI}.\end{aligned} \tag{8}$$

Then,

$$\begin{aligned}\Pi^* =& \int_0^l w''^2(1+w'^2)dx + \beta\int_0^l w''^4 dx - \sigma\int_0^l (4w'^2 + w'^4)dx \\ &+ \int_0^l (4w^2 + 4\alpha\sigma w^2 + \frac{1}{2}k_3 w^4)dx.\end{aligned} \tag{9}$$

At equilibrium, the variation of Π^* with respect to the virtual displacement δw must vanish. Thus,

$$\begin{aligned}\delta\Pi^* =& \int_0^l \left[2w''\frac{d^2(\delta w)}{dx^2} + 2w''^2 w'\frac{d(\delta w)}{dx} + 2w''w'^2\frac{d^2(\delta w)}{dx^2}\right]dx \\ &+ \beta\int_0^l 4w''^3\frac{d^2(\delta w)}{dx^2} - \sigma\int_0^l \left[8w'\frac{d(\delta w)}{dx} + 4w'^3\frac{d(\delta w)}{dx}\right]dx \\ &+ \int_0^l \left[8w\delta w + 8\alpha\sigma w\delta w + 2k_3 w^3\delta w\right]dx = 0.\end{aligned} \tag{10}$$

Integrating by parts and collecting terms, we obtain,

$$\delta\Pi^* = \int_0^l L^N(w)\delta w dx + \text{boundary terms} = 0. \tag{11}$$

Since δw is arbitrary in the interval $(0,l)$, we obtain the ordinary differential equation governing beam equilibrium,

$$L^N(w) = 0, \tag{12}$$

where,

$$\begin{aligned} L^N(w) =& L(w) + w'^2 w'''' + w''^3 + 4w'w''w''' \\ &+ 6\sigma w'^2 w'' + k_3 w^3 + \beta(6w''^2 w'''' + 12w''w'''^3). \end{aligned}$$

Here,

$$L(\ ) \equiv (\ )'''' + 4\sigma(\ )'' + 4(1+\alpha\sigma)(\ ).$$

Clearly, $w(x) = 0$ is a solution of (12) for all values of σ. The value of P corresponding to the smallest value of σ(designated σ_0) for which non-trivial solutions $w(x) \neq 0$, of arbitrarily small amplitude, exist, is identified as the buckling load. Let us perturb (12) about the state $(\sigma = \sigma_0, w(x) = 0)$. Thus, let

$$\begin{aligned} \sigma &= \sum_{n=0}^{\infty} \epsilon^n \sigma_n \\ w(x) &= \sum_{n=1}^{\infty} \epsilon^n w_n(x) \end{aligned} \tag{13}$$

Here, ϵw_1 is the buckling mode with shape $w_1(x)$ and small amplitude ϵ. Substituting (13) into (12) and grouping together terms according to ascending powers of ϵ, we obtain,

$$\begin{aligned} \epsilon^1 & \ldots\ldots\ldots\ldots L(w_1) = 0 \\ \epsilon^2 & \ldots\ldots\ldots\ldots L(w_2) = -4\sigma_1 w_1'' - 4w_1\alpha\sigma_1 \\ \epsilon^3 & \ldots\ldots\ldots\ldots L(w_3) = -w_1'^2 w_1'''' - 4\sigma_1 w_2'' - 4\sigma_2 w_1'' - 4w_1\alpha\sigma_2 \\ & \qquad - 4w_2\alpha\sigma_1 - w_1''^3 - 4w_1' w_1'' w_1''' - k_3 w_1^3 \\ & \qquad - 6\sigma_0 w_1'^2 w_1'' - \beta(6w_1''^2 w_1'''' + 12w_1'' w_1'''^2 \\ & \ldots\ldots\ldots etc. \end{aligned} \tag{14}$$

The first of these equations correspond to classical buckling. Let us assume a solution of the form $w_1 = \sin(\mu x)$, where we presume the beam to be of sufficient length so that the boundary conditions at the two ends of the beam are not accounted for. Then, the classical buckling load σ_0 and the corresponding wave number are found to be,

$$\sigma_0 = \frac{\mu^2}{2}, \qquad \mu^2 = \sqrt[2]{4+\alpha^2} + \alpha. \tag{15}$$

This reduces to $\sigma_0 = 1, \mu^2 = 2$ for $\alpha = 0$. The rest of the sequence of equations, (14), can be solved by appealing to the boundedness of $w(x)$ and suppressing secular terms. In so doing we obtain successively (σ_1, w_2), (σ_2, w_3),...etc. Thus, we have,

$$\begin{aligned} w &= \epsilon \sin\left(\sqrt[2]{\sqrt[2]{4+\alpha^2}+\alpha x}\right) + \epsilon^3 w_3(x) + \ldots.. \\ \sigma &= \frac{1}{2}\left(\sqrt[2]{4+\alpha^2}+\alpha\right) + \epsilon^2 \sigma_2 + \ldots\ldots., \end{aligned} \tag{16}$$

where,

$$w_2 = 0, \sigma_1 = 0 \quad \text{and} \quad \sigma_2 = \frac{\frac{3}{4}k_3 - \frac{1}{4}\mu^6 + \frac{3}{2}\beta\mu^8}{4(\sqrt[2]{4+\alpha^2})}.$$

Imperfect beams

When the beam contains an initial imperfection $\hat{w}(x)$, then the expression for the potential energy needs to be modified. In the present analysis it is assumed that the imperfection is 'small' in the sense that, terms of higher powers of $\hat{w}(x)$ or its derivatives entering in the expression for the potential energy will be omitted, but terms with $\hat{w}(x)$ (or its derivatives) by itself or as products with $w(x)$ (and/or its derivatives) will be retained. If we denote by Π_I^*, the potential energy corresponding to the case with an initial imperfection, then,

$$\begin{aligned} \Pi_I^* = \Pi^* + \int_0^l 2w''^2 w'\hat{w}' + w''w'^2\hat{w}'' \\ \sigma \int_0^l (2w'\hat{w}' + w'^3\hat{w}')dx. \end{aligned} \tag{17}$$

In the above we note that the expression for Π^* is given by (9) and in (17), $w(x)$ is understood to be the *additional displacement* measured from the initial configuration containing the imperfection. As before, by setting the first variation of $\Pi^*{}_I$ to zero, we obtain the ordinary differential equation governing beam equilibrium;

$$L_I{}^N(w) = 0, \tag{18}$$

where,

$$L_I{}^N(w) = L^N(w) + f_1(w)\hat{w}'''' + f_2(w)\hat{w}''' + f_3(w)\hat{w}'' + f_4(w)\hat{w}',$$

and,

$$\begin{aligned} f_1(w) &= \frac{1}{2}w'^2 \\ f_2(w) &= 3w'w'' + w' \\ f_3(w) &= 3w'^2 + 4w'w''' + w'' + \sigma(4 + 6w'^2) \\ f_4(w) &= 2(w''''w' + 2w''w''' + 6\sigma w'w''). \end{aligned}$$

Let the initial imperfection have non-dimensional amplitude τ (normalized by λ) and shape $v(x)$. Thus, $\hat{w}(x) = \tau v(x)$, with, $0 \le \tau \ll 1$. Unlike before, (18) is a non-homogeneous equation in w, due to the terms containing the initial imperfection. Thus, $w(x) = 0$ is not a solution of (18) for any value of σ. Since we already have some information regarding the behavior of the perfect system, it suffices for us to characterize the imperfect system with respect to the perfect system in terms of the imperfection amplitude τ and shape $v(x)$. Thus, let,

$$\hat{\sigma} = \frac{\sigma - \sigma_0}{\phi(\tau)} \quad \text{and} \quad w(x) = \psi_1(\tau)w_1(x) + \psi_2(\tau)w_2(x) + ...$$

Then, upon substituting for σ and $w(x)$ in (18) and considering the dominant terms, we find that,

$$\psi_1(\tau) = \tau^{\frac{1}{3}}, \psi_2(\tau) = \tau, \phi(\tau) = \tau^{\frac{2}{3}}$$

Next, we group together terms in (18), according to ascending powers of τ, to find,

$$\begin{aligned} \tau^{\frac{1}{3}}..........&L(w_1) = 0 \\ \tau..........&L(w_2) = -w_1'^2 w_1'''' - 4\hat{\sigma}w_1'' - w_1''^3 - 4w_1' w_1'' w_1''' \\ &\qquad - 4\alpha\hat{\sigma}w_1 - k_3 w_1^3 - 6w_1'^2 w_1'' - 4v''\hat{\sigma}. \end{aligned} \tag{20}$$

The first of these equations produces the results we already know; the classical buckling load and the associated mode number given by (15) with $w_1(x) = A\sin(\mu x)$, where A is as yet an arbitrary amplitude. Assuming the initial imperfection to have the same shape as the classical buckling mode [$v(x) = \sin(\mu x)$], and imposing the boundedness of $w(x)$ via suppression of the secular terms arising in the second equation of the sequence (20), we obtain,

$$\hat{\sigma} = \frac{-4\mu^2\sigma_0 - A^3 C_1}{4A(\mu^2 - \alpha)},$$

where,

$$C_1 = -\frac{1}{2}\mu^6 + \frac{3}{2}\mu^4 - \frac{3}{4}k_3.$$

Redefining $\epsilon = \tau^{\frac{1}{3}}A$ and noting the values of σ_0 and μ in (15), we arrive at a relation that characterizes the response of the imperfect system, in terms of the imperfection amplitude τ, near the buckling load;

$$\sigma\epsilon = \sigma_0\epsilon - \frac{\mu^4\tau}{2\sqrt[3]{4+\alpha^2}} - \frac{C_1\epsilon^3}{4\sqrt[3]{4+\alpha^2}}. \tag{21}$$

Discussion of Results

Figure 1 shows a graph of the dependence of σ_0 on the softening (hardening) parameter α of the foundation (corresponding to equation (15) in the text). Notice that as $\alpha \to 0, \mu^2 \to 2$ and $\sigma_0 \to 1$ and thus, we recover the result for classical buckling of a beam on a linear elastic foundation ([2],[3],[6]). As $\alpha \to \infty, \sigma_0 \to \infty$ as expected. As $\alpha \to -\infty, \mu^2 \to 0$ and $\sigma_0 \to 0$, and this shows that increasing amounts of softening tends to have a highly undesirable destabilising effect on the unbuckled beam. In the context of fiber composites, these results show quantitatively how fiber buckling loads are affected due to the softening behavior of the matrix material.

In figure 2, we have presented results corresponding to equation (16) for a fixed value of k_3, which characterizes the response of a perfect beam on a non-linear elastic foundation. These σ vs ϵ plots show the response of the beam in the initial stages of post buckling. Notice the negative slope of the $\sigma - \epsilon$ curve immediately proceeding buckling, indicative of the unstable post buckling behavior. In the case when k_3=0 and $\alpha = 0$, and noting that $\mu = \sqrt[4]{2}$, we obtain σ_2=-0.25+3β. For the present study $\beta = 0.04\left(\frac{h}{\lambda}\right)^2$. Since the buckle mode shape is $\sin(\mu x)$, we can express β in terms of the buckle wave length l_w as, $\beta = 0.04\left(\frac{4\pi^2}{\mu^2}\right)\left(\frac{h}{l_w}\right)^2$. Thus, $\sigma_2 = -0.25 + 0.78\left(\frac{h}{l_w}\right)^2$. Therefore, the influence of the term containing β on the slope of the $\sigma - \epsilon$ plot is negligible. In figure 3, we have presented results showing the variation of σ_2 as a function of the parameter α. The corresponding expression in the text is given in (16) (note that σ_2 controls the slope of the $\sigma - \epsilon$ plots). From this figure, it is seen that when $k_3 = 0$, σ_2 becomes more negative with increasing α, while, for $k_3 = -100$, σ_2 is a maximum at $\alpha = 0.21$. Inferring from figure 2, the effects of α on the buckling load, we note here that the buckling load is independent of the value of k_3, with k_3 influencing only the postbuckling response.

The influence of α on the postbuckling paths for imperfect beams is displayed in figure 4. From this figure, it is seen that with decreasing α, the load maximum is seen to occur at decreasing values of ϵ. Further, the $\sigma - \epsilon$ plots tend to 'flatten' out for decreasing values of α, indicating a gradual drop in the load, beyond the maximum load point. Such a gradual change is very desirable from a structural stand point. The shallowness of the $\sigma - \epsilon$ plot gives an indication of the severity of the imperfection sensitivity of the system. Thus, there is a compromise to be achieved in increasing the buckling load (increasing α), and reducing the severity of the imperfection sensitivity (decreasing α). The influence of k_3 on the postbuckling paths can be inferred from equation (21). It is noted that a positive k_3 has a tendency to stabilize the postbuckling response, while a negative k_3 contributes further to the destabilising effect in the postbuckling response.

To summarize, the effects of foundation softening on the initial postbuckling behavior of a beam resting on a non-linear elastic foundation have been studied. It was found that compression induced softening (hardening) of the foundation resulted in a decrease (increase) of the buckling load while simultaneously decreasing (increasing) the imperfection sensitivity of the postbuckling response. Within the context of fiber composites, these features may be related to the observed compression failure modes([7]). In brittle systems (with respect to the matrix), the initiation of compression failure occurs at loads which are relatively higher than their ductile counterparts. Beyond this point, the transition to final failure occurs suddenly. For systems in which the matrix material tends to soften with increasing load, the buckling point is not as sharply defined. Further, the transition to final failure occurs gradually. This study shows that the softening (hardening) characteristics of the foundation have an appreciable effect on the buckling and postbuckling response of the beam structure.

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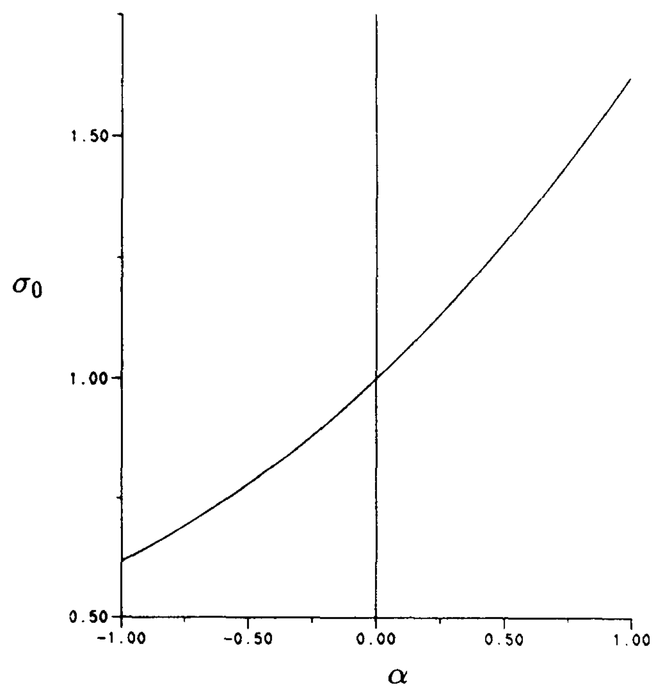

FIG 1. The critical buckling load (σ_0) vs. the parameter α.

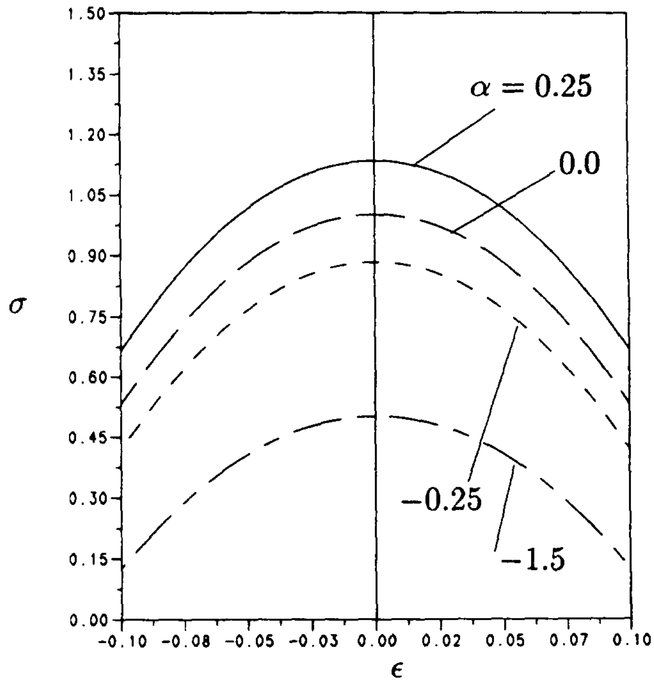

FIG 2. Postbuckling paths for k_3=-500 and for various values of α.

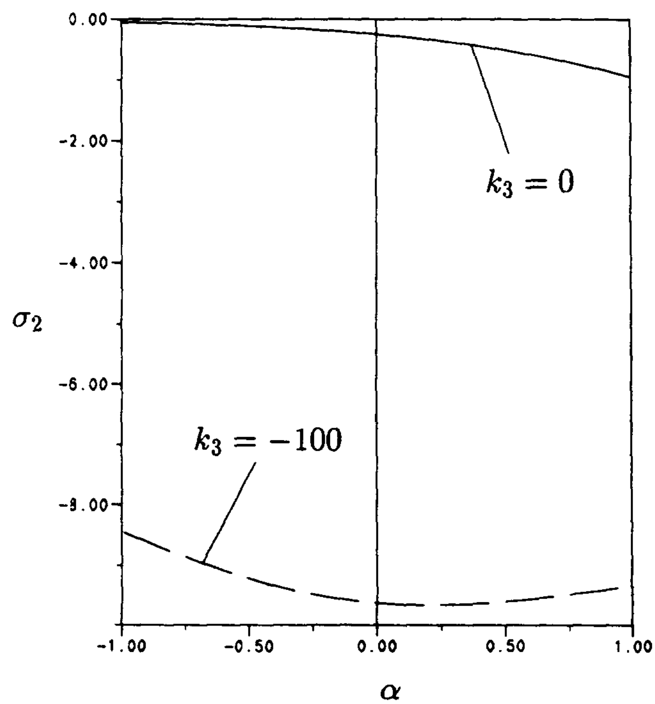

FIG 3. σ_2 vs. α for k_3=0 and $k_3 = -100$.

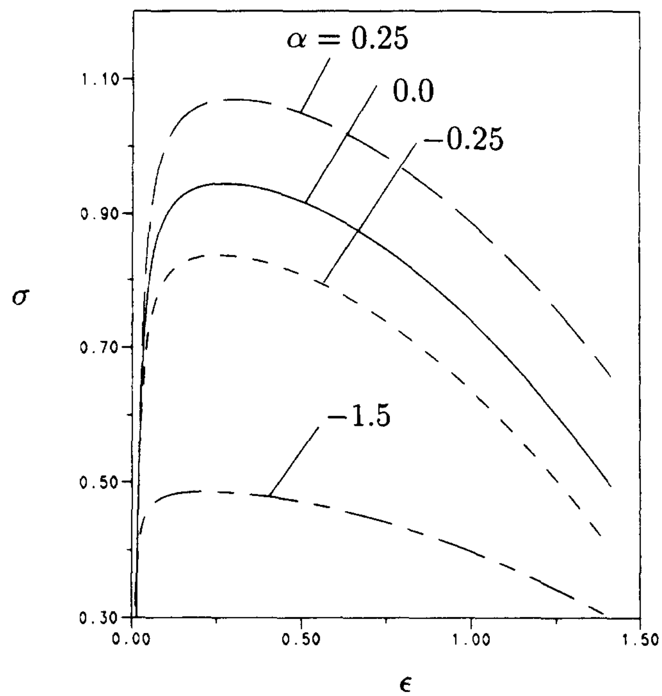

FIG 4. Influence of α on the postbuckling paths for imperfect beams ($\tau = 0.01$, $k_3 = 0$).