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On the Diophantine Equation $ap^x + bq^y = c + dp^zq^w$

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In this paper, the existence of effectively computable bounds on the solutions to the diophantine equation

$$ap^x + bq^y = c + dp^zq^w \qquad (*)$$

is shown. In this equation p, q are taken to be fixed relatively prime positive integers and a, b, c, d positive integers. The methods involve the application of linear forms in both real and p-adic logarithms. Also, a result on an inequality involving S-integers is used. All constants involved can be explicitly computed in terms of the parameters a, b, c, d, p, q, conceivably allowing one to list all solutions to (*) for any set of parameters. It is also indicated how the bounds in a particular case can be reduced to allow the practical solution of the equation. Finally, the methods are demonstrated by the solving of the equations $2^x + 3^y = 1 + 2^z3^w$ and $5 \cdot 2^x + 7 \cdot 3^y = 11 + 2^z3^w$. © 1990 Academic Press, Inc.

1. INTRODUCTION

Brenner and Foster remarked in [1] that the equation $1 + (pq)^a = p^b + q^c$ (p, q fixed primes) did not lend itself to being solved by their elementary congruential methods. This would indicate that more advanced methods are necessary to solve this equation. The purpose of this paper is not only to demonstrate a method for effectively solving the equation of Brenner and Foster, but also to solve a more general equation. Specifically, we show that there exist effectively computable upper bounds on the exponents x, y, z, w of the title equation when p, q are fixed relatively prime positive integers and a, b, c, d are positive integers. The equation of Brenner and Foster arises when a, b, c, d equal 1 and $z = w$. By effectively solving

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the equation, we mean a method is given by which all solutions could be determined.

The methods used to obtain the upper bounds on x, y, z, w are a combination of linear forms in logarithms, archimedean estimates, and some inequalities involving S-integers (used to show the effective computability of solutions that are below assumed lower bounds on certain cases involving the logarithmic arguments.) In Section 2, using these methods we are able to demonstrate the existence of effectively computable upper bounds and to show how these bounds can be explicitly computed. We do this by combining some archimedean estimates with bounds derived by the application of some results from linear forms in p-adic logarithms. We make a distinction between cases where a or b equals c and when they do not. If equality holds, then by using congruential arguments we can arrive at much better bounds. By virtue of the generality of the lemmas used in determining the upper bounds in the case a or b not equal to c, the resulting constants are very large. This makes the explicit determination of solutions impractical. Hence, in Section 4 we indicate methods by which these bounds can be reduced for specific cases by the application of a reduction algorithm dependent upon the computation of p-adic logarithms to hundreds of places. In Section 5 we combine these methods to solve the equations $2^x + 3^y = 1 + 2^z 3^w$ and $5 \cdot 2^x + 7 \cdot 3^y = 11 + 2^z 3^w$.

2. The Equation $ap^x + bq^y = c + dp^z q^w$

From the following theorem, it follows that the equation

$$ap^x + bq^y = c + dp^z q^w \tag{1}$$

has only finitely many non-trivial solutions (x, y, z, w).

Main Theorem on S-Unit Equations (Evertse [2, Corollary 1, p. 225]). *Let c, d be constants with $c > 0$, $0 < d < 1$. Let S be a finite set of prime numbers. Let n be an integer. There are only finitely many n-tuples of rational integers $\mathbf{x} = (X_0, X_1, ..., X_n)$ composed of primes taken from S such that*

$$X_0 + X_1 + \cdots + X_n = 0;$$

$$X_{i_1} + X_{1_2} + \cdots + X_{1_s} \neq 0$$

for each proper non-empty subset $\{i_0, i_1, ..., i_s\}$ of $\{0, 1, 2, ..., n\}$;

$$\gcd(X_0, X_1, ..., X_n) = 1;$$

$$\prod_{k=0}^{n} |X_k| \prod_{p \in S} |X_k|_p \leqslant c \cdot \max(X_i)^d.$$

However, by the nature of the theorem, this result is existential rather than effective. To achieve our effective result, we consider two different cases for Eq. (1). In Part A of this section we assume a or $b \neq c$. In Part B we show how to adapt the methods of Part A to a case where equality holds.

A. From now on we can assume without loss of generality that $p < q$. We first derive some archimedean estimates on $\max(z, w)$ in terms of $\max(x, y)$.

LEMMA 1. $\operatorname{Max}(z, w) \geqslant C_1 \max(x, y) + C_2$ *where* C_1 *and* C_2 *are effectively computable constants depending on* $p, q, a, b, c,$ *and* d.

Proof. If $d \geqslant c$, then

$$2(d/c)(pq)^{\max(z,w)} \geqslant (d/c)\, p^z q^w + 1 \geqslant (1/c) \max(ap^x, bq^y)$$
$$\geqslant (1/c) \min(a, b)\, p^{\max(x, y)}.$$

If $d < c$, then

$$2(pq)^{\max(z,w)} \geqslant (d/c)\, p^z q^w + 1 \geqslant (1/c) \min(a, b)\, p^{\max(x, y)}.$$

Let

$$C_1 = \log(p)/\log(pq)$$

and

$$C_2 = (1/\log(pq)) \min(\log(\min(a, b)/2d), \log(\min(a, b)/2c)),$$

then

$$\max(z, w) \geqslant C_1 \max(x, y) + C_2.$$

LEMMA 2. $\operatorname{Max}(x, y) \geqslant C_3 \max(z, w) + C_4$ *where* C_3 *and* C_4 *are effectively computable constants.*

Proof. It is easily seen from (1) that

$$2 \max(a, b)\, q^{\max(x, y)} \geqslant c + dp^{\max(z, w)} \geqslant dp^{\max(z, w)}.$$

Let

$$C_3 = \log(p)/\log(q) \qquad \text{and} \qquad C_2 = (\log(d) - \log(2 \max(a, b)))/\log(q),$$

then

$$\max(x, y) \geqslant C_3 \max(z, w) + C_4.$$

We will also need the following result from linear forms of p-adic logarithms.

LEMMA 3 (van der Poorten [4, Theorem 1]). *Let $\alpha_1, \alpha_2, ..., \alpha_n$ be non-zero algebraic integers belonging to a field* $\mathbf{K}$ *of degree D. Let a_i denote the height of α_i and $A_i \geqslant a_i$ be a number such that* $\log(\log(A_i)) > 0$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbf{K}$ *lying above the rational prime p, and $f_\mathfrak{p}$ be the residue class degree of $\mathfrak{p}$. Let $e_\mathfrak{p}$ be the ramification index of $\mathfrak{p}$, and define $g_\mathfrak{p} = (1/2 + e_\mathfrak{p}/(p-1))$. Let $b_1, b_2, ..., b_n$ be rational integers with* $B \geqslant \max(|b_i|)$, $b_n \not\equiv 0 \pmod{p}$ *Let*

$$N = \log(A_1)\log(A_2)\cdots\log(A_{n-1}), \qquad G_\mathfrak{p} = p^{e_\mathfrak{p} g_\mathfrak{p}}(p^{f_\mathfrak{p}} - 1).$$

If $\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_n^{b_n} - 1 \neq 0$, *then*

$$\operatorname{ord}_\mathfrak{p}(\alpha_1^{b_1}\alpha_2^{b_2}\cdots\alpha_n^{b_n} - 1) \leqslant (16(n+1)D)^{12(n+1)} G_\mathfrak{p} N \log(N)\log(A_n)\log(B).$$

It is easily seen that

$$\min(x, z) \leqslant \operatorname{ord}_{p_1}(bq^y - c) = \operatorname{ord}_{p_1}((b/c)\, q^y - 1) + \operatorname{ord}_{p_1}(c)$$

for some prime p_1 dividing p. Applying Lemma 3 with $n = 2$, $B = y$, we see that

$$\min(x, z) \leqslant C_5 \log(y) + C_6, \tag{2}$$

where C_5 is an effectively computable constant and $C_6 = \operatorname{ord}_{p_1}(c)$. Similarly,

$$\min(y, w) \leqslant C_7 \log(x) + C_8. \tag{3}$$

We now consider the following four cases.

Case 1. $x \leqslant z$, $y \leqslant w$. It follows that $x \leqslant C_5 \log(y) + C_6$ and $y \leqslant C_7 \log(x) + C_8$. Combining these two inequalities we find

$$x \leqslant C_5 \log(C_7 \log(x) + C_8) + C_6 \quad \text{and} \quad y \leqslant C_7 \log(C_5 \log(y) + C_6) + C_8.$$

Hence, $x \leqslant C_9$ and $y \leqslant C_{10}$, and it follows that z and w are also bounded.

Case 2. $z \leqslant x$, $w \leqslant y$.

(i) If $z \geqslant w$, then combining $z \leqslant C_5 \log(y) + C_6$ with the inequality found in Lemma 1 yields

$$C_1 \max(x, y) + C_2 \leqslant z \leqslant C_5 \log(\max(x, y)) + C_6.$$

Hence $\max(x, y) \leqslant C_{11}$ and it follows from Lemma 2 that z and w are also bounded.

(ii) If $w > z$, then

$$C_1 \max(x, y) + C_2 \leqslant w \leqslant C_7 \log(\max(x, y)) + C_8.$$

This results in bounds for the exponents, $\max(x, y) < C_{12}$ and $z < w \leqslant C_{13}$.

For the last two cases we will need the following results.

LEMMA 4 (Waldschmidt [7, Main Result]). *Let $\alpha_1, \alpha_2, ..., \alpha_n$ be rational numbers, and $b_1, b_2, ..., b_n$ be rational integers with $B = \max(|b_i|)$. Put*

$$V_i = \max(1, \log(\alpha_i)) \quad (i = 1, 2, ..., n), \qquad \Omega = V_1 V_2 \cdots V_n$$
$$C_{14} = 2^{9n+39} n^{2n} \Omega \log(eV_{n-1}), \qquad C_{15} = C_{14} \log(eV_n).$$

If $\Lambda = b_1 \log(\alpha_1) + \cdots + b_n \log(\alpha_n)$ then

$$|\Lambda| > \exp(-(C_{14} \log(B) + C_{15})).$$

Note. The statement of this lemma has been simplified to when the α_i and the b_i are rational. The original result is stated for the case when the α_i and the b_i are algebraic (see [7]).

LEMMA 5. *Let S be a set of a finite number of primes, and let T be the set of all positive integers composed of these primes. Let $0 < \delta < 1$ be a fixed real number and α, β fixed positive integers, then the inequality*

$$0 < \alpha x - \beta y < (\beta y)^{\delta}, \qquad x, y \in T$$

has only finitely many solutions, all of which can be effectively computed.

Proof. The proof of this lemma follows the same lines as that found in Theorem 5.1 of [8]. Obviously $\beta y > (1/2)\, \alpha x$ else

$$\beta y < \alpha x - \beta y < (\beta y)^{\delta}.$$

But this implies $\beta y < 1$ which is impossible. Let $\Lambda = \log(\alpha x / \beta y)$. Then

$$0 < |\Lambda| < (\alpha x / \beta y) - 1 < (\beta y)^{\delta - 1}.$$

If $X = \max(\mathrm{ord}_p(xy))$ as p runs through all the elements of S, then

$$0 < |\Lambda| < \beta P^{-(1-\delta) X},$$

where P is the product of all the elements of S. Applying Lemma 4 we see that X is bounded and hence so are x, y.

Case 3. $x \leqslant z, w \leqslant y$. Combining these inequalities it follows from (2) and (3) that $w \leqslant C_7 \log(C_5 \log(y) + C_6) + C_8$. Now, from (1) we see that

$$bq^{y-w} - dp^z = q^{-w}(c - ap^x).$$

If $c \geqslant ap^x$, then $x \leqslant C_{16}$, whence w and y are bounded and it follows from Lemma 2 that z is bounded. If $c \leqslant ap^x$, then it follows from Lemma 5 that for any fixed δ, $0 < \delta < 1$, the inequality

$$(bq^{y-w})^\delta \leqslant dp^z - bq^{y-w} \leqslant ap^x$$

holds for all x, y, z, w except for the finitely many determined by Lemma 5. This inequality implies

$$(bq^{y - C_7\log(C_5\log(y) + C_6) - C_8})^\delta \leqslant ap^{C_5\log(y) + C_6}.$$

Hence, $y \leqslant C_{17}$, and x, z, w are also bounded.

Case 4. $z \leqslant x, y \leqslant w$. From (2) and (3) it is easily seen that $z \leqslant C_5 \log(C_7 \log(x) + C_8) + C_6$. Thus, just as in Case 3, if $c \geqslant bq^y$ we have bounds on the exponents, and if $c < bq^y$ we have the inequality

$$(ap^{x - C_5\log(C_7\log(x) + C_8) + C_6})^\delta \leqslant bq^{C_7\log(x) + C_8}.$$

Hence, $x \leqslant C_{18}$, and y, z, w are also bounded.

B. In Part A we assumed a or $b \neq c$. If a/c or $b/c = 1$ then we can actually determine constants which are much better than C_5 or C_7. To do this we make use of the following lemma.

LEMMA 6. *Let p be a prime number, a an integer $a > 1$, $a \equiv 1 \pmod p$, and $n = \operatorname{ord}_p(a-1)$. Then p is the smallest positive number m such that $a^m \equiv 1 \pmod{p^{n+1}}$. Moreover, $a^p \not\equiv 1 \pmod{p^{n+2}}$.*

Proof. By hypothesis, $a = kp^n + 1$ $(p \nmid k)$. Then

$$a^m \equiv mkp^n + 1 \qquad \pmod{p^{n+1}}.$$

However, $mkp^n + 1 \equiv 1 \pmod{p^{n+1}}$ only if $mk \equiv 0 \pmod p$. Obviously $m = p$ is the smallest positive solution. Also,

$$a^p \equiv kp^{n+1} + 1 \not\equiv 1 \qquad \pmod{p^{n+2}}.$$

Let $p = p_1^{e_1} \cdots p_i^{e_i}$ and $q = q_1^{f_1} \cdots q_j^{f_j}$ be the prime factorizations of p and q. From (1) we see that if $a/c = 1$

$$p_1^{\min(x,z)} \mid q^y - 1. \tag{4}$$

Let m, n be integers such that

$$q^m \equiv 1 \pmod{p_1^n} \qquad \text{and} \qquad q^m \not\equiv 1 \pmod{p_1^{n+1}}.$$

It follows from applying Lemma 6 repeatedly that

$$y \geqslant m p_1^{\min(x,z)-n},$$

and thus

$$\min(x, z) \leqslant (\log(p_1))^{-1} \log(y) - (\log(m)/\log(p_1)) + n. \tag{5}$$

Similarly, if $b/c = 1$ we have

$$q_1^{\min(y,w)} \mid p^x - 1$$

and hence

$$\min(y, w) \leqslant \log(q_1))^{-1} \log(x) - (\log(m')/\log(q_1)) + n'. \tag{6}$$

Now it is possible for (1) to be solved as in Part A by substituting (5) and (6) for those inequalities involving C_5 and C_7.

C. From the results derived in Parts A and B, we can conclude

THEOREM 1. *Let p, q be relatively prime positive integers. Let a, b, c, d be positive integers prime to p and q. Then there exists an effectively computable constant C_{19} such that the solutions to*

$$ap^x + bq^y = c + dp^z q^w$$

with x, y, z, w positive integers satisfy $\max(x, y, z, w) \leqslant C_{19}$. *Moreover, for given a, b, c, d, p, and q, C_{19} can be explicitly computed.*

3. AN AUXILIARY LEMMA

Since all of the lemmas used to obtain our bounds provide explicitly computed constants, the computation of the bounds on x, y, z, w is relatively straightforward. This is made particularly easy by the application of the following lemma.

LEMMA 7 (Petho and de Weger [3]). *Let $a > 0$, $h > 0$, $b > (e^2/h)^h$, and let x satisfy $x < a + b(\log(x))^h$. Then,*

$$x < 2^h (a^{1/h} + b^{1/h} \log(h^h b))^h.$$

4. Reduction of Bounds

The key to the reduction of the bounds in the general case (a and $b \neq c$) is as follows. Let p be a rational prime and Ω_p be the algebraic closure of $\mathbf{Q}_p$. Let $\beta, \theta \in \Omega_p$ such that $\beta/\theta \in \mathbf{Q}_p$, and $x \in \mathbf{Z}$, $x > 0$. Put $\zeta = \beta/\theta$. Consider

$$\mathrm{ord}_p(\zeta - x) \geqslant c_1 + c_2 \cdot x, \tag{7}$$

where c_1 and c_2 are small constants. The following lemma can be applied.

Lemma 8. *Let $\zeta = \sum u_i p^i$ be the p-adic expansion of ζ. Let X_1 be a positive constant. Let r be the smallest index such that $p^r > X_1$ and $u_r \neq 0$. Then* (7) *has no solutions with*

$$(r - c_1)/c_2 < x \leqslant X_1.$$

Proof (de Weger). Let $x \leqslant X_1$ satisfy (7). Suppose $\mathrm{ord}_p(\zeta - x) \geqslant r + 1$. Then

$$x \equiv \sum_0^r u_i p^i \pmod{p^{r+1}}.$$

By $x \geqslant 0$, it follows that

$$x \geqslant u_r p^r \geqslant p^r > X_1,$$

which contradicts $x \leqslant X_1$. Hence $\mathrm{ord}_p(\zeta - x) \leqslant r$, and the result follows immediately.

Remark 1. If $\mathrm{ord}_p(\zeta - 1) \leqslant 1/(p-1)$, then $\mathrm{ord}_p(\zeta - 1) \neq \mathrm{ord}_p(\log_p \zeta)$. In our upcoming application of Lemma 8 to the reduction of upper bounds we assume solutions are such that x and $y \geqslant 1$. The cases for x or $y = 0$ must be solved independently using different methods. Our experience shows us that this is not very difficult.

We apply Lemma 8 to the cases considered in Part A of Section 2 as follows.

Case 1. Since $\max(x, y) \leqslant \max(w, z)$, Lemma 2 can be written as

$$\max(x, y) \geqslant C_3 \max(x, y) + C_4.$$

(i) If $x \geqslant y$, then, since $x \leqslant \mathrm{ord}_{p_1}((b/c)\, q^y - 1)$, it follows that

$$C_3 \cdot y + C_4 - C_6 \leqslant \mathrm{ord}_{p_1}((b/c)\, q^{y-1}) = \mathrm{ord}_{p_1}(\log_{p_1}(b/c) + y \cdot \log_{p_1}(q)),$$

which is equivalent to

$$C_3 \cdot y + C_4 - C_6 \leqslant \mathrm{ord}_{p_1}(\zeta - y),$$

where $\zeta = -\log_{p_1}(b/c)/\log_{p_1}(q)$. Lemma 8 can be applied to this inequality to reduce existing upper bounds on y, from which we can derive reduced bounds on x, w, z.

(ii) If $y > x$, then

$$C_3 \cdot x + C_4 - C_8 \leqslant \operatorname{ord}_{q_1}(\log_{q_1}(a/c) + x \cdot \log_{q_1}(p))$$

and so

$$C_3 \cdot x + C_4 - C_8 \leqslant \operatorname{ord}_{q_1}(\zeta - x),$$

where $\zeta = -\log_{q_1}(a/c)/\log_{q_1}(p)$. Applying Lemma 8 to this inequality reduces previously computed upper bounds on x. In practice, successive application of Lemma 8 reduces the upper bound on solutions until the upper bound is very close to the largest solution.

Case 2. Applying Lemma 1 to the cases $z \geqslant w$ and $w > z$ we find

$$C_3 \cdot x + C_4 - C_8 \leqslant \operatorname{ord}_{p_1}(\log_{p_1}(b/c) + x \cdot \log_{p_1}(q))$$

$$C_3 \cdot y + C_4 - C_6 \leqslant \operatorname{ord}_{q_1}(\log_{q_1}(a/c) + y \cdot \log_{q_1}(p)),$$

respectively. In each subcase, Lemma 8 can be applied to the ensuing inequalities to reduce existing bounds.

Case 3. (i) If $x \geqslant w$, then it follows from the arguments in Section 1 that

$$(bq^{y-x})^{\delta} \leqslant ap^{x}$$

and hence

$$x \geqslant (y \cdot \delta \log(q) + \delta \log(b) - \log(a))/(\log(p) + \delta \log(q)).$$

Combining this inequality with $x \leqslant \operatorname{ord}_{p_1}((b/c)\, q^{y} - 1) + C_6$, Lemma 8 can be applied as in the cases before to reduce upper bounds.

(ii) If $x < w$, then since $w \leqslant \operatorname{ord}_{q_1}((a/c)\, p^{x} - 1) + C_8$, we again have an inequality to which Lemma 8 can be applied.

Case 4. If $y \geqslant z$, then we must have

$$(ap^{x-y})^{\delta} \leqslant bq^{y}$$

and so

$$y \geqslant (x \cdot \delta \log(p) + \delta \log(a) - \log(b))/(\log(q) + \delta \log(p)).$$

Combining this with $y \leqslant \text{ord}_{p_1}((a/c)\,p^x - 1)$, we have an inequality to which Lemma 8 can be applied, yielding a reduced upper bound for x from which reduced bounds for y, z, and w can be derived.

Remark 2. At this point we would like to note that all solutions to the inequality of Lemma 5 corresponding to those values of x, y, z, w not satisfying the inequalities of Cases 3 and 4 can be readily computed by methods utilizing the well-known continued-fraction reduction algorithm (see [6, Chap. 5]).

5. The Equations $5 \cdot 2^x + 7 \cdot 3^y = 11 + 2^z 3^w$ and $2^x + 3^y = 1 + 2^z 3^w$

In this section we use the methods of Sections 2 and 3 to solve two diophantine equations.

Example 1. We first solve the equation

$$5 \cdot 2^x + 7 \cdot 3^y = 1 + 11 \cdot 2^z 3^w. \tag{8}$$

From Lemmas 1, 2, and 3 we find

$$\max(z, w) \geqslant 0.386 \cdot \max(x, y) - 0.826,$$

$$\max(x, y) \geqslant 0.630 \cdot \max(z, w) - 2.4,$$

and

$$\min(x, z) \leqslant 7.6 \times 10^{59} \log(y),$$

$$\min(y, w) \leqslant 2.3 \times 10^{60} \log(x).$$

To apply Lemma 8 we will need the following information

$(\log_2 11 - \log_2 7)/\log_2 3$

= 0.10011 11010 01111 00101 01111 11001 10001 10100
01110 10100 01101 10001 00001 10001 01100 11001
01011 01000 00010 01101 00101 11010 11001 01111
11110 00110 00100 00101 01101 01011 00011 11001
00001 11010 10100 10001 01111 10001 11101 01100
10111 00101 01101 10011...

$(\log_3 11 - \log_3 5)/\log_3 2$

= 0.22100 21020 22101 11110 02010 10212 01112 22201
22121 02122 11020 01222 20011 01201 11220 10100
22221 21111 11100 20021 10112 21002 22220 20221
12001 12212 02011 00112 02102 02101 20001 01022....

Also the only ordered pairs (x, y) satisfying

$$0 < 2^x - 7 \cdot 3^y < (7 \cdot 3^y)^{1/2} \tag{9}$$

are $(3, 0)$ and $(6, 2)$.

There are no ordered pairs satisfying

$$0 < 3^x - 5 \cdot 2^y < (5 \cdot 2^y)^{1/2}.$$

Case 1. Applying Lemma 7 we can deduce that

$$x \leqslant 4.21 \times 10^{62} \quad \text{and} \quad y \leqslant 6.37 \times 10^{62}.$$

(i) If $x \geqslant y$ then we see that

$$0.630 \cdot y - 2.4 \leqslant \mathrm{ord}_2(\log_2 7 - \log_2 11 + y \cdot \log_2 3).$$

We apply Lemma 8 with $\zeta = (\log_2 11 - \log_2 7)/\log_2 3$. We list the upper bound and the value for r for each time the lemma was applied.

(1) $X_1 = 6.37 \times 10^{62}$ $\quad r = 210$
(2) $X_1 = 338$ $\quad r = 9$
(3) $X_1 = 19$ $\quad r = 5$
(4) $X_1 = 12$ $\quad r = 4$
(5) $X_1 = 11$ $\quad r = 4.$

Thus all the solutions satisfy $y \leqslant 11$ and $x \leqslant 15$. The only solution satisfying these bounds and the given conditions is $(2, 2, 3, 2)$.

(ii) If $x < y$, then

$$0.630 \cdot x - 2.4 \leqslant \mathrm{ord}_3(\log_3 5 - \log_3 11 + x \cdot \log_3 2).$$

Applying Lemma 8 with $\zeta = (\log_3 11 - \log_3 5)/\log_3 2$ we find $x \leqslant 7$ and $y \leqslant 4$. There are no solutions satisfying the given conditions and these bounds.

Case 2. Following the arguments of Section 2 it is easily seen that

(i) If $z \geqslant w$, then $\max(x, y) \leqslant 5.47 \times 10^{62}$. From Lemma 2 we derive the inequality

$$0.630 \cdot y - 2.4 \leqslant \mathrm{ord}_2(\log_2 7 - \log_2 11 + y \cdot \log_2 3).$$

Applying Lemma 8 we see that $w \leqslant y \leqslant 11$ and $z \leqslant 16$. There are no solutions satisfying these bounds and the given conditions.

(ii) If $z < w$, then $\max(x, y) \leqslant 1.7 \times 10^{63}$ and, applying Lemma 8 as before, $x \leqslant 7$ and $z < w \leqslant 3$. This case provides no solutions.

Case 3. We consider two subcases.

(i) If $x \geqslant w$, then

$$0.44 \cdot y - 0.513 \leqslant x.$$

Applying Lemma 8, we see that $w \leqslant y \leqslant 11$ and $x \leqslant 16$. The only solution in these ranges is (2, 4, 6, 2).

(ii) If $x < w$, then $x \leqslant 1$. It is easily checked that there are no solutions in this range.

Case 4. As with Case 3 there are two subcases.

(i) If $y \geqslant z$, then

$$0.23 \cdot x - 0.79 \leqslant y.$$

Hence $x \leqslant 11$ and $y \leqslant 6$. The only solution in these ranges and satisfying the given bounds is (8, 3, 1, 6).

(ii) If $y < z$, then $y \leqslant 11$. It is easily checked that there are no solutions in this range satisfying the given conditions.

Note. If $x = 0$, then $\min(y, w) \leqslant 1$. This yields the solution (0, 0, 0, 0). If $y = 0$ then $\min(x, z) \leqslant 2$. This yields the solutions (0, 0, 0, 0), (2, 0, 4, 0), (1, 0, 1, 1), and (3, 0, 2, 2). Also, the solutions provided by the ordered pairs satisfying (9) are (2, 2, 3, 2) and (2, 4, 6, 2) corresponding to the ordered pairs (3, 0) and (6, 2).

THEOREM 2. *The diophantine equation*

$$5 \cdot 2^x + 7 \cdot 3^y = 11 + 2^z 3^w$$

has only seven solutions in non-negative integers x, y, z, w. *These solutions are* $(x, y, z, w) = (0, 0, 0, 0)$, (1, 0, 1, 1), (2, 0, 4, 0), (2, 2, 3, 2), (2, 4, 6, 2), (3, 0, 2, 2) *and* (8, 3, 1, 6).

EXAMPLE 2. Consider the equation

$$2^x + 3^y = 1 + 2^z 3^w. \tag{10}$$

Following the reasoning in Part B of Section 2 we find

$$\min(x, z) \leqslant 1.5 \log(y) + 0.12$$

and

$$\min(y, w) \leqslant \log(x) - 0.8.$$

Obviously these inequalities are much better than those derived in the previous example by the application of Lemma 3. These allow us to do away with the reduction algorithm. The equation can be solved by combining the methods of Part A of Section 2 with the following result. The inequality

$$|2^x - 3^y| < \min(2^x, 3^y))^{0.5}$$

has only the solutions $(x, y) = (3, 2)$, $(5, 3)$, and $(8, 5)$.

Finally, we may conclude that

THEOREM 3. *The diophantine equation*

$$2^x + 3^y = 1 + 2^z 3^w$$

has only ten non-trivial solutions when x, y, z, w are non-negative integers. These solutions are $(x, y, z, w) = (0, 0, 0, 0)$, $(1, 1, 2, 0)$, $(2, 1, 1, 1)$, $(2, 2, 2, 1)$, $(3, 2, 4, 0)$, $(4, 1, 1, 2)$, $(4, 2, 3, 1)$, $(4, 4, 5, 1)$, $(6, 2, 3, 2)$, *and* $(6, 4, 4, 2)$.

6. DISCUSSION

In the previous sections we demonstrated the existence of effectively computable upper bounds on the solutions to (1), indicated methods for the practical determination of solutions in a given case, and applied these methods to two equations representing the general (a or $b \neq c$) and the special (a or $b = c$) cases.

Equation (1) is, as indicated before, a generalization of one that appeared in [1]. Since it was noted that this particular equation did not lend itself to being solved by elementary means, it was reasonable to assume that such advanced methods as linear forms in logarithms would be needed to solve it. For a survey of similar equations, some having elementary solutions and others requiring very advanced methods, the reader is referred to [1, 5, 6].

The restriction of a, b, c, d to positive integers is probably unnecessary and equally strong results can most likely be derived using methods similar to those found in Section 2. Another loosening of restrictions would be to allow p, q and a, b, c, d to be integral elements of an arbitrary algebraic field. The lemmas used in this paper extend almost completely to algebraic fields and so it would seem that many of the same results would apply. The exception might be Lemma 5. This would probably have to be replaced with an argument on the heights of the numbers involved.

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