European Journal of Operational Research 46 (1990) 109–112
North-Holland

Theory and Methodology

The growth of *m*-constraint random knapsacks

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Abstract: The author computes the asymptotic value of a particular m-constraint, n-variable 0–1 random integer programming problem as n increases, m remaining fixed. This solves a problem of Frieze and Clarke (1984).

1. Introduction

Consider the m-constraint 'random knapsack' problem

$$\begin{aligned} V_n = \max\ & X_1\delta_1 + X_2\delta_2 + \cdots + X_n\delta_n \\ \text{subject to}\ & W_{11}\delta_1 + W_{12}\delta_2 + \cdots + W_{1n}\delta_n \leqslant 1, \\ & W_{21}\delta_1 + W_{22}\delta_2 + \cdots + W_{2n}\delta_n \leqslant 1, \\ & \vdots \\ & W_{m1}\delta_1 + W_{m2}\delta_2 + \cdots + W_{mn}\delta_n \leqslant 1, \\ & \delta_i \in \{0, 1\}, \end{aligned} \tag{1.1}$$

where the random variables X_j and W_{ij}, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, are mutually independent, and all uniformly distributed on the interval (0, 1). In a paper in this Journal, Frieze and Clarke (1984) raised the question of computing the asymptotic value of the random variables V_n, for fixed m, as $n \to \infty$; that is, finding a sequence (x_n) of numbers such that

$$P(x_n(1 - o(1)) \leqslant V_n \leqslant x_n(1 + o(1))) \to 1 \quad \text{as } n \to \infty. \tag{1.2}$$

(As usual, o(1) denotes a sequence which tends to 0 as $n \to \infty$.)

In this paper we solve this problem. To be precise, let $V_n \sim x_n$ be an abbreviation for (1.2). We shall prove

Theorem 1. $V_n \sim (m + 1)\,(n/(m + 2)!)^{1/(m+1)}$.

2. Proof of Theorem 1

Let m be a fixed positive integer, and let V_n be defined by (1.1).

Received June 1988; Revised December 1988

Lemma 2. *Suppose* $t_i > 0$ *for* $i = 1, 2, \ldots, m$. *For* $j = 1, 2, \ldots, n$, *let* I_j *be the indicator of the event* $\{X_j \geqslant t_1 W_{1j} + t_2 W_{2j} + \cdots + t_m W_{mj}\}$. *If* $W_{i1} I_1 + W_{i2} I_2 + \cdots + W_{in} I_n \geqslant 1$ *for* $i = 1, 2, \ldots, m$, *then* $X_1 I_1 + X_2 I_2 + \cdots + X_n I_n \geqslant V_n$.

Proof. Let $(\delta_1, \ldots, \delta_n)$ be an optimal solution to (1.1), that is $\sum_{j=1}^n W_{ij} \delta_j \leqslant 1$ for $i = 1, \ldots, m$, and $\sum_{j=1}^n X_j \delta_j = V_n$. In the sums below, let k range over all $j \in \{1, \ldots, n\}$ such that it is not the case that $I_j = \delta_j = 1$. Then

$$\begin{aligned}
\sum_k X_k I_k &\geqslant \sum_k \sum_{i=1}^m t_i W_{ik} I_k \quad \text{by definition of } I_k \\
&\geqslant \sum_k \sum_{i=1}^m t_i W_{ik} \delta_k \quad \text{since, for all } i, \sum_j W_{ij} I_j \geqslant 1 \geqslant \sum_j W_{ij} \delta_j, \\
&\qquad\qquad \text{so } \sum_k W_{ik} I_k \geqslant \sum_k W_{ik} \delta_k \\
&\geqslant \sum_k X_k \delta_k \quad \text{since, whenever } \delta_k = 1,\ I_k = 0, \\
&\qquad\qquad \text{so } X_k < \sum_i t_i W_{ik}.
\end{aligned}$$

Now, adding in $X_j I_j = X_j \delta_j$ for j such that $I_j = \delta_j = 1$, we have $\sum_{j=1}^n X_j I_j \geqslant \sum_{j=1}^n X_j \delta_j = V_n$, which completes the proof. □

We now proceed with the proof of Theorem 1. For $i = 1, 2, \ldots, m$, let

$$\hat{W}_{i;n}(t_1, t_2, \ldots, t_m) = \sum_{j=1}^n W_{ij} \cdot 1_{\{X_j \geqslant t_1 W_{1j} + t_2 W_{2j} + \cdots + t_m W_{mj}\}},$$

and let

$$\hat{X}_n(t_1, t_2, \ldots, t_m) = \sum_{j=1}^n X_j \cdot 1_{\{X_j \geqslant t_1 W_{1j} + t_2 W_{2j} + \cdots + t_m W_{mj}\}}.$$

A computation shows that, if $t_i > 1$ for $i = 1, \ldots, m$, then

$$\begin{aligned}
E\big(\hat{W}_{i;n}(t_1, t_2, \ldots, t_m)\big) &= n\, E\big(W_{i1} \cdot 1_{\{X_1 \geqslant t_1 W_{11} + t_2 W_{21} + \cdots + t_m W_{m1}\}}\big) \\
&= n/\big((m+2)!\, t_i \cdot t_1 t_2 \cdots t_m\big),
\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}
\operatorname{Var}\big(\hat{W}_{i;n}(t_1, t_2, \ldots, t_m)\big) &\leqslant n\, E\big(W_{i1}^2 \cdot 1_{\{X_1 \geqslant t_1 W_{11} + t_2 W_{21} + \cdots + t_m W_{m1}\}}\big) \\
&\leqslant \frac{n}{t_i} \cdot E\big(W_{i1} \cdot 1_{\{X_1 \geqslant t_1 W_{11} + t_2 W_1 + \cdots + t_m W_{m1}\}}\big) \\
&\qquad (\text{since } X_1 \leqslant 1 \text{ and } X_1 \geqslant t_i W_i \text{ imply } W_i \leqslant 1/t_i) \\
&= n/\big((m+2)!\, t_i^2 \cdot t_1 t_2 \cdots t_m\big).
\end{aligned} \tag{2.2}$$

Now let

$$\tau_n = \big(n/((m+2)!\,(1+\varepsilon_n))\big)^{1/(m+1)} \quad \text{and} \quad \upsilon_n = \big(n/((m+2)!\,(1-\varepsilon_n))\big)^{1/(m+1)}, \tag{2.3}$$

where $\varepsilon_n = n^{-1/(2m+3)}$. We shall write $\hat{W}_{i;n}(t)$ to abbreviate $\hat{W}_{i;n}(t, t, \ldots, t)$. For $i = 1, 2, \ldots, m$, we have

$$\begin{aligned} P\big(\hat{W}_{i;n}(\tau_n) < 1\big) &= P\big(\hat{W}_{i;n}(\tau_n) - E\big(\hat{W}_{i;n}(\tau_n)\big) < 1 - E\big(\hat{W}_{i;n}(\tau_n)\big)\big) \\ &= P\big(\hat{W}_{i;n}(\tau_n) - E\big(\hat{W}_{i;n}(\tau_n)\big) < -n^{-1/(2m+3)}\big) \qquad \text{(by (2.1) and (2.3))} \\ &\leqslant \mathrm{Var}\big(\hat{W}_{i;n}(\tau_n)\big) \cdot n^{2/(2m+3)} \quad \text{(by Chebyshev's inequality)}. \end{aligned}$$

It follows from (2.2) and (2.3) that this last expression converges to 0 as $n \to \infty$; thus, for $i = 1, 2, \ldots, m$, $P(\hat{W}_{i;n}(\tau_n) \geqslant 1) \to 1$ as $n \to \infty$. Applying Lemma 2, we may infer

$$P\big(\hat{X}_n(\tau_n) \geqslant V_n\big) \to 1 \quad \text{as } n \to \infty. \tag{2.4}$$

An argument similar to that just completed shows that $P(\hat{W}_{i;n}(\upsilon_n) \leqslant 1) \to 1$ as $n \to \infty$. But whenever $\hat{W}_{i;n}(\upsilon_n) \leqslant 1$ for $i = 1, \ldots, m$, by definition of $\hat{W}$ the assignment

$$\delta_j = 1_{\{X_j \geqslant \upsilon_n W_{1j} + \upsilon_n W_{2j} + \cdots + \upsilon_n W_{mj}\}}, \quad j = 1, 2, \ldots, n,$$

is feasible for problem (1.1); thus we have

$$P\big(\hat{X}_n(\upsilon_n) \leqslant V_n\big) \to 1 \quad \text{as } n \to \infty, \tag{2.5}$$

so, from (2.4) and (2.5)

$$P\big(\hat{X}_n(\tau_n) \geqslant V_n \geqslant \hat{X}_n(\upsilon_n)\big) \to 1 \quad \text{as } n \to \infty. \tag{2.6}$$

A computation shows that if $t_i > 1$ for $i = 1, \ldots, m$, then

$$\begin{aligned} E\big(\hat{X}_n(t_1, t_2, \ldots, t_m)\big) &= n \cdot E\big(X_1 \cdot 1_{\{X_1 \geqslant t_1 W_{11} + t_2 W_{21} + \cdots + t_m W_{m1}\}}\big). \\ &= n/\big((m+2)m!\, t_1 \cdot t_2 \cdots t_m\big), \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} \mathrm{Var}\big(\hat{X}_n(t_1, t_2, \ldots, t_m)\big) &\leqslant n \cdot E\big(X_1^2 \cdot 1_{\{X_1 \geqslant t_1 W_{11} + t_2 W_{21} + \cdots + t_m W_{m1}\}}\big) \\ &\leqslant n \cdot E\big(X_1 \cdot 1_{\{X_1 \geqslant t_1 W_{11} + t_2 W_{21} + \cdots + t_m W_{m1}\}}\big) \\ &= E\big(\hat{X}_n(t_1, t_2, \ldots, t_m)\big) \\ &= n/\big((m+2)m!\, t_1 \cdot t_2 \cdots t_m\big). \end{aligned} \tag{2.8}$$

From (2.3), (2.7) and (2.8), we have

$$\begin{aligned} &E\big(\hat{X}_n(\tau_n)/\alpha_n\big) = (1 + \varepsilon_n)^{1/(m+1)} \to 1 \quad \text{and} \\ &\mathrm{Var}\big(\hat{X}_n(\tau_n)/\alpha_n\big) \leqslant \big((1 + \varepsilon_n)/\alpha_n\big)^{1/(m+1)} \to 0 \quad \text{as } n \to \infty, \end{aligned}$$

where $\alpha_n = (m+1)(n/(m+2)!)^{1/(m+1)}$.

Therefore, by Chebyshev's inequality,

$$\hat{X}_n(\tau_n) \sim \alpha_n. \tag{2.9}$$

By an identical argument we also have

$$\hat{X}_n(\upsilon_n) \sim \alpha_n, \tag{2.10}$$

and from (2.6), (2.9), and (2.10), we conclude that $V_n \sim \alpha_n$, which completes the proof of our theorem. □

Acknowledgement

The result in Theorem 1 was announced in Mamer and Schilling (preprint). I wish to acknowledge John Mamer's help and encouragement in this work.

References

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