COMMUNICATION

A NEW DERIVATION OF THE GENERATING FUNCTION FOR THE MAJOR INDEX

Sumanta GUHA and Sriram PADMANABHAN
EECS Department, The University of Michigan, Ann Arbor, MI 48109, USA

Communicated by I. Gessel
Received 7 November 1989

We present a new proof of the well-known combinatorial result \[ \sum_{w} q^{\text{Maj}(w)} \text{ where } w \text{ is a permutation of } 0^k1^{n-k} \text{ having } \text{Maj}(w) = m. \]

1. Introduction

Consider the sequence \( w = w_1w_2 \cdots w_n \) where \( w_i \in \mathbb{R}, 1 \leq i \leq n. \) The Descent Set of \( w, D(w) = \{i \mid 1 \leq i < n, w_i > w_{i+1}\}, \) and the major index, \( \text{Maj}(w) \) is the sum of all the elements (possibly zero) of \( D(w). \) MacMahon \[4, 5\] showed that the major indices of the set of all permutations of \( w \) has the same generating function as the inversion numbers, \( \text{Inv}(w) \) of these permutations. A combinatorial proof of this correspondence between \( \text{Maj}(w) \) and \( \text{Inv}(w) \) was obtained by Foata \[1\]. Rawlings \[7\] uses a statistic called \( r \)-major index to describe a bijection that takes the major index to the inversion number.

This paper considers the case where the \( w \)'s are permutations of \( 0^k1^{n-k}. \) A new combinatorial proof for the generating function of the major indices of these sequences is derived by showing a bijection between the set of partitions of a positive integer \( m \) that fit inside a \( k \times n - k \) rectangle and the set consisting of all permutations \( w \) of \( 0^k1^{n-k} \) having \( \text{Maj}(w) = m. \) The bijections in \[1, 7\] can be shown, with some effort, to reduce to the bijection described in this paper in this case.

2. Notation and definitions

A partition of a positive integer \( m \) into \( k \) parts is a sequence \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k) \) such that \( \lambda_1 \geq \cdots \geq \lambda_k > 0 \) and \( \lambda_1 + \cdots + \lambda_k = m. \) The size of the partition, denoted by \( |\lambda|, \) is \( m, \) and the length, \( L(\lambda), \) is \( k. \)

The diagram of the partition is the set of lattice points \( \{(i, j) \in \mathbb{Z}^2 : 1 \leq i \leq L(\lambda), 1 \leq j \leq \lambda_i\} \) and is denoted by \( D_\lambda. \) The diagram \( D_\lambda \) and the partition \( \lambda \) may be used interchangeably in this paper. The rank of a partition \( \lambda \) is the length of the largest square subdiagram of \( D_\lambda. \) Suppose the rank of \( \lambda \) is \( r. \) Then \( \lambda \) can be...
expressed as \((\alpha_1, \ldots, \alpha_r \mid \beta_1, \ldots, \beta_r)\) [Frobenius notation, see Macdonald [3] page 3], where \(\alpha_i\) is the number of nodes in the \(i\)th row of \(D_i\) to the right of \((i, i)\), and \(\beta_i\) is the number of nodes in the \(i\)th column of \(D_i\) below \((i, i)\). A list of standard notation used to describe partitions and \(q\)-combinations is presented below.

- \(p(m) = \) Number of partitions of \(m\).
- \(p_{\lambda}(m) = \) Number of partitions, \(\lambda\), of \(m\), where \(L(\lambda) \leq k\).
- \(p_{k,n}(m) = \) Number of partitions of \(m\) that fit inside a \(k \times n\) rectangle.
- \(\lfloor k \rfloor = 1 + q + \cdots + q^{k-1}, \quad k \geq 1; \lfloor 0 \rfloor = 1\).
- \([n]! = \lfloor 1 \rfloor \lfloor 2 \rfloor \cdots \lfloor n \rfloor\).
- \([n]_q! = \lfloor k \rfloor ! \lfloor n - k \rfloor !\).

\[ \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]!}{[k]! [n-k]!}. \]

3. **Theorem and proof**

**Theorem.** There is a bijection between the set of partitions of \(m\) \((>0)\) that fit inside a \(k \times n - k\) rectangle and the set consisting of permutations \(w\) of \(0^k1^{n-k}\) having \(\text{Maj}(w) = m\).

This theorem and the well-known result \(2^k = \sum_{m=0}^k p_{k,n-k}(m)q^m\) (Stanley [8], page 29) directly imply that the generating function for the major index of permutations \(w\) of \(0^k1^{n-k}\) is given by

\[ \begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_w q^{\text{Maj}(w)}. \]

**Proof.** Let \(w = w_1w_2\cdots w_n\) be a permutation of \(0^k1^{n-k}\) such that \(\text{Maj}(w) = m\). Suppose \(D(w) = \{i_1, \ldots, i_j, \ldots, i_r\}\) where \(i_1 < i_2 < \cdots < i_r\). For each \(j, 1 \leq j \leq r\), let \(\text{Zero}(j)\) denote the number of zeros, and \(\text{One}(j)\) denote the number of ones, respectively, found in the substring \(w_1w_2\cdots w_{i_{j-1}}\). Let partition \(\lambda_w\) be defined as \((\text{One}(r), \text{One}(r-1), \ldots, \text{One}(1) \mid \text{Zero}(r), \text{Zero}(r-1), \ldots, \text{Zero}(1))\). The height of \(\lambda_w\) is \((\text{Zero}(r)+1)\), which is at most \(k\). Likewise, the width of \(\lambda_w\) is \((\text{One}(r)+1)\), which is at most \(n-k\). Also, \(|\lambda_w| = \sum_{j=1}^r (\text{One}(j) + \text{Zero}(j) + 1) = \sum_{j=1}^r i_j = m\). Thus, \(w \mapsto \lambda_w\) defines a mapping from the set of partitions of \(0^k1^{n-k}\) having \(\text{Maj}(w) = m\) to the set of partitions \(p_{k,n-k}(m)\).

**Example.** Let \(n = 10, k = 6,\) and, \(w = 0010011010\). Then \(D(w) = \{3, 6, 9\}, \text{Zero}(1) = 2, \text{One}(1) = 0, \text{Zero}(2) = 4, \text{One}(2) = 1, \text{Zero}(3) = 5, \text{One}(3) = 3\). Fig. 1 shows the partition diagram for the example.

The inverse function can be obtained by using the following method. Let \(\lambda\) be a partition of \(m\) that fits inside a \(k \times n - k\) rectangle. Suppose \(\lambda = \)}
Generating function for the major index

Fig. 1. $D_4$ for $w = 0010010110$.

$(\alpha_r, \alpha_{r-1}, \ldots, \alpha_1 \mid \beta_r, \beta_{r-1}, \ldots, \beta_1)$. The string $w_\lambda = w_1 w_2 \cdots w_n$ can be constructed as follows. Place $\beta_1$ zeros and $\alpha_1$ ones before the first 10 descent pair. All these zeros must precede all the ones since they cannot give rise to any descents. Thus, the partial string $w_\lambda$ up to and including the first descent pair is

$$00 \cdots 0 1 \cdots 1 0 \cdots 1 0.$$

Place $\beta_2 - \beta_1 - 1$ zeros and $\alpha_2 - \alpha_1 - 1$ ones between the first and second descent pairs, such that the zeros precede the ones. The partial string $w_\lambda$ up to the second descent pair is therefore

$$0 \cdots 0 1 \cdots 1 0 0 \cdots 0 1 \cdots 1 0 \cdots 1 0.$$

Continuing similarly, the string $w_\lambda$ can be compiled up to the last descent pair. Placing $k - \beta_r - 1$ zeros and $(n - k) - \alpha_r - 1$ ones in sequence after the last descent pair completes string $w_\lambda$. Note that, $\text{Maj}(w_\lambda) = \sum_{i \in D(w)} i_j = \sum_{j=1}^{\lambda} (\alpha_j + \beta_j + 1) = |\lambda| = m$. Thus, the bijection has been established. \qed

Consider permutations of $0^k 1^{n-k}$ which have exactly $r$ descents. The following Corollary establishes the generating function for the major indices of these permutations. MacMahon [6] (pp. 169–170) describes a different bijection for the case $q = 1$. Goulden [2] provides a bijective proof of Stanley’s shuffling theorem, which in a special case, provides an alternate proof, similar to MacMahon’s, of this corollary.

**Corollary.**

$$q^r \binom{k}{r} \binom{n-k}{r} = \sum_w q^{\text{Maj}(w)},$$

summed over permutations $w$ of $0^k 1^{n-k}$ having $r$ descents.
Subpartitions that fit in a $n-k-r \times r$ rectangle

Subpartitions that fit in a $r \times k - r$ rectangle

Fig. 2. Partitions that fit inside a $k \times n - k$ rectangle with rank $r$.

Proof. Use the bijection technique of the theorem. Each permutation, $w$, of $0^n n^{-k}$ having $r$ descents is mapped to a partition with rank $r$. The generating function for partitions of rank $r$ that fit inside a $k \times n - k$ rectangle is

$$q^r \binom{k}{r} \binom{n-k}{r}$$

as is evident from Fig. 2. □

Acknowledgement

We wish to thank Professor John Stembridge for suggesting this problem, and for the helpful discussions we had with him regarding this paper. We would also like to thank the referee for many valuable comments and suggestions.

References


