JOURNAL OF FUNCTIONAL ANALYSIS 89, 364–409 (1990)

Symmetry, Degeneracy, and Universality in Semilinear Elliptic Equations. Infinitesimal Symmetry-Breaking

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Communicated by R. B. Melrose

Received September 28, 1988

1. INTRODUCTION

In this paper we study the bifurcation of radial solutions of semilinear elliptic equations on n-balls,

$$\Delta u(x) + f(u(x)) = 0, \qquad x \in D^n \tag{1.1}$$

$$\alpha u(x) - \beta\, du(x)/dn = 0, \qquad x \in \partial D^n, \tag{1.2}$$

to asymmetric solutions. We shall show that for some fairly broad classes of nonlinear functions f (see (1.3)), "infinitesimal" symmetry-breaking occurs in the sense that there must exist infinitely many degenerate radial solutions of (1.1), (1.2) (on distinct balls), the kernel of whose linearized operators are non-trivial and contain asymmetric elements. In fact, if we write an element w in the kernel in its spherical harmonic decomposition $w = \sum_{N \geq 0} a_N(r)\, \Phi_N(\theta)$, where Φ_N lies in the Nth eigenspace of the Laplacian on S^{n-1}, then each summand is in the kernel, and these kernels contain asymmetric elements, $a_N \Phi_N$, $a_N \not\equiv 0$, of arbitrarily high modes; i.e., $N \geq N_0$. This means that on each such degenerate solution there exists the possibility of symmetry-breaking in the sense of bifurcation of a radial solution into an asymmetric one.[1]

We assume throughout this paper that the nonlinear function f satisfies the following hypotheses:

* Research supported in part by the NSF Contract MCS-830123.

[1] Indeed, using the Conley Index [8], we shall show in a forthcoming paper that actual symmetry-breaking must occur.

0022-1236/90 $3.00

There exist points $b<0<\gamma$ such that

$$\begin{aligned}
&\text{(i)} && f(\gamma)=0, \quad f'(\gamma)<0 \\
&\text{(ii)} && F(\gamma)>F(u) \quad \text{if} \quad b<u<\gamma \\
& && \quad (\text{here } F'=f \text{ and } F(0)=0) \\
&\text{(iii)} && F(b)=F(\gamma) \\
&\text{(iv)} && \text{if } f(b)=0, \quad \text{then} \quad f'(b)<0 \\
&\text{(v)} && uf(u)+2(F(\gamma)-F(u))>0 \quad \text{if} \quad b<u<\gamma.
\end{aligned} \tag{1.3}$$

Some remarks on these conditions are in order. The condition (ii) is a necessary condition for the existence of radial solutions belonging to a given nodal class with $u(0)$ near γ; see [6]. Condition (iii) is likewise a necessary condition for the existence of radial solutions in a given nodal class, but $b=-\infty$ is possible; this condition is also sufficient for the existence of such solutions (if, e.g., (i) holds); cf. Proposition 2.2. Condition (v) is a disguised version of a transversality condition. It implies that the level curves of the Hamiltonian function $H(u, v)=v^2/2+F(u)$ (associated to radial solutions, so $v=u'$) meet every boundary line transversally—it is needed to rule out certain degeneracies, thereby giving us an optimal result; cf. Proposition 3.4. Finally, conditions (i) and (iv) allow us to prove the existence of infinitesimal symmetry-breaking.

To be more precise, we show that if k is a fixed non-negative integer (k represents a given nodal class of radial solutions), then there is a positive integer N_0 with the following property: If $N \in \mathbb{Z}_+$, and $N \geqslant N_0$, one can find k distinct degenerate radial solutions of (1.1), (1.2) on which the symmetry breaks infinitesimally in the Nth mode. That is, for each $N \geqslant N_0$, there are solutions $u_N^1, ..., u_N^k$, of (1.1), (1.2), for which the associated linearized operators admit solutions of the form $a_N^j(r)\, \Phi_N(\theta)$, $j=1, ..., k$. Thus infinitesimal symmetry breaking must occur, in a k-fold way, for all sufficiently high modes. If we assume somewhat stronger assumptions on f, then this result takes on a universal flavor in a sense which can be described as follows. Assume that f satisfies these stronger hypotheses: There exist points $b<0<\gamma$ such that

$$\begin{aligned}
&\text{(i)} && f(\gamma)=0, \quad f'(\gamma)<0 \\
&\text{(ii)} && uf(u)>0 \quad \text{if} \quad b<u<0 \text{ or } \gamma>u>0 \\
&\text{(iii)} && F(b)=F(\gamma) \\
&\text{(iv)} && \text{if } f(b)=0, \quad \text{then} \quad f'(b)<0 \\
&\text{(v)} && f'(0)>0.
\end{aligned} \tag{1.4}$$

(Note that (1.4) (ii) implies (1.3) (ii) and (1.3) (v).) As before, let k denote a given nodal class of radial solutions. Our theorem is that if f satisfies (1.4), then we may choose the above integer N_0 *independent of* f. It is in this sense that we consider our result to be universal.

We point out that conditions (i) through (iv) of (1.4) ensure that the problem (1.1) and (1.2) admits in every nodal class, and for every p in $(0, \gamma)$, a radial solution $u(r)$ satisfying $u(0) = p$. On the other hand, if the weaker hypotheses (i)–(iv) of (1.3) hold, we can only assert that for p near γ, there are solutions of (1.1), (1.2) which satisfy $u(0) = p$.

In view of these last remarks, it is thus natural to consider the quantity p as a parameter for the radial solutions; we write $u = u(\cdot, p)$. Having made this choice, we must then allow the radii R of the balls to vary with p, $R = T(p)$; see [3–7]. Observe that by rescaling, we can rewrite (1.1), (1.2) as

$$\Delta u(x) + \lambda^2 f(u(x)) = 0, \qquad |x| < 1 \tag{1.5}$$

$$\alpha u(x) - \lambda\beta\, du(x)/dn = 0, \qquad |x| = 1, \tag{1.6}$$

whereby λ, instead of p, can be considered as the parameter. Note that λ appears explicitly in the boundary conditions only if both α and β are non-zero (so that Dirichlet or Neumann conditions are independent of λ). This causes a minor technical problem in studying the actual bifurcation of radial solutions, but in the next publication in this series, we shall show how to overcome this difficulty.

This paper is divided into four sections. In the next section we formulate the problem, and give the background material; we also prove the required existence theorems. In Section 3 we obtain the main technical result needed to prove our infinitesimal symmetry-breaking results. We show that the linearized operator is the direct sum of certain operators, the union of whose spectra is the spectrum of the full linearized operator. We prove that these operators have k positive eigenvalues if $p = u(0)$ is near γ.[2] The proof of this statement is quite long, and involves some delicate estimates which are fairly interesting in their own right. In Section 4, we show how these technical results are used to prove the infinitesimal symmetry-breaking theorems.

2. Formulation of the Problem

We consider the boundary-value problem

$$\Delta u(x) + f(u(x)) = 0, \qquad x \in D_R^n, \tag{2.1}$$

$$\alpha u(x) - \beta\, du(x)/dn = 0, \qquad x \in \partial D_R^n. \tag{2.2}$$

[2] It is here where condition (1.3)(v) is used; without it we can only assert the existence of $(k-1)$ positive eigenvalues.

Here D^n_R denotes an n-ball of radius R, $f \in C^1$, d/dn denotes the outward-pointing normal derivative on ∂D^n_R, and α and β are constants, with $\alpha^2+\beta^2=1$. If this problem admits radial solutions $u(r)$, $(r=|x|)$, then u solves the boundary-value problem $('=d/dr)$

$$u''(r)+\frac{n-1}{r}u'(r)+f(u(r))=0, \qquad 0<r<R, \tag{2.3}$$

$$u'(0)=0=\alpha u(R)-\beta u'(R). \tag{2.4}$$

We write this as the first-order system

$$u'=v,\ v'=-\frac{n-1}{r}v-f(u), \tag{2.5}$$

together with the boundary conditions

$$v(0)=0=\alpha u(R)-\beta v(R). \tag{2.6}$$

The solution of the initial-value problem for (2.5) with $u(0)=p$, $u'(0)=0$, will be denoted by $u(r,p)$, and p will be considered as a parameter throughout this paper. Define

$$\theta_0=\tan^{-1}(\alpha/\beta), \qquad \frac{-\pi}{2}\leqslant\theta_0<\frac{\pi}{2}.$$

If $k\geqslant 0$ is a given non-negative integer, and $f(p)>0$, we define the "time map"; i.e., the function $p\mapsto T(p)$ (whenever it exists; see Proposition 2.2 below), by the following two conditions:

$$\text{if } \theta(r,p)=\tan^{-1}(v(r,p)/u(r,p)), \text{ then } \theta(T(p),p)=\theta_0-k\pi. \tag{2.7}$$

Thus $T(p)$ plays the role of R, and R varies with p (see [3–5]). A solution of (2.5), (2.6), satisfying (2.7) will be said to belong to the "kth *nodal class*" of the function f (relative to the given boundary conditions), on $[0, T(p)]$; see Fig. 1. We define a function $H(u,v)$, the energy associated with (2.5), by $H(u,v)=v^2/2+F(u)$; then along any orbit of (2.5), $H'=-(n-1)v^2/r$, so H decreases on these orbits. We will use this fact throughout the paper.

We assume that the function f satisfies hypotheses (1.3); cf. Fig. 2, where we have taken $f(b)\neq 0$. The following proposition gives some properties of the time map which we shall need.

LEMMA 2.1. *Assume that f satisfies hypotheses* (i)–(iii) *of* (1.3). *Then for any $p\in\mathrm{dom}(T)$, $b<u(r,p)<\gamma$, and $u'(r,p)^2\leqslant 4M_0^2$, for all $r\geqslant 0$, where*

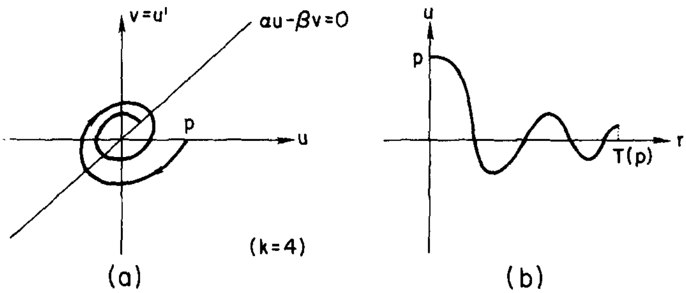

FIGURE 1

$M_0^2 = \sup\{F(u): b \leqslant u \leqslant \gamma\}$. *If for* $p \in \text{dom}(T)$, $T_N(p)$ *is defined by* $u'(T_N(p), p) = 0$, *and* $\alpha(p)$ *is defined by* $u(T_N(p), p) = \alpha(p) > 0$, *then* $\alpha(p) \to \gamma$ *as* $p \to \gamma$.

Proof. For $p \in \text{dom}(T)$, $u'(r, p)^2 \leqslant 2F(\gamma) - 2F(u(r, p))$ so that $u'(r, p)^2 \leqslant 4M_0^2$. For such p, if $u(\bar{r}, p) = b$ or $u(\bar{r}, p) = \gamma$, then $H(u(\bar{r}, p), v(\bar{r}, p)) = F(\gamma) + v'(\bar{r}, p)^2/2 > F(p) = H(p, 0)$, and this is impossible, since H decreases along orbits. Finally, suppose $\alpha(p_n) \geqslant \gamma - 2\varepsilon$ for some $\varepsilon > 0$, for a sequence $p_n \in \text{dom}(T)$, $p_n \to \gamma$. Then since $f(u) \geqslant \delta > 0$ on $\gamma - 2\varepsilon \leqslant u < \gamma - \varepsilon$, for some $\delta > 0$, then for p_n near γ,

$$\begin{aligned}
\varepsilon\delta &\leqslant \int_{\gamma - 2\varepsilon}^{\gamma - \varepsilon} f(s)\, ds \leqslant \int_{\alpha(p_n)}^{p} f(s)\, ds \\
&= F(p_n) - F(\alpha(p_n)) = H(p_n) - H(\alpha(p_n)) \\
&= \int_{T_N(p_n)}^{0} -\frac{v^2}{r}\, dr = \int_{p_n}^{\alpha(p_n)} \frac{v}{r}\, du \\
&\leqslant \frac{1}{T_N(p_n)} \int_{\alpha(p_n)}^{p_n} -v\, du \leqslant \text{const}/T_N(p_n).
\end{aligned}$$

But this is impossible since $T_N(p_n) \to \infty$ as $p_n \to \gamma$; thus $\alpha(p) \to \gamma$ and the proof is complete. ∎

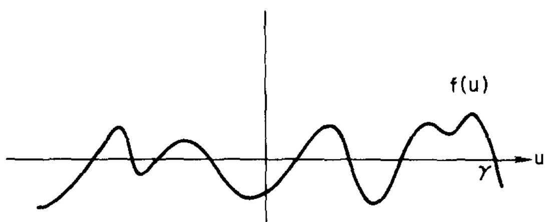

FIGURE 2

PROPOSITION 2.2. *Let f satisfy hypotheses* (i)–(iv) *of* (1.3). *Then*

(i) *There is a point $\bar{p}$, $0<\bar{p}<\gamma$, such that the open interval $(\bar{p}, \gamma)\subset \operatorname{dom}(T)$,*

(ii) $\lim_{p \nearrow \gamma} T(p)=+\infty$;

(iii) *Given any $\varepsilon>0$, $\bar{p}$ may be chosen so that if $\bar{p}<p<\gamma$, the total energy loss along the entire orbit segment $(u(r, p), u'(r, p))$, $0 \leqslant r \leqslant T(p)$ is less than ε.*

Proof. We have given a proof for positive solutions of the Dirichlet problem, under the same hypotheses, in [6]. We indicate here how this result can be extended to give an existence theorem for solutions of (2.3), (2.4) in the kth nodal class.

Since f is positive near γ and negative near b, there exist points E and B near γ and b, respectively, such that $F(B)=F(E)$ and $F(E)>F(u)$ if $B<u<E$. (Since b is the first negative value for which $F(b)=F(\gamma)$, we can find $\delta>0$ such that $f|_{(b, b+\delta)}<0$, $f|_{(\gamma-\delta, \gamma)}>0$, and $F|_{[b+\delta, \gamma-\delta]}<F(\gamma)$. Thus F is uniformly bounded away from $F(\gamma)$ on $b+\delta \leqslant u \leqslant \gamma-\delta$. The existence of the points E and B follows easily from this.) Now let $3\varepsilon=F(\gamma)-F(E)$, and take p_0, $E<p_0<\gamma$, such that $F(p_0)=F(\gamma)-\varepsilon$; then $F(\gamma)-F(p)<\varepsilon$ if $p_0 \leqslant p<\gamma$. From Lemma 2.1, there is a constant $M>0$ such that along any solution, $|v(r, p)|<M$. Let $T_0(p)$ be defined by $u(T_0(p), p)=p_0$. (T_0 is defined if p is near γ; this follows from our aforementioned result in [6] for the Dirichlet problem. In fact, the orbit also gets to the line $u=E$.) Let $L=(n-1) M(\gamma-b)(k+1)/\varepsilon$, and choose p_1 so close to γ that $T_0(p)>L$ if $p>p_1$. Set $\bar{p}=\max(p_0, p_1)$, and let $p>\bar{p}$. Let z be any point such that $B \leqslant z \leqslant A$ and suppose $u(\bar{r})=z$, $v(r) \leqslant 0$, $0 \leqslant r<\bar{r}$. Now with $H(u, v)= F(u)+v^2/2$, we have

$$
\begin{aligned}
& H(u(T_0(p), p), v(T_0(p), p))-H(u(\bar{r}), v(\bar{r})) \\
& \quad=(n-1) \int_{T_0(p)}^{\bar{r}} \frac{v^2}{r} d r=\int_{p_0}^z \frac{v}{r} d u<\frac{\varepsilon}{k+1} .
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\frac{v^2(\bar{r})}{2}+F(z) & \geqslant F(p_0)+\frac{v^2(T_0(p), p)}{2}-\varepsilon/(k+1) \\
& \geqslant F(\gamma)-\varepsilon-\varepsilon/(k+1),
\end{aligned}
$$

so

$$
\begin{aligned}
v^2(\bar{r}) & \geqslant 2[F(\gamma)-\varepsilon-F(z))-2 \varepsilon/(k+1) \\
& \geqslant 2(F(\gamma)-\varepsilon-F(A))-2 \varepsilon/(k+1) \\
& =4 \varepsilon-2 \varepsilon/(k+1)>\varepsilon .
\end{aligned}
$$

Thus $v(\bar{r}) < -\sqrt{\varepsilon}$, and since z was arbitrary, we see that the orbit $(u(\cdot, p), v(\cdot, p))$ gets to the line $u = B$ and thus cuts the line $v = 0$ at a point $\tilde{u}$, $b < \tilde{u} < B$. Now repeat the argument to show that the orbit segment lying in $v \geqslant 0$ meets the line $v = 0$ once again at a point $\bar{\bar{u}} > E$, since H has lost $2\varepsilon/(k+\ )$-amount of energy. Repeating this argument k times shows that $p \in \operatorname{dom}(T)$. Furthermore, given $\varepsilon > 0$, we see that by taking E close to γ, we can make the total energy loss less than ε. Finally (ii) follows easily, since $(\gamma, 0)$ is a rest point. ∎

Now for $\bar{p} < p < \gamma$, the orbit $(u(\cdot, p), u'(\cdot, p))$ lies inside the closed level curve $H(u, v) = F(\gamma)$. We use this fact if f satisfies (1.4) to strengthen the last proposition as follows.

PROPOSITION 2.3. *Assume that f satisfies hypotheses* (i)–(iii), *and* (v) *of* (1.4). *Then*

$$\text{(i)} \qquad \operatorname{dom}(T) = (0, \gamma);$$

$$\text{(ii)} \qquad \lim_{p \searrow 0} T(p) \text{ exists and is finite;}$$

$$\text{(iii)} \qquad \lim_{p \nearrow \gamma} T(p) = +\infty.$$

Proof. For any $r > 0$, set $(u(r, p), v(r, p)) = (u, v)$; then $F(u) \leqslant F(u) + v^2/2 = H(u, v) < H(u(0, p), v(0, p)) = F(p)$, so that $u < p$. Since there exists a unique point β_p, $b < \beta_p < 0$, satisfying $F(\beta_p) = F(p)$, we see $\beta_p < u$.

Now for any $p \in (0, \gamma)$, define $\theta(r, p)$ by

$$\theta(r, p) = \tan^{-1}(u'(r, p)/u(r, p)).$$

Write $f(u) = ug(u)$, where $g > 0$ on (b, γ). Then there is a $\sigma > 0$ such that $g(u(r, p)) \geqslant \sigma$ for all $r \geqslant 0$. Thus, if $\delta = \min(1, \sigma)$,

$$g(u) \cos^2 \theta + \sin^2 \theta \geqslant \sigma \cos^2 \theta + \sin^2 \theta \geqslant \delta$$

along the entire orbit $(u(r, p), v(r, p))$, $r \geqslant 0$. So for large r, say $r \geqslant r_0$,

$$\begin{aligned} \theta' &= -\frac{n-1}{2r} \sin 2\theta - g(u) \cos^2 \theta - \sin^2 \theta \\ &\leqslant -\frac{n-1}{2r} \sin 2\theta - \delta \\ &< -\delta/2. \end{aligned}$$

Thus $\theta(r, p) - \theta(r_0, p) < -\delta(r - r_0)/2$, if $r \geqslant r_0$, so that $\theta(r, p) \to -\infty$ as $r \to \infty$. This shows that $p \in \operatorname{dom}(T)$, and proves (i). Part (ii) is a consequence of the usual linearization techniques, and (iii) follows from Proposition 2.2.

From now on, we fix f satisfying conditions (1.3), *we fix an integer* $k \in \mathbb{Z}_+$, *and we fix the boundary conditions* (*i.e., we fix* α *and* β). *Moreover, radial solutions of* (1.1), (1.2) *will be assumed to lie in the* kth-*nodal class of* f; *i.e., to satisfy* (2.7).

Consider now the simple Bessel-type linear equation

$$u''(r) + \frac{n+1}{r} u'(r) + Au(r) = 0, \tag{2.8}$$

where A is a positive constant. Concerning this equation, we have the following result.

PROPOSITION 2.4. *Given any solution* $u(r, p)$ *of* (2.8), (2.4), *with* $u(0) = p > 0$, $u'(0) = 0$, *and any integer* $k \geq 0$, *there is a number* $\rho_{k,n}^{\alpha,\beta}(A)$, *for which* $T(p) \equiv \rho_{k,n}^{\alpha,\beta}(A)$; *that is, the associated time map is constant, so that*

$$\alpha u(\rho_{k,n}^{\alpha,\beta}(A), p) - \beta u'(\rho_{k,n}^{\alpha,\beta}(A), p) = 0, \tag{2.9}$$

and $u(\cdot, p)$ *belongs to the* kth *nodal class of the function* $f(u) = Au$, *on* $[0, \rho_{k,n}^{\alpha,\beta}(A)]$.

Proof. This follows from the proof of Proposition 2.3, part (i), with $g(u) = A$. ∎

We define the numbers $c_{k,n}^D$ and $c_{k,n}^N$ by

$$c_{k,n}^D = \rho_{k,n}^{1,0}(1) \qquad \text{and} \qquad c_{k,n}^N = \rho_{k,n}^{0,1}(1). \tag{2.10}$$

Thus $c_{k,n}^D$ (resp. $c_{k,n}^N$) is the radius of that n-ball for which the particular Bessel's equation

$$u'' + \frac{n-1}{r} u' + u = 0, \tag{2.11}$$

admits a solution u satisfying the boundary conditions

$$u'(0) = 0 = u(c_{k,n}^D) \text{ (resp. } u'(0) = 0 = u'(c_{k,n}^N)), \tag{2.12}$$

and u belongs to the kth nodal class of the $f(u) = u$ on the interval $[0, c_{k,n}^D]$ (resp. $[0, c_{k,n}^N]$).

We will later need the following result.

LEMMA 2.5. *Let* $A > 0$; *then* $\rho_{k,n}^{1,0}(A) = A^{-1/2} c_{k,n}^D$, *and* $\rho_{k,n}^{1,0}(A) = A^{-1/2} c_{k,n}^N$.

Proof. This follows from a scaling argument; namely, if u is a radial solution of $\Delta_x u + Au = 0$, on $|x| < \rho_{k,n}^{1,0}(A)$, so (2.8) holds, set $x = \lambda y$, and $w(y) = u(\lambda y)$. Then if $u(x) = 0$ for $|x| = \rho_{k,n}^{1,0}(A)$,

$$\Delta_y w(y) + \lambda^2 A w(y) = 0, \qquad |y| < \rho_{k,n}^{1,0}(A)/\lambda,$$
$$w'(0) = 0 = w(\rho_{k,n}^{1,0}(A)/\lambda).$$

Now set $A = \lambda^{-2}$; then w satisfies (2.11) on $|y| < \rho_{k,n}^{1,0}(A)\, A^{1/2}$, $w(\rho_{k,n}^{1,0}(A)\, A^{1/2}) = 0$, so that $c_{k,n}^D = \rho_{k,n}^{1,0}(A)\, A^{1/2}$, as desired. The proof for Neumann boundary conditions is similar. ∎

As a final result along these lines we note that if $u(x)$ is a solution of (1.1), (1.2) on the n-ball $|x| < R$, if $\lambda > 0$ and $w(y) = u(\lambda y)$, then w solves

$$\Delta w(y) + \lambda^2 f(w(y)) = 0, \qquad |y| < R/\lambda,$$
$$\alpha w(y) - \lambda\beta\, dw(y)/dn = 0, \qquad |y| = R/\lambda.$$

Thus there is a 1–1 correspondence between solutions u of (1.1), (1.2), and solutions w of the above boundary-value problem (corresponding to the function $\bar{f}(w) = \lambda^2 f(w)$). Hence as far as existence of symmetry-breaking solutions is considered, *there is no loss in generality if we*

$$\textit{assume } f'(\gamma) < -1 \tag{$*$}$$

throughout the paper. This will prove technically useful in Section 3.

Before stating our theorems, we must recall some standard results concerning the eigenvalues of the Laplacian on the $(n-1)$-sphere, S^{n-1}; (see [2] for details). These eigenvalues are given by

$$\lambda_N = -N(N+n-2), \tag{2.13}$$

and if E_N is the corresponding eigenspace, then

$$l_N \equiv \dim E_N = \binom{N+n-2}{N}\left(\frac{n+2N-2}{n+N-2}\right). \tag{2.14}$$

Now suppose that $u(\cdot, p)$ is a radial solution of (2.1), (2.2) lying in the kth nodal class of f. Then there is the possibility of u being a bifurcation point only if u is degenerate; i.e., only if there exists a non-trivial solution w of the linearized equations

$$\Delta w(x) + f'(u(|x|, p))\, w(x) = 0, \qquad |x| < T(p) \tag{2.15}$$

$$\alpha w(T(p)) - \beta\, dw(T(p))/dn = 0. \tag{2.16}$$

(This is an easy consequence of the implicit function theorem.) Note that the above scaling yields a 1–1 correspondence between the two sets of linearized equations. Furthermore, the symmetry can break on $u(\cdot, p)$ only if these last two equations admit an asymmetric solution (see [1]). In this case we say that the "*symmetry breaks infinitessimally*" on $u(\cdot, p)$. The main results in Sections 3 and 4 are stated in terms of this notion.

Now any solution of (2.15), (2.16) can be expressed in terms of spherical harmonics,

$$w(r, \theta) = \sum_{N=0}^{\infty} a_N(r)\, \Phi_N(\theta), \qquad 0 \leqslant r \leqslant T(p), \theta \in S^{n-1},$$

where $\Phi_N \in E_N$. Thus, the symmetry will break *infinitesimally* on $u(\cdot, p)$ if and only if for some $N \geqslant 1$, $a_N(r) \not\equiv 0$. However, as was shown in [3], for $N \geqslant 1$, these a_N's satisfy the equations and boundary conditions

$$a_N''(r) + \frac{n-1}{r} a_N'(r) + \left(f'(u(r, p)) + \frac{\lambda_N}{r^2}\right) a_N(r) = 0, \qquad 0 < r < T(p), \tag{2.18}$$

$$a_N(0) = 0 = \alpha a_N(T(p)) - \beta a_N'(T(p)). \tag{2.19}$$

It follows that the symmetry will break infinitesimally on $u(\cdot, p)$ if and only if Eqs. (2.18), (2.19) admit a non-trivial solution for some $N \geqslant 1$.

In order to study solutions of (2.18), (2.19), we let p be a point in $(\bar{p}, \gamma)$ (cf. Proposition 2.2), and let N be a non-negative integer. Define the function $q_N^p(r)$ by

$$q_N^p(r) = f'(u(r, p)) + \frac{\lambda_N}{r^2}, \qquad 0 < r \leqslant T(p)$$

(where u solves (2.3), (2.4) with $R = T(p)$, and lies in the kth nodal class of f), and let Ψ_p be the space of functions defined by

$$\Psi_p = \{\phi \in C^2(0, T(p)) \colon \phi(0) = 0 = \alpha\phi(T(p)) - \beta\phi'(T(p))\}. \tag{2.20}$$

For $N \geqslant 1$, we define the operator L_N^p on Ψ_p into $C(0, T(p))$ by

$$L_N^p \phi = \phi'' + \frac{n-1}{r} \phi' + q_N^p \phi. \tag{2.21}$$

Note that Proposition 2.1, (i) implies that these operators are well defined if $p \in \operatorname{dom}(T)$. If $N = 0$, we let

$$\Psi_p = \{\phi \in C^2(0, T(p)) \colon \phi'(0) = 0 = \alpha\phi(T(p)) - \beta\phi'(T(p))\},$$

and we denote by L_0^p the operator on Ψ_p into $C(0, T(p))$, defined by (2.21), with $N=0$. Finally, define the operator L^p, on

$$\{w \in C^2(|x| < T(p)): \alpha w(x) - \beta\, dw(x)/dn = 0,\ |x| = T(p)\}$$

into $C(|x| < T(p))$, by

$$L^p w = \Delta w + f'(u(\cdot, p))w. \tag{2.22}$$

Concerning these operators, we have the following result relating their spectra.

PROPOSITION 2.6. *For each* $p \in (\bar{p}, \gamma)$, $\operatorname{sp}(L^p) = \bigcup_{N \geq 0} \operatorname{sp}(L_N^p)$.

Proof. Since each of the operators L^p, L_N^p are self-adjoint (in the metric induced by $|\phi|^2 = \int_0^{T(p)} \phi(r)^2 r^{n-1}\, dr$), their spectra are real and discrete, with the only limit point being at $-\infty$.

If $\Phi_N \in E_N$ then if a is a radial function, $L^p(a\Phi_N) = (L_N^p a)\, \Phi_N$. Thus if $L_N^p a = \mu a$, then $L^p(a\Phi_N) = \mu(a\Phi_N)$ for any $\Phi_N \in E_N$. Conversely, if $L^p w = \mu w$, then $w = \sum a_N \Phi_N$ with $a_{N_0} \not\equiv 0$, for some N_0. Thus $\mu \Sigma a_N \Phi_N = L^p(\sum a_N \Phi_N) = \sum L^p(a_N \Phi_N) = \sum (L_N^p a_N)\, \Phi_N$, so $\mu a_{N_0} = L_{N_0}^p a_{N_0}$ and the proof is complete. ∎

3. The Positive Spectra of L_N^p

This section is devoted to the proof of the following theorem. (Recall that L_N^p is defined by (2.21); namely,

$$L_N^p \phi = \phi'' + \frac{n-1}{r} \phi' + q_N^p \phi,\ \phi \in \Psi_p.$$

THEOREM 3.1. *Let f satisfy hypothesis* (1.3). *For every* $N \in \mathbb{Z}_+$, $N \geq 1$, *there is a point* s_N, $0 < s_N < \gamma$, *such that if* $s_N < p < \gamma$, *then the spectrum of* L_N^p *contains k-positive eigenvalues.*

The proof is quite long, and to assist the reader we present here a summary of the ideas involved. The key point is to show that for p near γ, the operator L_1^p has k-distinct positive eigenvalues; this is done by first observing that $v = u'$ satisfies (2.18) with $N = 1$, and then showing that for p near γ, and $0 \leq r \leq T(p)$, the orbit $(v(r, p), v'(r, p))$ crosses the line $\alpha v = \beta v$ in (at least) k distinct points. Thus, e.g., if $0 < \theta_0 < \pi/2$, and ϕ_1 is defined by $\phi_1(r, p) = \tan^{-1}(v'(r, p)/v(r, p))$, then ϕ_1 satisfies the estimate $\theta_0 - (k+1)\pi < \phi_1(T(p), p) < \theta_0 - k\pi$. On the other hand, the positive spectrum of each operator L_1^p is bounded above, [8, Chap. 11]. Thus if

$L_1^p a_\mu = \mu a_\mu$, and $\phi_1^\mu(r, p) = \tan^{-1}(a_\mu'(r, p)/a_\mu(r, p))$, then for large μ, $\theta_0 - \pi < \phi_1^\mu(T(p), p) < 0$. Since $\phi_1^\mu(T(p), p)$ is a continuous function of μ, there must be k positive numbers μ_j for which $\phi_1^{\mu_j}(T(p), p) = -j\pi + \theta_0$, $j = 1, ..., k$, and to each such μ_j there corresponds an eigenfunction a_j of L_1^p. Now as the operators L_N^p and L_1^p differ only by the term $|\lambda_N - \lambda_1|/r^2$ (which is small if r is large), it should be possible to conclude the existence of k positive eigenvalues for L_N^p, at least if p is near γ. The difficulty is that one must show that orbits of $L_1^p a_1 = 0$ and $L_N^p a_N = 0$ which start close at $r = 0$, stay close at $r = T(p)$, but $T(p) \to \infty$ as $p \to \gamma$. To overcome this difficulty, we prove the decay and estimate its rate as a function of p, of the difference in angular variation of these orbits. It is here where the most difficult analysis is made. We proceed now with the details.

Since $-\lambda_1 = n - 1$ (see (2.12)), we find, upon differentiating (2.3) with respect to r, that $v(r, p) = u'(r, p)$ satisfies the equation $L_1^p(v) = 0$ and the initial condition $v(0, p) = 0$. Moreover, as was shown in [5], the only smooth solutions of $L_1^p(\phi) = 0$ are multiples of v. The key step in the proof of Theorem 3.1 is the next proposition (recall that θ_0 is defined by

$$\theta_0 = \tan^{-1}(\alpha/\beta), \qquad -\pi/2 \leqslant \theta_0 < \pi/2.)$$

PROPOSITION 3.2. *For each p sufficiently near γ, $0 < p < \gamma$, the following hold*: (a) *If $\theta_0 \geqslant 0$, the operator L_1^p has exactly k positive eigenvalues*; (b) *if $0 > \theta_0$, L_1^p has at least k, and no more than $(k+1)$, positive eigenvalues.*

The proof of this proposition will follow from a series of lemmas. We first need some notation. Let Λ (resp. $\tilde{\Lambda}$) denote the line $\alpha u = \beta v$ (resp. $\alpha v = \beta v'$) and define[3] $\phi_1(r, p)$ by

$$\phi_1(r, p) = \tan^{-1}(v'(r, p)/v(r, p)). \tag{3.1}$$

Recall that u being in the kth nodal class means that (cf. (2.7)),

$$\theta(T(p)) = \theta_0 - k\pi.$$

We remark that k is the number of zeros of $u(\cdot, p)$, except in the case of Dirichlet boundary conditions ($\theta_0 = -\pi/2$), in which case u has $(k+1)$ zeros. This is straightforward to verify, using the fact that orbits cross the axes $u = 0$ and $v = 0$ in a clockwise direction; see the proof of the following result.

LEMMA 3.3. *Let $\theta(r_s) = \theta_0 - s\pi$, $s = 0, 1, ..., k$; $r_k = T(p)$. Then if p is near γ there is exactly one zero of u and one zero of v in each interval $(r_s, r_{s+1}]$.*

Proof. The orbit (u, v) always crosses the line $u = 0$ transversally (see Fig. 3) and in a clockwise direction since $u' = v$. Also, we claim that the

[3] We will often drop the dependence of ϕ_1 on p where there is no chance of confusion.

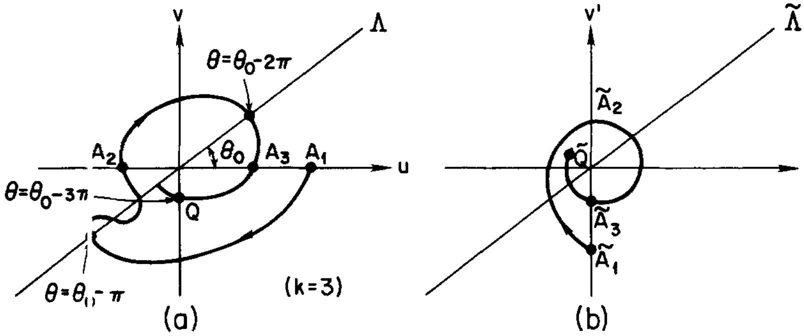

FIGURE 3

orbit crosses $v=0$ clockwise and transversally. To see this, note that if $v=0$, then $v' = -f(u) \neq 0$, so the orbit crosses the line $v=0$ transversally. It remains to show that $uf(u)>0$ at such points. Now the orbit cannot go through the point $(0, 0)$, if p is near γ (Proposition 2.2) and, since the orbit cuts $v=0$ at points near b or near γ (cf. the proof of Propositions 2.2), it meets $v=0$ at points where $uf(u)>0$. Thus our claim holds and the result follows.

PROPOSITION 3.4. *If $\theta_0>0$ and p is near γ, then v has k zeros in $[0, T(p))]$ and $\theta_0-(k+1)\pi<\phi_1(T(p), p)<-k\pi$.*

Proof. By the last lemma, v has $(k-1)$ zeros in $(r_1, T(p)]$ and one zero in $[0, r_1]$, so that v has k zeros on $[0, T(p)]$; cf. Fig. 3a.

Now $\phi_1(0)=-\pi/2$, and $\phi_1(r)$ changes by $-\pi$ at every zero of u; thus

$$-\pi/2-k\pi<\phi_1(T(p))<-\pi/2-(k-1)\pi.$$

Since $-\pi/2>\theta_0-\pi$, we have $\phi_1(T(p))>\theta_0-(k+1)\pi$. To obtain the upper bound on $\phi_1(T(p))$, we let $A_j=(u(s_j), 0)$, $0=s_1<s_2<\cdots<s_k=T(p)$ be those points on the orbit segment $(u(r), v(r))$, $0\leqslant r\leqslant T(p)$, which cross $v=0$, and denote the corresponding point on the orbit segment $(v(r), v'(r))$, $0\leqslant r\leqslant T(p)$, by $\tilde{A}_j=(0, v'(s_j))$. (Since p is fixed, we drop the dependence of v and v' on p.) We have shown that the latter orbit cuts $\tilde{\Lambda}$ in at least $(k-1)$ points. We claim that if $v(T(p))<0$ then $\alpha v(T(p))-\beta v'(T(p))<0$, and if $v(T(p))>0$, then $\alpha v(T(p))-\beta v'(T(p))>0$; these will imply that the orbit (v, v') crosses $\tilde{\Lambda}$ in k-points, since the function $\alpha v-\beta v'$ is negative above the line, and positive below it. To see this, note that along the level curve $H(u, v)=v^2/2+F(u)=H(\gamma, 0)$, we have $v^2=2(F(\gamma)-F(u))$; thus (1.3)(v) implies that $uf(u)+v^2>0$ along this level curve. This implies that the boundary line Λ meets this level curve transversally. Indeed, if we compute the derivative $dH/d\mu$ of H in the direction tangent to Λ, we find $dH/d\mu = (\beta, \alpha) \cdot \nabla H = \beta f(u) + \alpha v$. Thus on the line $\beta/\alpha = u/v$ (say $\alpha \neq 0$), $(v/\alpha)\, dH/d\mu = uf(u) + v^2 > 0$ along this level curve; this proves the transverse intersection (a similar proof works if $\beta \neq 0$). Thus, e.g., if $v(T(p)) < 0$ (as in Fig. 3a), then $\alpha u - \beta v$ is strictly decreasing at $r = T(p)$; hence $\alpha v(T(p)) - \beta v'(T(p)) < 0$ as desired. (A similar crossing would occur if $v(T(p)) \leqslant 0$). The orbit (v, v') thus crosses $\tilde{\Lambda}$ k times on $0 \leqslant r \leqslant T(p)$; it clearly cannot cross $\tilde{\Lambda}$ again on this range since this would force v to have $(k+1)$ zeros. Thus $\phi_1(T(p), p) < -k\pi$, and the proof is complete. ∎

We remark that if (1.3)(v) is replaced by the stronger condition (1.4)(ii), then one can give an easier proof of the above claim. Thus if $\bar{r}$ is the largest zero of $u(r)$ ($= u(r, p)$), on $[0, T(p)]$, let $Q = (0, v(\bar{r}))$, $\tilde{Q} = (v(\bar{r}), v'(\bar{r}))$; cf. Fig. 3. Then $v'(\bar{r}) + (n-1)v(\bar{r})/\bar{r} = 0$ so $v(\bar{r})$ and $v'(\bar{r})$ have opposite signs. Thus if $v(\bar{r}) < 0$ (as in Fig. 3a), then $v'(\bar{r}) > 0$ so the orbit (v, v') crosses $\tilde{\Lambda}$ once more going from $\tilde{A}_k$ to $\tilde{Q}$; a similar crossing occurs if $v(\bar{r}) > 0$.

Note that if we do not assume hypothesis (1.3)(v), then we can only assert that (v, v') crosses $\tilde{\Lambda}$ $(k-1)$ times, if $\theta_0 > 0$.

PROPOSITION 3.5. *If $\theta_0 \leqslant 0$, and p is near γ, then v has $(k+1)$ zeros on $[0, T(p)]$, and $\theta_0 - (k + 3/2)\pi < \phi_1(T(p), p) < -(k + \frac{1}{2})\pi$. If hypothesis (1.3)(v) is replaced by (1.4)(ii), then $\theta_0 - (k+1)\pi < \phi_1(T(p), p)$. If $\theta_0 = -\pi/2$, then $-(k + \frac{3}{2})\pi < \phi_1(T(p), p) < -k\pi$.*

Proof. By Lemma 3.3, v has k zeros on $(r_0, T(p)]$ (note that if $\theta_0 > 0$, r_0 does not occur). Since v has one zero in $[0, r_0]$, we see that v has $(k+1)$ zeros on $[0, T(p)]$. Thus $\phi_1(T(p)) \leqslant -(k + \frac{1}{2})\pi$, with equality holding only for the Neumann problem, $\theta_0 = 0$. If $\theta_0 = -\pi/2$, then the argument given in the proof of Proposition 3.3 shows that $\phi_1(T(p), p) < -k\pi$.

Now $\phi_1(T(p), p) > -\pi/2 - (k+1)\pi \geqslant \theta_0 - (k + \frac{3}{2})\pi$, since $-\pi/2 \geqslant \theta_0 - \pi/2$. Assuming (1.4)(ii) to hold, if p is near γ, we can find an $\bar{r} < r_0$ (r_0 is defined

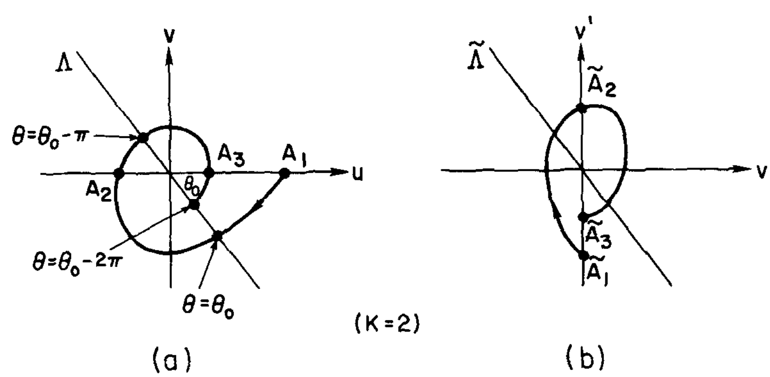

FIGURE 4

in Lemma 3.3), such that $\alpha/\beta > (n-1)/\bar{r}$. We claim that for p near γ, $\phi_1(T(p), p) > \theta_0 - (k+1)\pi$. For, otherwise, we could find an $\tilde{r} = \tilde{r}(p) > \bar{r}$ such that $\alpha v(\tilde{r}) - \beta v'(\tilde{r}) = 0$, and $\tilde{r}$ is maximal with respect to this equality. Thus

$$\left(\frac{\alpha}{\beta} + \frac{n-1}{\tilde{r}}\right) v(\tilde{r}) + f(u(\tilde{r})) = 0.$$

But $\theta_0 < 0$ implies $v(\tilde{r})u(\tilde{r}) < 0$ (cf. Fig. 5) which violates the last equality since $uf(u) > 0$. Similarly if $\theta_0 = 0$, then $v(\tilde{r}) = 0$ and the last equality implies that $f(u(\tilde{r})) = 0$, so $u(\tilde{r}) = 0$, which is again a contradiction since p is near γ. Thus $\phi_1(T(p), p) > \theta_0 - (k+1)\pi$, as asserted. ∎

Remark. Proposition 3.4 shows that if $\theta_0 \geq 0$, and p is near γ, $(v(r, p), v'(r, p))$ crosses $\tilde{\Lambda}$ in k points on $0 \leq r \leq T(p)$. Proposition 3.5 shows that if $\theta_0 \leq 0$, then for p near γ, the orbit crosses $\tilde{\Lambda}$ in at most $(k+1)$ points and exactly k points if hypothesis (1.3)(v) holds.

Now let $N \in \mathbb{Z}$, $N \geq 1$, and consider the eigenvalue equation

$$L_N^p b_\mu = \mu b_\mu, \qquad 0 \leq r \leq T(p),\ \mu \geq 0. \tag{3.2$_N$}$$

Concerning this equation, we have the following lemma.

LEMMA 3.6. *Fix* $N \in \mathbb{Z}$, $N \geq 1$. *Then*:

(i) *if* $\mu > 0$ *is sufficiently large*, μ *cannot be an eigenvalue of* L_N^p;

(ii) *if* b_μ *is a non-zero solution of* $(3.2)_N$ *then for small* $r > 0$, $b_\mu(r)\, b'_\mu(r) > 0$;

(iii) $b_\mu(r) = r^N d(r)$, *where* $d \in C^2$ *and* $d(0) \neq 0$. *Thus if* $N > 1$, $b'_\mu(0) = 0$

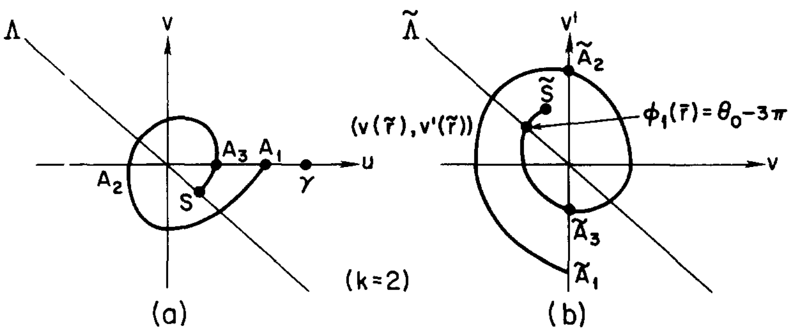

FIGURE 5

Proof. The statement (i) is well known; cf. [8, Chap. 11]. Now, since $(r^{n-1}b'_\mu)' = -r^{n-1}(f'(u)+\lambda_N r^{-2}-\mu)\,b_\mu$, we have $b_\mu(r^{n-1}b'_\mu) = -r^{n-1}(f'(u)+\lambda_N r^{-2}-\mu)\,b_\mu^2$. Integrating from 0 to $r<\varepsilon$ gives

$$r^{n-1}b_\mu(r)\,b'_\mu(r) = \int_0^r s^{n-1}b'_\mu(s)^2\,ds - \int_0^r s^{n-1}\left(f'(u)+\frac{\lambda_N}{s^2}-\mu\right)b_\mu(s)^2\,ds,$$

where the right-hand side is positive if ε is small, since $\lambda_N s^{-2}$ is the dominant term in the second integral. For (iii), writing $b_\mu(r) = r^N d(r)$, we find that d satisfies the equation

$$d'' + \frac{2N+n-1}{r}\,d' + (f'(u)-\mu)\,d = 0,$$

and it is a standard result that this equation has a unique smooth solution satisfying $d(0)=1$, $d'(0)=0$ (see the appendix to [3] for a related result). ∎

Now let $N \geqslant 1$, and let $a_{\mu,N}$ be a (smooth) solution of $(3.2)_N$. We define ϕ_N^μ by

$$\phi_N^\mu(r,p) = \tan^{-1}(a'_{\mu,N}(r,p)/a_{\mu,N}(r,p)), \qquad \text{and set } \phi_N(r,p) = \phi_N^0(r,p). \tag{3.3$_N$}$$

Then if μ is sufficiently large, the last lemma implies that (cf. Fig. 6)

$$0 > \phi_N^\mu(T(p),p) > \theta_0 - \pi. \tag{3.4$_N$}$$

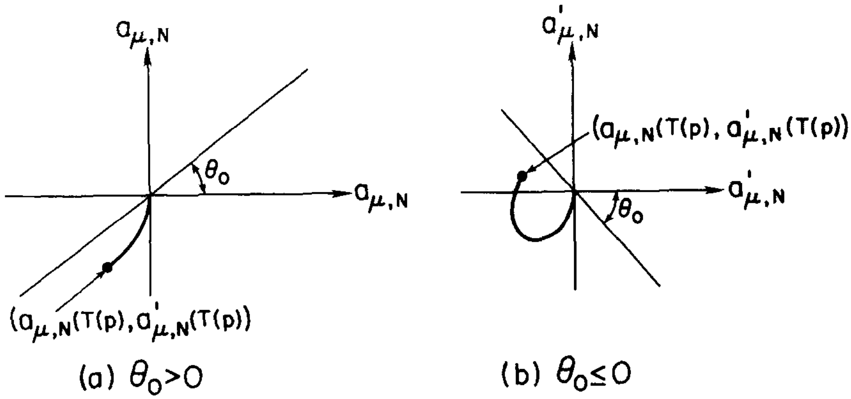

FIGURE 6

The last result we need before giving the proof of Proposition 3.2 is the following.

LEMMA 3.7. *For each fixed r*, $0 < r \leqslant T(p)$, ϕ_N^μ *is a monotone increasing function of* μ.

Proof. Since p is fixed, we will suppress the dependence of ϕ_N^μ on p, and since N is fixed, we shall write ϕ_N^μ simply as ϕ_μ. Now let $\mu > \nu \geqslant 0$, and set $z(r) = \phi_\mu(r) - \phi_\nu(r)$. We are to show that $z(r) > 0$ if $0 < r \leqslant T(p)$. Since N is fixed, we abbreviate $a_{\mu, N}$ by a_μ in this proof.

We have $(r^{n-1}a'_\mu)' = -r^{n-1}(f'(u) + \lambda_N/r^2 - \mu)\, a_\mu$, and $(r^{n-1})\, a'_\nu)' = -r^{n-1}(f'(u) + \lambda_N/r^2 - \nu)\, a_\nu$; then multiply the first equation by a_ν, the second by a_μ and subtract to get

$$[r^{n-1}(a_\nu a'_\mu - a_\mu a'_\nu)]' = r^{n-3} a_\mu a_\nu (\mu - \nu). \tag{3.5}$$

Since we may assume (from Lemma 3.6) that the right-hand side is positive for r near 0, we have after integrating from $r = 0$ to $r = \varepsilon > 0$, $\varepsilon^{n-1}(a_\nu a'_\mu - a_\mu a'_\nu)(\varepsilon) > 0$ so $\varepsilon^{n-1}(a'_\mu/a_\mu - a'_\nu/a_\nu)(\varepsilon) > 0$, or $\operatorname{Tan}\phi_\mu > \operatorname{Tan}\phi_\nu$ on $0 \leqslant r \leqslant \varepsilon$, and so $\phi_\mu > \phi_\nu$ on this range. Thus, $z(r) > 0$ for r near 0. Now let $\bar{r}$ be the first point (if it exists), where $z(\bar{r}) = 0$. Since ϕ_μ satisfies the equation

$$\phi'_\mu = -\sin^2\phi_\mu - \frac{n-1}{2r}\sin^2\phi_\mu - \left(f'(u) + \frac{\lambda_N}{r^2} - \mu\right)\cos^2\phi_\mu,$$

and ϕ_ν satisfies a similar equation (with μ replaced by ν), we have

$$z' = (\sin^2\phi_\mu - \sin^2\phi_\nu) + \frac{n-1}{2r}(\sin 2\phi_\mu - \sin 2\phi_\nu)$$
$$+ \left(f'(u) + \frac{\lambda_N}{r^2}\right)(\cos^2\phi_\mu - \cos^2\phi_\nu) + \mu\cos^2\phi_\mu - \nu\cos^2\phi_\nu.$$

Thus, $z'(\bar{r}) = (\mu - \nu)\cos^2\phi_\mu(\bar{r}) > 0$ if $\cos^2\phi_\mu(\bar{r}) \neq 0$. If $\cos^2\phi_\mu(\bar{r}) = 0$, we compute $z''(\bar{r}) = 0$, and $z'''(\bar{r}) = (\phi'_\mu(\bar{r}))^2 (2\cos 2\phi_\mu(\bar{r}))(\nu - \mu) = (1)^2(-2)(\nu - \mu) > 0$. We conclude no such $\bar{r}$ can exist and the lemma is proved. ∎

Proof of Proposition 3.2. We suppose first that $\theta_0 \geqslant 0$. Since Lemma 3.7 shows that $\phi_1^\mu(T(p), p)$ is a montone continuous function of μ, it follows from Proposition 3.4, and Eq. $(3.4)_1$, that there exist exactly k positive numbers $\mu_1, \mu_2, ..., \mu_k$ such that

$$\phi_1^{\mu_j}(T(p), p) = \theta_0 - j\pi, \qquad j = 1, 2, ..., k;$$

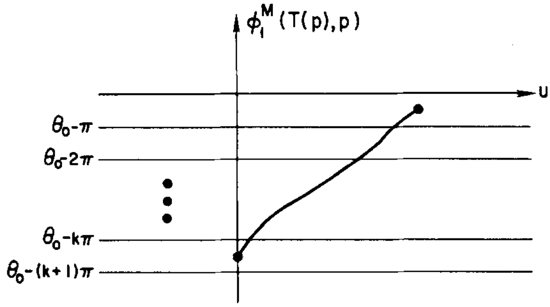

FIGURE 7

cf. Fig. 7. To each such μ_j, there corresponds a function $B_j \not\equiv 0$ such that the following inequalities hold:

$$L_1^p B_j = \mu_j B_j, \qquad 0 < r < T(p)$$
$$B_j(0) = 0 = \alpha B_j(T(p)) - \beta B_j'(T(p)).$$

That is, the μ_j's comprise the set of positive eigenvalues of L_1^p. This proves the theorem if $\theta_0 \geqslant 0$. The same argument works for the case $\theta_0 < 0$; thus the proof of Proposition 3.2 is complete.

We now turn our efforts to showing that $\phi_N(T(p), p) \equiv \phi_N^0(T(p), p)$ (cf. $(3.3)_N$) is close to $\phi_1(T(p), p) \equiv \phi_1^0(T(p), p)$, if p is near γ. This will imply, just as in the proof of Proposition 3.2, that for p near γ, L_N^p has k positive eigenvalues.

From hypothesis (1.3)(i), we know that $f'(u) < 0$ if $A \leqslant u \leqslant \gamma$, for some $A > 0$. Proposition 2.2 says that for p near γ, p lies in the domain of T. Thus the function $p \to T^A(p)$, where

$$u(T^A(p), p) = A \tag{3.6}$$

(here $T^A(p)$ is minimal with respect to this property), is well defined on $A < p < \gamma$, if p is near γ.

LEMMA 3.8. *$T^A(p) \to \infty$ as $p \nearrow \gamma$.*

Proof. This is similar to [5, Lemma 4.2].

We remind the reader that a_N is a (smooth) solution of (2.18), (2.19), and, according to Lemma 3.6, we may assume that $a_N(r) < 0$ on some interval $0 < r < \varepsilon < T^A(p)$. Concerning such functions, we have the following lemma.

LEMMA 3.9. *If $0 < r \leqslant T^A(p)$, then for any $N \geqslant 1$, $a_N'(r) < 0$, and $a_N(r) < 0$.*

Proof. If $0<r<\varepsilon$, we have

$$(r^{n-1}a_N')' = -\left(f'(u(r,p))+\frac{\lambda_N}{r^2}\right)r^{n-1}a_N<0,$$

since $\varepsilon < T^A(p)$. Hence $a_N'<0$ on $0<r<\varepsilon$. Were there a first point $\bar{r}$, $\varepsilon \leqslant \bar{r} \leqslant T^A(p)$, for which $a_N'(\bar{r})=0$, then $a_N(r)<0$ on $0<r\leqslant\bar{r}$ so $(r^{n-1}a_N'(r))'<0$ on this interval. Integrating from 0 to $\bar{r}$ gives the contradiction $a_N'(\bar{r})<0$. Thus $a_N'(r)<0$ if $0<r\leqslant T^A(p)$, and so $a_N(r)<0$ on this interval. ∎

Recall that the angles $\phi_N(r,p)$, $N\geqslant 1$, $0<p<\gamma$ are defined by (cf. $3.3)_N$ with $\mu=0$)

$$\phi_N(r,p)=\tan^{-1}(a_N'(r,p)/a_N(r,p)).$$

LEMMA 3.10. *Let $N>M\geqslant 1$; then*

$$0<\phi_N(T^A(p),p)-\phi_M(T^A(p),p)\leqslant\frac{c}{T^A(p)^2}, \tag{3.7}$$

where $c>0$ is independent of p.

Proof. Since $\phi_N(T^A(p),p) - \phi_M(T^A(p),p) \leqslant \phi_N(T^A(p),p) - \phi_1(T^A(p),p)$, it suffices to prove that (3.7) holds if $M=1$. We have

$$(r^{n-1}a_N')'+\left(f'(u(r,p))+\frac{\lambda_N}{r^2}\right)r^{n-1}a_N=0,$$

$$(r^{n-1}a_1')'+\left(f'(u(r,p))+\frac{\lambda_1}{r^2}\right)r^{n-1}a_1=0.$$

Multiply the first equation by a_1, the second by a_N, and subtract to get

$$[r^{n-1}(a_N'a_1-a_1'a_N)]'=(-\lambda_N+\lambda_1)\,r^{n-3}a_Na_1$$

or

$$[r^{n-1}a_Na_1(\operatorname{Tan}\phi_N-\operatorname{Tan}\phi_1)]'=(\lambda_1-\lambda_N)\,r^{n-3}a_Na_1.$$

If we integrate this from $r=0$ to $r=T^A(p)$, we find

$$(T^A)^{n-1}\,a_N(T^A)\,a_1(T^A)[\operatorname{Tan}\phi_N(T^A)-\operatorname{Tan}\phi_1(T^A)]$$
$$=\int_0^{T^A}(\lambda_1-\lambda_N)\,a_1a_Nr^{n-3}\,dr;$$

this shows that $\operatorname{Tan}\phi_N(T^A(p), p) - \operatorname{Tan}\phi_1(T^A(p), p) > 0$, so that the mean-value theorem gives $\phi_N(T^A(p), p) - \phi_1(T^A(p), p) > 0$. Moreover, dividing this last equation by $(T^A)^{n-1} a_N(T^A) a_1(T^A)$ and using Lemma 3.8, we have (assuming $n > 2$),

$$\begin{aligned}
&\operatorname{Tan}\phi_N(T^A(p), p) - \operatorname{Tan}\phi_1(T^A(p), p) \\
&\quad = \frac{\lambda_1 - \lambda_N}{(T^A)^{n-1}} \int_0^{T^A} \frac{a_N(r)}{a_N(T^A)} \frac{a_1(r)}{a_1(T^A)} r^{n-3}\, dr \\
&\quad \leqslant \frac{\lambda_1 - \lambda_N}{(T^A)^{n-1}} \int_0^{T^A} \frac{r^{n-3}}{a_1(T^A)} a_1(r)\, dr \\
&\quad \leqslant \frac{(\lambda_1 - \lambda_N)}{(T^A)^{n-1}} c_1 \int_0^{T^A} -r^{n-3} a_1(r)\, dr \\
&\quad \leqslant \frac{\lambda_1 - \lambda_N}{(T^A)^2} c_1 \int_0^{T^A} -a_1(r)\, dr
\end{aligned} \tag{3.8}$$

since $-a_1(T_1(p), p) \geqslant 1/c_1$ for p near γ, where $c_1 > 0$ is independent of p. Hence since a_1 satisfies the u' equation (and is a C^2-function), $a_1\, dr = du$,

$$\begin{aligned}
&\operatorname{Tan}\phi_N(T^A(p), p) - \operatorname{Tan}\phi_1(T^A(p), p) \\
&\quad \leqslant c_1 \frac{(\lambda_1 - \lambda_N)}{(T^A)^2} \int_p^A -du \\
&\quad \leqslant \frac{c_1(\lambda_1 - \lambda_N)}{(T^A)^2} (\gamma - A).
\end{aligned}$$

Again using the mean-value theorem, we see that (3.7) (with $M = 1$), holds.

If $n = 2$, write $a_N(r) = r b_N(r)$ ($a_N(0) = 0$ if $N \geqslant 1$ by Lemma 3.6) and note that b_N satisfies the equation

$$(r^3 b_N')' + r^3 \left(\frac{\lambda_N - \lambda_1}{r^2} + f'(u(r, p)) \right) b_N = 0.$$

This shows that $b_N' < 0$ if $0 < r < T^A(p)$. Then as in (3.8) we have

$$\begin{aligned}
&\operatorname{Tan}\phi_N(T^A(p), p) - \operatorname{Tan}\phi_1(T^A(p), p) \\
&\quad = \frac{\lambda_1 - \lambda_N}{(T^A)^2} \int_0^{T^A} \frac{b_N(r)}{b_N(T^A)} \frac{a_1(r)}{a_N(T^A)}\, dr \\
&\quad \leqslant \frac{c_1(\lambda_1 - \lambda_N)}{(T^A)^2} \int_p^A -du \leqslant \frac{c_1(\lambda_1 - \lambda_N)}{(T^A)^2} (A - \gamma),
\end{aligned}$$

and this completes the proof of the lemma. ∎

LEMMA 3.11. *For all* $r>0$, $\phi_N(r, P)>\phi_M(r, p)$ *if* $N>M$.

Proof. For $r \leqslant T^A(p)$, the result is a consequence of the last lemma. If there is a point r with $\phi_N(r, p)=\phi_M(r, p)$ then let $\bar{r}$ be the first such point and set $z(r, p)=\phi_N(r, p)-\phi_M(r, p)$. Then since

$$\phi_i' = -\frac{n-1}{2r}\sin 2\phi_i - \left(f'(u(r, p))+\frac{\lambda_i}{r^2}\right)\cos^2\phi_i - \sin^2\phi_i, \qquad i=M \text{ or } N,$$

we have

$$z' = -\frac{n-1}{2r}(\sin 2\phi_N - \sin 2\phi_M) - f'(u)(\cos^2\phi_N - \cos^2\phi_M) - (\sin^2\phi_N - \sin^2\phi_M) - \frac{\lambda_N}{r^2}\cos^2\phi_N + \frac{\lambda_M}{r^2}\cos^2\phi_M, \tag{3.9}$$

and

$$z'(\bar{r}, p) = \frac{\lambda_M - \lambda_N}{\bar{r}^2}\cos^2\phi_M(\bar{r}, p).$$

It follows that $z'(\bar{r}, p)>0$ if $\phi_M(\bar{r}, p)=\phi_N(\bar{r}, p)$ is not an odd multiple of $\pi/2$, and in this case we see that no such $\bar{r}$ can exist. It remains to show that $\phi_M(\bar{r}, p)=\phi_N(\bar{r}, p)$ cannot be an odd multiple of $\pi/2$. Thus, suppose that $\phi_M(\bar{r}, p)=\phi_N(\bar{r}, p)=-(2j+1)(\pi/2)$; see Fig. 8. Define $\bar{R}$ by $a_N'(\bar{R})=0$; then by hypothesis, $\phi_N(\bar{R}, p)>\phi_M(\bar{R}, p)$. As above, we have

$$[r^{n-1}(a_N' a_M - a_M' a_N)]' = (\lambda_M - \lambda_N)\, r^{n-3} a_N a_M,$$

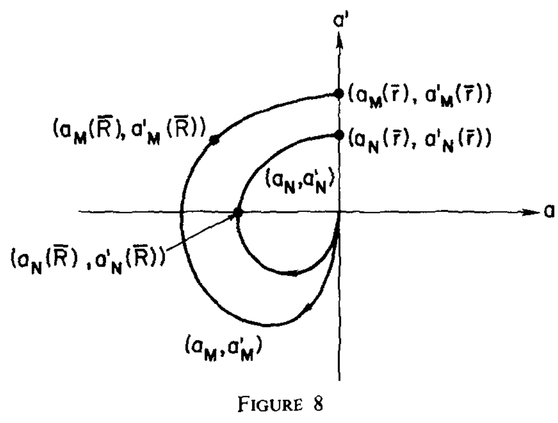

FIGURE 8

so if we integrate this from $r=\bar{R}$ to $r=\bar{r}$, we get

$$r^{n-1}(a_N' a_M - a_M' a_N)|_{\bar{R}}^{\bar{r}} = (\lambda_M - \lambda_N) \int_{\bar{R}}^{\bar{r}} a_M(r)\, a_N(r)\, r^{n-3}\, dr.$$

But as the left-hand side is negative and the right-hand side is positive, we get a contradiction. This completes the proof. ▮

We next show that $\phi_N(T(p), p)$ is close to $\phi_1(T(p), p)$ if p is near γ. In proving this, we shall assume $f(b)<0$; the modifications necessary if $f(b)=0$ will be left to the reader. We consider the region $W=W_+ \cup W_-$ in (u, v)-space, determined by the line $u=A$, and the level curve $H(u, v)=F(\gamma)$, as depicted in Fig. 9.

For orbits $(u(\cdot, p), v(\cdot, p))$ with p near γ, let $T_2(P)$ be defined by $u(T_2(p), p)=A$, $T_2(p)>T_1(p)$, and T_2 is minimal with respect to these properties. (Thus $T_2(p)$ is the second "time" that the orbit meets $u=A$, while $T_1(p)$ denotes the first "time" that it meets this line; cf. (3.6).) Then $T_2(p)-T_1(p)$ denotes the "time" that the orbit spends outside of W as it goes from the line $u=A$ back to itself. The next lemma states that this quantity is bounded independently of p; we defer the proof to the appendix of this paper.

LEMMA 3.12. *There exists a constant $k>0$ and a point p_0, $A<p_0<\gamma$, such that if $p_0 \leqslant p<\gamma$, then $T_2(p)-T_1(p)<k$.*

In what follows, the quantities c, c_i, k_i, and k will denote constants independent of p. We remind the reader that we are assuming with no loss of generality that $f'(\gamma)<-1$; cf. $(*)$ in Section 2.

Now if p is near γ, we see from Lemma 3.10 that $z(r, p)=\phi_N(r, p)-\phi_1(r, p)$ is small, for $0 \leqslant r \leqslant T_1(p)$. Using (3.9), we can write

$$z' = -\frac{n-1}{2r}(\cos 2\xi)z + (1-f'(u))(\sin 2\eta)z$$
$$-\frac{\lambda_N}{r^2}\cos^2\phi_N + \frac{\lambda_1}{r^2}\cos^2\phi_1, \qquad (3.10)$$

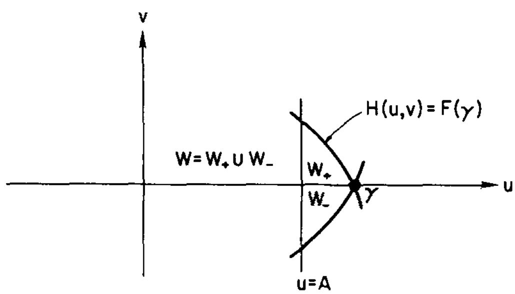

FIGURE 9

for some intermediate points ξ and η. Thus since $z>0$ (Lemma 3.11), we have for $T_1(p)\leqslant r\leqslant T_2(p)$ (cf. Lemma 2.1),

$$z'\leqslant\frac{M_1}{T_1(p)}z+M_2z+\frac{M_3}{T_1(p)^2}\leqslant Mz+\frac{c}{T_1(p)^2}$$

for some constants M_i and M, independent of p. Multiplying this last inequality by e^{-Mr} and integrating from $T_1(p)$ to $r\leqslant T_2(p)$ gives

$$z(r,p)\leqslant z(T_1)\,e^{M(r-T_1)}+\frac{c}{MT_1^2}\,(e^{M(r-T_1)}-1)$$
$$\leqslant\frac{d}{T_1^2}\,e^{Mk}+\frac{c}{MT_1^2}\,(e^{Mk}-1),$$

where we have used Lemmas 3.10 and 3.12. Thus if $T_1(p)\leqslant r\leqslant T_2(p)$,

$$z(r,p)\leqslant\frac{k}{T_1(p)^2},\qquad k \text{ independent of } p. \tag{3.11}$$

Hence ϕ_1 and ϕ_N are close at least until the orbit $(u(r,p),u(r,p))$ enters the region W_+.

Now the "time" that the orbit (u,u') spends in W_+ is *not* uniformly bounded in p, since for p close to γ, the orbit again comes close to γ, and in fact, z increases in W_+. However, we can still control z, as we shall show. In order to do this, we need some notation. Let $\alpha_0(p),\alpha_1(p),\ldots,\alpha_s(p)$ be the u coordinates of the intersection points of the orbit $(u(\cdot,p),u'(\cdot,p))$ with the positive u axis, $\gamma<\alpha_0(p)=p>\alpha_1(p)>\cdots>\alpha_s(p)>A$, and let $T_i^N(p)$ be defined by $u(T_i^N(p),p)=\alpha_i(p)$, $0\leqslant i\leqslant s\leqslant k$. Finally, let $T_j^A(p)$ be defined by $u(T_j^A(p),p)=\alpha_j(p)$, $1\leqslant j\leqslant k$, $T_1(p)=T_1^A(p)<T_2^A(p)=T_2(p)<\cdots<T_s(p)$; see Fig. 10.

Now we shall use the differential equation (3.10) for z and show that although z grows in W_+, $z(T_1^N(p))\leqslant c(\gamma-\alpha_1(p))^{2-\varepsilon}$ for some $\varepsilon<1/k$. There is then a small time (uniformly bounded in p for p near γ), from $T_1^N(p)$ to the time $\tau(p)$, for which $\phi_N(\tau(p))=-\pi/2$, and so $z(\tau)\leqslant c(\gamma-\alpha_1(p))^{2-\varepsilon}$. Then in W_-, z decreases from τ to T_3^A, and we prove an estimate of the form $z(T_3^A(p))\leqslant c(\gamma-\alpha_1(p))^{4-\varepsilon}$. Using the fact that $(\gamma-\alpha_i(p))$ is of the same order as $(\gamma-\alpha_j(p))$ if $i\neq j$, we can repeat the above argument to show that $z(T(p))$ is uniformly small for p near γ. We proceed now with the details.

Let $f'(\gamma)=-\mu$, $\mu>1$ (recall from Section 2 that this entails no loss of generality); then given $\delta>0$, we take u so close to γ that

$$(\mu-\delta)(\gamma-u)\leqslant f(u)\leqslant(\mu+\delta)(\gamma-u). \tag{3.12}$$

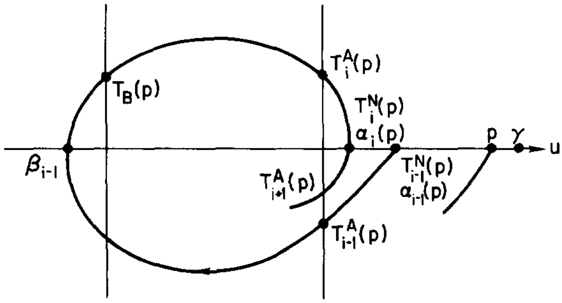

FIGURE 10

We assume that A is so close to γ as to make $f'(u)-f'(\gamma)<\delta$ on $A\leqslant u\leqslant\gamma$. *We also assume from now on, unless otherwise stated, that p is chosen so close to γ as to render all of our previous results valid.*

LEMMA 3.13. *There is a constant $c_1>0$, independent of p such that*

$$(\gamma-\alpha_i(p))^2\,T_{i-1}^A(p)\geqslant c_1.$$

Proof. Let $F(\gamma)-F(A)=3\eta$; choose $p_0<\gamma$ such that $p_0<p<\gamma$ implies $F(p_0)-F(A)\geqslant 2\eta$, and such that the total energy loss along the entire orbit segment $(u(r,p),u'(r,p))$, $0\leqslant r\leqslant T(p)$ is at most η (cf. Proposition 2.2). Finally, let $B<0$ be chosen so that $F(B)=F(A)$. Then

$$F(\alpha_{i-1})-F(\alpha_i)=(n-1)\int_{T_{i-1}^N}^{T_i^N}\frac{v^2}{r}\,dr$$

$$\geqslant\int_{T_{i-1}^A}^{T_B}\frac{v^2}{r}\,dr\geqslant\frac{1}{T_B}\int_{T_{i-1}^A}^{T_B}v^2\,dr,$$

where $T_B=T_B(p)>T_{i-1}^A(p)=T_{i-1}^A$ is the first time that the orbit meets $u=B$ on its journey from $u=A$ (cf. Fig. 10). Thus

$$F(\alpha_{i-1})-F(\alpha_i)\geqslant\frac{1}{T_B}\int_B^A -v\,du\geqslant\int_B^A\frac{\sqrt{2(F(p)-F(u))-\eta}}{T_B}\,du$$

$$\geqslant\frac{(A-B)\sqrt{3\eta}}{T_B}\geqslant\frac{(A-B)\sqrt{3\eta}}{2T_{i-1}^A},$$

since for p near γ, $T_B(p)-T_{i-1}^A(p)\leqslant$ const., independent of p (cf. the appendix). Next

$$F(\alpha_{i-1})-F(\alpha_i)=\int_{\alpha_i}^{\alpha_{i-1}}f(u)\,du\leqslant\int_{\alpha_i}^{\alpha_{i-1}}(\mu+\delta)(\gamma-u)\,du$$

$$=\frac{(\mu+\delta)}{2}\,[(\gamma-\alpha_i)^2-(\gamma-\alpha_{i-1})^2].$$

It follows that

$$\frac{(\mu+\delta)}{2}\left[(\gamma-\alpha_i)^2-(\gamma-\alpha_{i-1})^2\right] \geqslant \frac{(A-B)\sqrt{3\eta}}{2T_{i-1}^A},$$

so that

$$(\gamma-\alpha_i)^2\, T_{i-1}^A \geqslant \frac{(A-B)\sqrt{3\eta}}{(\mu+\delta)},$$

and this proves the lemma. ∎

LEMMA 3.14. *There is a constant c_2, independent of p such that*

$$T_i^N(p)-T_i^A(p) \leqslant c_2 - \frac{1}{\sqrt{\mu-\delta}} \ln(\gamma-\alpha_i(p)). \tag{3.13}$$

Proof.

$$T_i^N(p)-T_i^A(p)=\int_A^{\alpha_i(p)} \frac{du}{v},$$

and as $F(u)+v^2/2 \geqslant F(\alpha_i(p))$, we have

$$v \geqslant \sqrt{2(F(\alpha_i)-F(u))} \geqslant \sqrt{(\mu-\delta)\left[(\gamma-u)^2-(\gamma-\alpha_i)^2\right]}.$$

Thus

$$\begin{aligned} T_i^N(p)-T_i^A(p) &\leqslant \int_A^{\alpha_i} \frac{du}{\sqrt{\mu-\delta}\sqrt{(\gamma-u)^2-(\gamma-\alpha_i)^2}} \\ &= \frac{1}{\sqrt{\mu-\delta}} \cosh^{-1}\left(\frac{\gamma-A}{\gamma-\alpha_i}\right) \\ &= \frac{1}{\sqrt{\mu-\delta}} \ln 2\left(\frac{\gamma-A}{\gamma-\alpha_i}\right) \\ &= c_2 - \frac{1}{\sqrt{\mu-\delta}} \ln(\gamma-\alpha_i), \end{aligned}$$

as desired. ∎

Now let $z(r,p)=\phi_N(r,p)-\phi_1(r,p)$; then from (3.10) with $M=1$, we have $z \geqslant 0$ and

$$\begin{aligned} z' = {} & \frac{n-1}{2r}(\sin 2\phi_1-\sin 2\phi_N)-\left(-1+f'+\frac{\lambda_1}{r^2}\right) \\ & \times(\cos^2\phi_N-\cos^2\phi_1)-\left(\frac{\lambda_N-\lambda_1}{r^2}\right)\cos^2\phi_N, \end{aligned} \tag{3.14}$$

where $f' = f'(u(r, p))$. The mean-value theorem gives

$$z' = \frac{n-1}{r}(\cos 2\tau)z + (1-f')(-\sin 2\xi)z + O\left(\frac{1}{r^2}\right).$$

Now for any $\delta > 0$, we can choose p so close to γ as to have $-f' < \mu + \delta$ and $(n-1)(\cos 2\eta)/r < \delta$ for $r > T_1(p)$ (cf. Fig. 11); δ will be chosen below. Then

$$z' \leqslant (\mu + 1 + 2\delta)(-\sin 2\xi)z + O(1/r^2). \tag{3.15}$$

In order to analyze this equation, we choose certain constants, as follows:

let $\varepsilon < 1/k$ (recall that k denotes the given nodal class);

let $\sigma < \frac{1}{2}$, and choose $\delta > 0$ so that the following hold:

$$\frac{2(\mu+\delta)}{\mu-\delta} > 2 - \varepsilon, \tag{3.16}$$

$\mu - \delta > 1$ (recall $\mu > 1$), and

$$2 + \varepsilon > \frac{(2+\varepsilon/2)(1+\mu+2\delta)}{1+\mu-\delta}. \tag{3.17}$$

(Then we choose A so close to γ that (3.12) is valid.)

We claim that for p near γ (cf. Fig. 11),

$$\sin(-2\xi) \leqslant \frac{\left(2+\frac{\varepsilon}{2}\right)\sqrt{\mu-\delta}}{1+\mu-\delta}, \qquad T_2^A(p) \leqslant r \leqslant T_1^N(p). \tag{3.18}$$

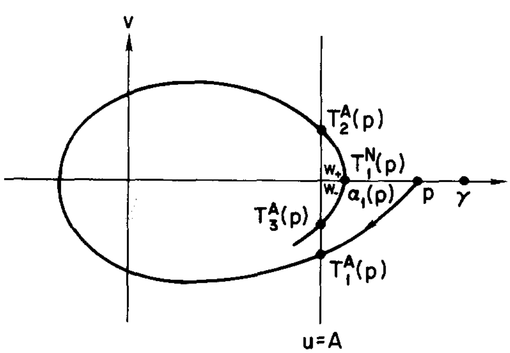

FIGURE 11

Now assuming (3.18), we shall show that

$$z(R) \leqslant c(\gamma - \alpha_1(p))^{2-\varepsilon}, \qquad T_2^A(p) \leqslant R \leqslant T_1^N(p). \tag{3.19}$$

Thus, (3.14) and (3.18) give

$$z' \leqslant Mz + \frac{k}{r^2}, \qquad M = \frac{(1+\mu+2\delta)(2+\varepsilon/2)\sqrt{\mu-\delta}}{1+\mu-\delta},$$

so that for $T_2^A(p) \leqslant R \leqslant T_1^N(p)$,

$$\begin{aligned} z(R) &\leqslant e^{M(R-T_2)} z(T_1) + e^{MR} \int_{T_2}^{R} e^{-Mr} \frac{k}{r^2}\, dr \\ &\leqslant e^{M(R-T_2)} z(T_2) + \frac{k_1 e^{MR}}{T_2^2} \frac{e^{-Mr}}{-M}\bigg|_{T_2}^{R} \\ &\leqslant e^{M(R-T_2)} z(T_2) + \frac{k_1}{MT_2^2} (e^{M(R-T_2)} - 1) \\ &\leqslant e^{M(R-T_2)} z(T_2) + \frac{k_1}{MT_2^2} e^{M(R-T_2)} \\ &\leqslant e^{M(R-T_2)} \frac{k_2}{T_2^2} + \frac{k_1}{MT_2^2} e^{M(R-T_2)}, \end{aligned}$$

where we have used (3.11) and we denoted T_2^A by T_2. Thus

$$\begin{aligned} z(R) &\leqslant \frac{k}{T_2^2} e^{M(T_1^N - T_2)} \leqslant \frac{c}{T_1^2} e^{M(T_1^N - T_2)} \\ &\leqslant \frac{c}{c_1^2} (\gamma - \alpha_1)^4 e^{M(c_2 - (1/\sqrt{\mu-\delta}) \ln(\gamma - \alpha_1))}, \end{aligned}$$

where we have used Lemma 3.13 and (3.13). It follows that

$$\begin{aligned} z(R) &\leqslant c_3(\gamma - \alpha_1)^{4 - (2+\varepsilon/2)(1+\mu+2\delta)/(1+\mu-\delta)} \\ &\leqslant c(\gamma - \alpha_1)^{2-\varepsilon}, \end{aligned}$$

in view of (3.17); this is (3.19). It remains to prove (3.18).

We have

$$\begin{aligned} \phi_1' &= -\frac{n-1}{2r} \sin 2\phi_1 - \left(f'(u) + \frac{\lambda_1}{r^2} \right) \cos^2 \phi_1 - \sin^2 \phi_1 \\ &= -\cos^2 \phi_1 \left[\tan^2 \phi_1 + \frac{n-1}{r} \tan \phi_1 + \left(f'(u) + \frac{\lambda_1}{r^2} \right) \right], \end{aligned}$$

so that if $T_2^A(p) \leqslant r < T_1^N(p)$, $\cos^2 \phi_1 \neq 0$ so $\phi_1' = 0$ when the above term in square brackets vanishes. That is, $\phi_1' = 0$ on $T_2^A(p) \leqslant r < T_1^N(p)$ if $\phi_1 = \bar{\phi}_1(r)$, where

$$-\operatorname{Tan} \bar{\phi}_1 = \frac{n-1}{2r} + \sqrt{((n-1)/2r)^2 - (f'(u) + \lambda_1/r^2)}$$
$$\geqslant \sqrt{-f'(u)} \geqslant \sqrt{\mu - \delta}.$$

Now if $-\operatorname{Tan} \phi_1 < \sqrt{\mu - \delta}$, then $\phi_1 > \bar{\phi}_1$ so $\phi_1' > 0$ and hence $\phi_1(r)$ could never reach $-\pi/2$. It follows that $-\operatorname{Tan} \phi_1 \geqslant \sqrt{\mu - \delta}$, on the interval.

Since we have assumed $\mu > 1$,

$$-\sin 2\phi_1 = \frac{-2 \operatorname{Tan} \phi_1}{1 + \operatorname{Tan}^2 \phi_1} \leqslant \frac{2\sqrt{\mu - \delta}}{1 + \mu - \delta}$$

because $\phi(x) = x(1 + x^2)^{-1}$ is decreasing if $x \geqslant 1$. Let

$$R(p) = \sup \left\{ r \in [T_2^A(p), T_1^N(p)]: -\sin 2\xi \leqslant \frac{(2 + \varepsilon/2)\sqrt{\mu - \delta}}{1 + \mu - \delta} \right\},$$

and note that the above set is non-void, since if $r = T_2^A(p)$ then z is small, so ξ is close to ϕ_1 for p near ξ. Now $\sin(-2\xi) \leqslant \sin(-2\phi_1) + 2(\xi - \phi_1) \leqslant \sin(-2\phi_1) + 2z$. If $R(p) < T_N(p)$, then on some subinterval of $[T_2^A(p), T_1^N(p))$,

$$\sin(-2\xi) > \frac{(2 + \varepsilon)\sqrt{\mu - \delta}}{1 + \mu - \delta}$$

so

$$2z \geqslant \frac{\left(2 + \frac{\varepsilon}{2}\right)\sqrt{\mu - \delta}}{1 + \mu - \delta} - \frac{2\sqrt{\mu - \delta}}{1 + \mu - \delta} = \frac{\varepsilon/2}{1 + \mu - \delta}.$$

But this is impossible as $z \to 0$ if p is near γ on any such subinterval, in view of (3.19) with $R = R(p)$. Thus $R(p) = T_N(p)$, and this proves that (3.18) holds so that (3.19) also holds.

We now consider $z(r, p)$ for $r > T_1^N(p)$; see Fig. 11. So, let $r > T_1^N(p)$; we claim that

$$\phi_N(T_1^N(p) + 2z(T_1^N(p))) < -\pi/2. \tag{3.20}$$

Thus

$$\phi_N(T_1^N + 2z(T_1^N)) - \phi_N(T_1^N) = 2z\phi_N'(\tau),$$

for some intermediate point τ, $T_1^N(p) \leqslant \tau \leqslant T_1^N(p) + 2z(T_1^N(p))$. Since

$$\phi_N'(r, p) = -\frac{\sin 2\phi_N}{2r} - \left(f' + \frac{\lambda_N}{r^2}\right)\cos^2\phi_N - \sin^2\phi_N,$$

and $z(T_1^N(p))$ is small if p is near γ, $\phi_N(\tau)$ is near $\phi_1(\tau)$, which in turn is approximately equal to $\phi_1(T_1^N(p)) = -\pi/2$. Thus, $\phi_N'(\tau, p) < -\frac{1}{2}$ if p is near γ. It follows that

$$\begin{aligned}\phi_N(T_1^N + 2z(T_1^N)) - \left(z(T_1^N) - \frac{\pi}{2}\right)\\ = \phi_N(T_1^N + 2z(T_1^N)) - \phi_N(T_1^N) < -z(T_1^N)\end{aligned}$$

and this proves (3.20). Hence $\phi_N(\tau(p), p) = -\pi/2$ for some $\tau(p)$, $T_1^N(p) \leqslant \tau(p) \leqslant T_1^N(p) + 2z(T_1^N(p))$. It follows that $\tau(p)$ is uniformly bounded in p. Moreover,

$$z(\tau(p), p) \leqslant c(\gamma - \alpha_1(p))^{2-\varepsilon}, \tag{3.21}$$

since $\tau(p) - T_1^N(p)$ is uniformly bounded (this is the same argument that led from (3.7) to (3.10)).

We claim that

$$z(T_3^A(p), p) < k(\gamma - \alpha_1(p))^{4-\varepsilon}. \tag{3.22}$$

First we show that $T_2^A(p) > \tau(p)$; this is easy since

$$T_3^A(p) - T_1^N(p) = \int_A^{\alpha_1} \frac{du}{-v} \geqslant \frac{\alpha_1 - A}{M},$$

where M is an upper bound for $-v$. Thus $T_3^A(p)$ is uniformly bounded away from $T_1^N(p)$; on the other hand, for p near γ, $\tau(p) \to T_1^N(p)$. It follows that $T_3^A(p) > \tau(p)$ if p is near to γ.

We now show (3.22). We have (cf.(3.9))

$$\begin{aligned}z' = -\frac{n-1}{2r}\left(\frac{\sin 2\phi_N - \sin 2\phi_1}{\phi_N - \phi_1}\right) z - \left(f' + \frac{\lambda_1}{r^2} - 1\right)\\ \times \left(\frac{\cos^2\phi_N - \cos^2\phi_1}{\phi_N - \phi_1}\right) z + \frac{\lambda_N - \lambda_1}{r^2}\cos^2\phi_N.\end{aligned}$$

If we define b by

$$b(r) = -\frac{n-1}{2r}\left(\frac{\sin 2\phi_N - \sin 2\phi_1}{\phi_N - \phi_1}\right) - \left(f' + \frac{\lambda_1}{r^2} - 1\right)\left(\frac{\cos^2\phi_N - \cos^2\phi_1}{\phi_N - \phi_1}\right),$$

we have

$$z' \leqslant b(r)z + k/r^2.$$

Let $B'(r) = b(r)$; then multiplying the z' equation by $e^{B(r)}$ and integrating from $r = T_1^N(p)$ to $r = T_3^A(p)$ gives

$$z(T_3^A(p), p) \leqslant e^{B(T_1^N) - B(T_3^A)} z(T_1^N) + k \int_{T_1^N}^{T_3^A} \frac{e^{B(r) - B(T_3^A)}}{r^2} \, dr. \tag{3.23}$$

We shall first show that

$$\exp[B(T_1^N(p)) - B(T_3^A(p))] \leqslant c(\gamma - \alpha_1(p))^{2-\varepsilon}. \tag{3.24}$$

To this end, note that we may write

$$b(r) = -\frac{n-1}{r} \cos 2\xi - \left(f' + \frac{\lambda_1}{r^2} - 1\right) [\sin 2\phi_1 - (\cos 2\eta) z],$$

for some intermediate points ξ, η. Also, if $T_1^N \leqslant r \leqslant T_3^A$,

$$\begin{aligned} v'(r) &= -\frac{n-1}{r} v - f(u) \\ &\leqslant -\frac{n-1}{T_1^N} v - (\mu - \delta)(\gamma - u) \\ &\leqslant -\frac{n-1}{T_1^N} v - (\mu - \delta)(\gamma - \alpha_1) \\ &= \frac{-(n-1)v - (\mu - \delta)(\gamma - \alpha_1)\, T_1^N}{T_1^N}. \end{aligned}$$

But from Lemma 3.13, $(\gamma - \alpha_1)\, T_1^N \geqslant (\gamma - \alpha_1)\, T_2^A \geqslant c_1/(\gamma - \alpha_1)$, so that $v' < 0$ on this interval, if p is near γ. Hence $\operatorname{Tan} \phi_1 > 0$ on this interval so that if $T_1^N \leqslant r \leqslant T_3^A$,

$$\sin 2\phi_1 = \frac{2 \operatorname{Tan} \phi_1}{1 + \operatorname{Tan}^2 \phi_1} > 0.$$

Also,

$$\begin{aligned} &\int_{T_1^N}^{T_3^A} -\frac{n-1}{r} \cos(2\xi)\, dr \\ &\geqslant \int_{T_1^N}^{T_3^A} -\frac{n-1}{r}\, dr = (1-n) \ln\left(\frac{T_3^A}{T_1^N}\right) \\ &= (1-n) \ln\left(\frac{T_1^N + D}{T_1^N}\right) \qquad (D = T_3^A - T_1^N) \\ &= (1-n) \ln(1 + D/T_1^N) \\ &\geqslant (1-n)\, D/T_1^N. \end{aligned}$$

But if $Q(p)=v(T_3^A(p),p)$, then $Q(p)<0$ and

$$D=\int_0^Q \frac{dv}{v'}=\int_0^Q dv \Big/ \left(-\frac{n-1}{r}v-f(u)\right)$$
$$=\int_Q^0 dv \Big/ \left(\frac{n-1}{r}v+f(u)\right)$$
$$\leqslant \int_Q^0 dv \Big/ \left(\frac{n-1}{r}v-(\mu-\delta)(\gamma-u)\right)$$
$$\leqslant \int_Q^0 dv \Big/ \left(\frac{1}{2}(\mu-\delta)(\gamma-u)\right),$$

if p is near γ (since v is bounded; cf. Lemma 2.1). Hence

$$D\leqslant \int_Q^0 dv \Big/ \left(\frac{1}{2}(\mu-\delta)(\gamma-\alpha_1)\right)=\frac{2|Q|}{(\mu-\delta)}\frac{1}{(\gamma-\alpha_1)}. \tag{3.25}$$

It follows from Lemma 3.13 that

$$\int_{T_1^N}^{T_3^A} -\frac{n-1}{r}\cos(2\xi)\,dr \geqslant (1-n)\frac{2|Q|}{(\mu-\delta)\,T_1^N(\gamma-\alpha_1)}$$
$$\geqslant \frac{2|Q|\,(1-n)}{(\mu-\delta)}\frac{1}{T_3^A(\gamma-\alpha_1)}$$
$$\geqslant \frac{2(1-n)\,|Q|\,c_1}{(\mu-\delta)}(\gamma-\alpha_1);$$

thus from Lemma 2.1 again,

$$\int_{T_3^A}^{T_1^N} -\frac{n-1}{r}\cos(2\xi)\,dr \leqslant \frac{4(n-1)\,M_0c_1}{(\mu-\delta)}(\gamma-\alpha_1). \tag{3.26}$$

Next,

$$\int_{T_3^A}^{T_1^N} -\left(f'+\frac{\lambda_1}{r^2}-1\right)\cos(2\eta)z\,dr$$
$$=\int_{T_1^N}^{T_3^A}\left(-f'-\frac{\lambda_1}{r^2}+1\right)\cos(2\eta)z\,dr$$
$$\leqslant \int_{T_1^N}^{T_3^A}\left(-f'-\frac{\lambda_1}{r^2}+1\right)z\,dr,$$
$$=\int_{T_1^N}^{\tau}+\int_{\tau}^{T_3^A},$$

where τ is defined as above by $\phi_N(\tau(p), p) = -\pi/2$. Now on $\tau \leqslant r \leqslant T_3^A$, if p is close to γ,

$$z' = -\frac{n-1}{r}\cos(2\xi) + \left(f' + \frac{\lambda_1}{r^2} + 1\right)\sin(2\Delta) + \frac{\lambda_N - \lambda_1}{r^2}\cos^2\phi_N,$$

where $-\pi \leqslant \phi_1 \leqslant \xi_1\ \Delta \leqslant \phi_N \leqslant -\pi/2$ (since

$$\tan\phi_1 = \frac{v'}{v} = -\frac{n-1}{r} - \frac{f(u)}{v} < 0$$

if r is large), we see that $\sin 2\Delta > 0$, and hence $z' < 0$ on this r interval. Hence

$$\begin{aligned}\int_\tau^{T_3^A}\left(-f' - \frac{\lambda_1}{r^2} - 1\right) z\,dr &\leqslant kz(\tau)(T_3^A - \tau)\\ &\leqslant c(\gamma - \alpha_1)^{2-\varepsilon} D\\ &\leqslant k(\gamma - \alpha_1)^{1-\varepsilon},\end{aligned}$$

where we have used (3.21) and (3.25). Moreover,

$$\int_{T_1^N}^{\tau}\left(-f' - \frac{\lambda_1}{r^2} + 1\right) z\,dr \leqslant k(\tau - T_1^N).$$

But as we have seen above, $T_1^N \leqslant \tau \leqslant T_1^N + 2z(T_1^N)$, so $\tau - T_1^N \leqslant 2z(T_1^N) \leqslant c(\gamma - \alpha_1)^{2-\varepsilon}$, from (3.19). Thus,

$$\begin{aligned}&\int_{T_3^A}^{T_1^N}\left(f' + \frac{\lambda_1}{r^2} - 1\right)\cos(2\eta) z\,dr\\ &\qquad \leqslant k_1(\gamma - \alpha_1)^{1-\varepsilon} + k_2(\gamma - \alpha_1)^{2-\varepsilon}. \qquad (3.27)\end{aligned}$$

Next, as $\sin 2\phi_1 > 0$ on $T_3^A \leqslant r \leqslant T_1^N$, if p is near γ,

$$\begin{aligned}&\int_{T_3^A}^{T_1^N}\left(-f' - \frac{\lambda_1}{r^2} + 1\right)\sin 2\phi_1\,dr\\ &\quad = \int_{T_1^N}^{T_3^A}\left(f' + \frac{\lambda_1}{r^2} + 1\right)\sin 2\phi_1\,dr\\ &\quad < \left(\frac{\lambda_1}{(T_3^A)^2} + 1 + \mu - \delta\right)\int_{T_1^N}^{T_3^A}\sin 2\phi_1\\ &\quad = \rho\int_{T_1^N}^{T_3^A}\frac{-2vv'}{v^2 + v'^2}\,dr \qquad \left(\rho = \frac{\lambda_1}{(T_3^A)^2} + 1 + \mu - \delta\right)\\ &\quad = 2\rho\int_{\alpha_1}^{A}\frac{-v'\,du}{v^2 + v'^2}\end{aligned}$$

$$= 2\rho \int_{\alpha_1}^{A} \frac{((n-1)/r)v + f(u)}{v^2 + (((n-1)/r)v)^2 + f^2 + (2(n-1)/r)\, vf} du$$

$$\leqslant 2\rho \int_{\alpha_1}^{A} \frac{(1+\sigma) f(u)\, du}{(v^2+f^2)/2} \qquad (\sigma \text{ is near } 0 \text{ for } p \text{ near } \gamma)$$

$$\leqslant 4\rho(1+\sigma) \int_{\alpha_1}^{A} \frac{f(u)\, du}{2(F(\alpha_1) - F(u)) + f(u)^2}$$

$$\leqslant 4\rho(1+\sigma) \int_{\alpha_1}^{A} \frac{(\mu+\delta)(\gamma-u)\, du}{(\mu-\delta)[(\gamma-u)^2 - (\gamma-\alpha_1)^2] + (\mu-\delta)^2 (\gamma-u)^2}$$

$$= 4\rho(1+\sigma) \frac{(\mu+\delta)}{(\mu-\delta)} \int_{\alpha_1}^{A} \frac{(\gamma-u)\, du}{(1+\mu-\delta)(\gamma-u)^2 - (\gamma-\alpha_1)^2}$$

$$= -4\rho(1+\sigma) \frac{(\mu+\delta)}{(\mu-\delta)} \frac{1}{2(1+\mu-\delta)} \ln[(\mu+1-\delta)(\gamma-u)^2 - (\gamma-\alpha_1)^2]\Big|_{\alpha_1}^{A}$$

$$= k - 4\rho(1+\sigma) \frac{(\mu+\delta)}{(\mu-\delta)} \frac{1}{(1+\mu-\delta)} \frac{1}{2} \ln[(\mu+1-\delta)(\gamma-\alpha_1)^2 - (\gamma-\alpha_1)^2]$$

$$= k + 4\rho(1+\sigma) \frac{\mu+\delta}{\mu-\delta} \frac{1}{(1+\mu-\delta)} \frac{1}{2} \ln\, (\mu-\delta)(\gamma-\alpha_1)^2$$

$$= k' + 4\rho(1+\sigma) \frac{\mu+\delta}{\mu-\delta} \frac{1}{(1+\mu-\delta)} \ln(\gamma-\alpha_1)$$

$$= k' + 4(1+\sigma) \frac{\mu+\delta}{\mu-\delta} \frac{\lambda_1/(T_3^A)^2 + 1 + \mu - \delta}{(1+\mu-\delta)} \ln(\gamma-\alpha_1).$$

Now since $4(\mu+\delta)/(\mu-\delta) > 2-\varepsilon$ (by (3.16)), we see that for p near γ (so that σ is near 0),

$$4(1-\sigma) \frac{\mu+\delta}{\mu-\delta} \frac{\lambda_1/(T_3^A)^2 + 1 + \mu - \delta}{1+\mu-\delta} > 2-\varepsilon,$$

and thus

$$\int_{T_3^A}^{T_1^N} \left(-f' - \frac{\lambda_1}{r^2} + 1\right) \sin 2\phi_1\, dr \leqslant k + (2-\varepsilon) \ln(\gamma-\alpha_1).$$

If we now combine this with (3.26) and (3.27), we find, for p near γ,

$$B(T_1^N(p)) - B(T_3^A(p)) \leqslant k + \ln(\gamma-\alpha_1)^{2\varepsilon}, \tag{3.28}$$

so that (3.24) holds. Thus from (3.19) and (3.23)

$$z(T_3^A) \leqslant c(\gamma-\alpha_1)^{4-\varepsilon} + k \int_{T_1^N}^{T_3^A} \frac{\exp[B(r) - B(T_3^A)]\, dr}{r^2}. \tag{3.29}$$

But from arguments similar to those which led to (3.28), we have, for $T_1^N(p) \leqslant r \leqslant T_3^A(p)$,

$$\exp[B(r) - B(T_3^A)] \leqslant k(\gamma - \alpha_1)^{2-\varepsilon}$$

if p is near γ. Thus

$$\begin{aligned}\int_{T_1^N}^{T_3^A} \frac{\exp[B(r) - B(T_3^A)]}{r^2}\, dr &\leqslant k(\gamma - \alpha_1)^{2-\varepsilon} \int_{T_1^N}^{T_3^A} \frac{dr}{r^2} \\ &= k(\gamma - \alpha_1)^{2-\varepsilon} \frac{(T_3^A - T_1^N)}{T_3^A T_1^N} \\ &\leqslant k(\gamma - \alpha_1)^{1-\varepsilon} \frac{1}{T_3^A T_1^N},\end{aligned}$$

where we have used (3.25). But from Lemma 3.13, $T_4^A(p)(\gamma - \alpha_2(p))^2 \geqslant c_1$, and from Lemma 3.12, $T_4^A(p) - T_3^A(p)$ is uniformly bounded. Thus $T_3^A(p)(\gamma - \alpha_2(p))^2 \geqslant k$, so that

$$\int_{T_1^N}^{T_3^A} \frac{\exp[B(r) - B(T_3^A)]}{r^2} \leqslant k(\gamma - \alpha_1)^{1-\varepsilon} (\gamma - \alpha_2)^2 \frac{1}{T_1^N}.$$

But again from Lemma 3.13, $T_1^N(p)(\gamma - \alpha_1)^2 \geqslant T_2^A(p)(\gamma - \alpha_1)^2 \geqslant c_1$; thus $(\gamma - \alpha_1)^2 \geqslant c_1/T_1^N(p)$, and this gives

$$\int_{T_1^N}^{T_3^A} \frac{\exp[B(r) - B(T_3^A)]}{r^2}\, dr \leqslant k(\gamma - \alpha_1)^{3-\varepsilon} (\gamma - \alpha_2)^2.$$

If we show that

$$c \geqslant \left(\frac{\gamma - \alpha_2(p)}{\gamma - \alpha_1(p)}\right)^2 - 1, \tag{3.30}$$

this will imply

$$\int_{T_1^N}^{T_3^A} \frac{\exp[B(r) - B(T_3^A)]}{r^2}\, dr \leqslant k(\gamma - \alpha_1)^{5-\varepsilon},$$

which, together with (3.29), will prove (3.24).

To show (3.30), we have

$$\begin{aligned}F(\alpha_1) - F(\alpha_2) = \int_{\alpha_2}^{\alpha_1} f(u)\, du &\geqslant (\mu - \delta) \int_{\alpha_2}^{\alpha_1} (\gamma - u) \\ &= \frac{\mu - \delta}{2} [(\gamma - \alpha_2)^2 - (\gamma - \alpha_1)^2],\end{aligned}$$

and, on the other hand, from energy considerations,

$$F(\alpha_1)-F(\alpha_2)=(n-1)\int_{T_1^N}^{T_2^N}\frac{v^2}{r}\,dr\leqslant\frac{n-1}{T_1^N}\int_{T_1^N}^{T_2^N}v^2\,dr$$
$$=\frac{n-1}{T_1^N}\left[\int_{\alpha_1}^{\beta_1}v\,du+\int_{\beta_1}^{\alpha_2}v\,du\right],$$

where $\beta_1=\beta_1(p)$ is such that $v(\beta_1(p),p)=0$, $T_1^N(p)<\beta_1(p)<T_2^N(p)$; cf. Fig. 10. Thus, with b defined in (1.3),

$$F(\alpha_1)-F(\alpha_2)=\frac{n-1}{T_1^N}\left[\int_{\beta_1}^{\alpha_1}-v\,du+\int_{\beta_1}^{\alpha_2}v\,du\right]$$
$$\leqslant\frac{(n-1)}{T_1^N}k(\gamma-b).$$

It follows that

$$\frac{c}{T_2^A}\geqslant\frac{k}{T_1^N}\geqslant(\gamma-\alpha_2)^2-(\gamma-\alpha_1)^2\geqslant 0,$$

so that from Lemma 3.13,

$$\frac{c}{c_1}\geqslant\frac{c}{T_2^A(\gamma-\alpha_1)^2}\geqslant\left(\frac{\gamma-\alpha_2}{\gamma-\alpha_1}\right)^2-1\geqslant 0,$$

and this gives (3.30).

We summarize what has been proved so far. Referring to Fig. 11, there are four steps:

(i) From $u=p$ to $u=A$ $(0<r\leqslant T_1^A(p))$,

$$z(T_1^A(p),p)<\frac{\text{const}}{T_1^A(p)^2}\qquad\text{(by Lemma 3.10)}.$$

(ii) From $u=A$ to $u=A$ $(T_1^A(p)\leqslant r\leqslant T_2^A(p))$,

$$z(r,p)\leqslant\text{const}\,z(T_1^A(p),p)\qquad\text{(by (3.11))}.$$

(iii) In W_+ $(T_2^A(p)\leqslant r\leqslant T_1^N(p))$,

$$z(r,p)\leqslant\text{const}[z(T_2^A(p),p)+1/T_2^A(p)^2]\exp M(r-T_2^A(p))$$
$$\leqslant\text{const}(\gamma-\alpha_1(p))^{2-\varepsilon}\qquad\text{(by (3.19) ff.)}.$$

(iv) In W_- $(T_1^N(p) \leqslant r \leqslant T_3^A(p))$,

$$z(r, p) \leqslant \text{const}\ z(T_1^N(p), p)(\gamma - \alpha_1(p))^{2-\varepsilon}$$
$$\leqslant (\gamma - \alpha_1(p))^{4-\varepsilon} \qquad \text{(by (3.22)–(3.30))}.$$

Now we may repeat this procedure, namely (using (3.30)),

(i)′ From $u = A$ to $u = A$ $(T_3^A(p) \leqslant r \leqslant T_4^A(p))$,

$$z(r, p) \leqslant \text{const}\ z(T_3^A(p), p) \leqslant \text{const}(\gamma - \alpha_1(p))^{4-\varepsilon};$$

(ii)′ In W_+ $(T_4^A(p) \leqslant r \leqslant T_2^N(p))$,

$$z(r, p) \leqslant \text{const}[z(T_4^A(p), p) + 1/T_4^A(p)^2] \exp M(r - T_4^A(p))$$
$$\leqslant \text{const}(\gamma - \alpha_1(p))^{2-2\varepsilon};$$

(iii)′ In W_- $(T_2^N(p) \leqslant r \leqslant T_5^A(p))$,

$$z(r, p) \leqslant \text{const}\ z(T_2^N(p), p)(\gamma - \alpha_1(p))^{2-\varepsilon}$$
$$\leqslant \text{const}(\gamma - \alpha_1(p))^{4-2\varepsilon}.$$

Repeating this procedure k times gives

$$z(T(p), p) \leqslant \text{const}(\gamma - \alpha_1(p))^{2-k\varepsilon}$$
$$\leqslant \text{const}(\gamma - \alpha_1(p)).$$

We have thus proved the following proposition.

PROPOSITION 3.15. *Fix $N \in \mathbb{Z}_+$, $N \geqslant 1$, and let $\delta > 0$ be given. Then there is a point p_δ, $0 < p_\delta < \gamma$, such that if $p_\delta \leqslant p < \gamma$,*

$$0 < \phi_N(T(p), p) - \phi_1(T(p), p) < \delta. \tag{3.31}$$

We may use this last result to show that if p is near γ, $\phi_N(T(p), p)$ satisfies inequalities similar to those satisfied by $\phi_1(T(p), p)$; cf. Propositions 3.4 and 3.5.

PROPOSITION 3.16. *Given any $N > 1$, there is a point α_N, $0 < \alpha_N < \gamma$, for which*

$$\theta_0 - (k+1)\pi < \phi_N(T(p), p) < \theta_0 - k\pi, \qquad \text{if} \quad \alpha_N \leqslant p < \gamma. \tag{3.32}$$

Proof. From Propositions 3.4 and 3.5, there is a point β_N, $0 < \beta_N < \gamma$, such that if $\beta_N \leqslant p < \gamma$,

$$\theta_0 - (k+1)\pi < \phi_1(T(p), p) \begin{cases} \leqslant -k\pi, & \text{if} \quad \theta_0 > 0 \text{ or } \theta_0 = -\pi/2 \\ \leqslant -(k+\frac{1}{2})\pi, & \text{if} \quad \theta_0 \leqslant 0. \end{cases}$$

Thus there is an $\varepsilon > 0$ for which

$$\theta_0 - (k+1)\pi < \phi_1(T(p), p) < \theta_0 - k\pi - \varepsilon, \qquad \beta_N \leqslant p < \gamma.$$

Now choose $\delta = \varepsilon$ in Proposition 3.15, and let $\alpha_N = \max(\beta_N, p_\delta)$. If $\alpha_N \leqslant p < \gamma$, then from Lemma 3.11 and Proposition 3.15,

$$\begin{aligned}\theta_0 - (k+1)\pi &< \phi_1(T(p), p) < \phi_N(T(p), p) \\ &< \phi_1(T(p), p) + \delta < \theta_0 - k\pi,\end{aligned}$$

as desired. ∎

Now just as the proof of Proposition 3.2 follows from the inequalities in Proposition 3.4, the proof of Theorem 3.1 follows from the inequalities (3.32). Thus the proof of Theorem 3.1 is complete, in the case that $f'(\gamma) < -1$. If this condition is not fulfilled, we replace f by $\lambda^2 f$, where $\lambda > 0$ and $\lambda^2 f'(\gamma) < -1$. Since there is a 1–1 correspondence between the positive eigenvalues of the two problems

$$\begin{aligned} \Delta w(x) + f'(u) w = \mu w, \qquad & |x| < R, \\ \alpha w(x) - \beta\, dw(x)/dn = 0, \qquad & |x| = R \end{aligned} \tag{A}$$

(where u satisfies (1.1), (1.2) on $|x| \leqslant R$), and

$$\begin{aligned} \Delta z(y) + \lambda^2 f'(\tilde{u}) z(y) = \nu z, \qquad & |x| < R/\lambda, \\ \alpha w(y) - \lambda\beta\, dw(y)/dn = 0, \qquad & |y| = R/\lambda \end{aligned} \tag{B}$$

(where $\tilde{u}$ satisfies $\Delta\tilde{u}(y) + \lambda^2 f(\tilde{u}(y)) = 0$, on $|y| < R/\lambda$, $\alpha\tilde{u}(y) - \lambda\beta\, d\tilde{u}(y)/dn = 0$, on $|y| = R(\lambda)$), we see that Theorem 3.1 holds even if $f'(\gamma) \geqslant -1$. Thus Theorem 3.1 is completely proved.

4. Infinitesimal Symmetry-Breaking—The Universality Theorem

In this section we shall prove two theorems on infinitesimal symmetry breaking for radial solutions of (2.3), (2.4) which lie in the kth nodal class of f (i.e. which satisfy (2.7)). These theorems differ depending on whether f satisfies hypotheses (1.3) or the stronger hypotheses (1.4). Before giving the statements, we need some notation.

Let f satisfy (1.3) and set (cf. Lemma 2.1)

$$M = \sup\{|f'(u)|: b \leqslant u \leqslant \gamma\}.$$

Next, let μ_1 denote the principal eigenvalue of the radial Laplacian Δ_R on the unit ball in $\mathbb{R}^n$ (i.e., $\Delta_R = d^2/dr^2 + (n-1)\, r^{-1} d/dr$), with the boundary conditions $u'(0)=0=\alpha u(1)-\beta u'(1)$). Let $C^{\mathrm{D}}_{k,n}$, and $C^{\mathrm{N}}_{k,n}$ be the constants associated with Bessel's equation, defined by (2.10), and recall that λ_N denotes the Nth eigenvalue of the Laplacian on S^{n-1}; cf. (2.13).

THEOREM 4.1. *Assume f satisfies* (1.4), *and consider radial solutions of* (2.3), (2.4) *in the* kth *nodal class of f. Let N_0 be the integer defined by*

$$N_0=\begin{cases}\min\{N\in\mathbb{Z}_+ : -\lambda_N>(c^{\mathrm{D}}_{n,k})^2+\mu_1\},\\ \quad \text{if} \quad -\pi/2\leqslant\theta_0<0,\\ \min\{N\in\mathbb{Z}_+ : -\lambda_N>(c^{\mathrm{N}}_{n,k})^2+\mu_1\},\\ \quad \text{if} \quad -0\leqslant\theta_0<\pi/2.\end{cases} \tag{4.1}$$

Then for every integer $N\geqslant N_0$, there are k distinct points $p_1^N, ..., p_k^N$ in the open interval $(0,\gamma)$, for which Eq. (2.18), (2.19) (*with $p=p_m^N$, $m=1, ..., k$*), *have non-trivial solutions.*

Thus the symmetry must break infinitessimally on k distinct radial solutions for all sufficiently large modes, i.e., for all $N\geqslant N_0$. Furthermore, the integer N_0 is "universal" in the sense that it is independent of the particular function f.

If now f satisfies the weaker conditions (1.3), we also have the following theorem. (Recall the definition of $\bar{p}$: radial solutions of (2.3), (2.4) in the kth nodal class exist if $\bar{p}<p<\gamma$; cf. Proposition 2.2.)

THEOREM 4.2. *Assume f satisfies* (1.3), *and consider radial solutions of* (2.3), (2.4) *in the* kth *nodal class of f. Let N_1 be the integer defined by*

$$N_1=\min\{N_1(p) \colon \bar{p}<p<\gamma\},$$

where (4.2)

$$N_1(p)=\min\{N\in\mathbb{Z}_+ : -\lambda_N>MT(p)^2+\mu_1\}.$$

Then given any integer $N\geqslant N_1$, there are k distinct points $q_1^N, ..., q_k^N$ in the open interval $(\bar{p},\gamma)$ for which Eqs. (2.18), (2.19) (*with $p=q_m^N$, $m=1, ..., k$*) *have non-trivial solutions.*

Thus here again the symmetry breaks infinitesimally on k distinct radial solutions, for all $N\geqslant N_1$. But, however, in this case the integer N_1 depends on f. Note too that for positive solutions of the Dirichlet problem ($\beta=0$), it was shown in [4] that if $f(0)\geqslant 0$, the symmetry cannot break infinitesimally. This result is consistent with our theorems since for such solutions $k=0$.

We shall give the complete proof of Theorem 4.1; the reader should have no difficulty in supplying the details for proving Theorem 4.2. Having established Theorem 3.1, the proof of Theorem 4.1 is completed by the following result.

PROPOSITION 4.3. *Let f satisfy* (1.4), *and let N_0 be defined by* (4.1). *Then for p near* 0 *and* $N \geqslant N_0$, L_N^p *has negative spectrum.*

Proof. We shall give the details for the case $-\pi/2 \leqslant \theta_0 < 0$; the proof for the other case is similar. Now N_0 is the smallest positive integer for which

$$-\lambda_{N_0} > (c_{k,n}^{\mathrm{D}})^2 + \mu_1.$$

We define an inner product on Ψ_p (cf. (2.20)) by

$$\langle \phi, \psi \rangle = \int_0^{T(p)} \phi(r)\psi(r)\, r^{n-1}\, dr.$$

The spectrum of each operator L_N^p consists of real eigenvalues; let λ be one such, where $N \geqslant N_0$. Then writing $T \equiv T(p)$ and $q(r) = f'(u(r, p)) + \lambda_N/r^2$, we have, for some $\phi \in \Psi_p$,

$$\|\phi\|^2 \lambda = \langle L_N^p \phi, \phi \rangle = \int_0^T \left\{ \phi \left[r^{n-1}\phi'' + \frac{n-1}{r} \phi r^{n-1} \right] + q\phi^2 r^{n-1} \right\} dr. \tag{4.4}$$

Now let $r = sT$, $\psi(s) = \phi(sT)$; then as

$$\int_0^T \phi \left[\phi'' + \frac{n-1}{r} \phi' \right] r^{n-1}\, dr$$

$$= T^{n-2} \int_0^1 \psi(s)[s^{n-1}\psi'' + (n-1)\, s^{n-2}\psi']\, ds,$$

and

$$\|\phi\|^2 = T^n \int_0^1 \psi(s)^2\, s^{n-1}\, ds,$$

it follows that

$$\frac{\int_0^T \phi \left[\phi'' + \frac{n-1}{r} \phi' \right] r^{n-1}\, dr}{\|\phi\|^2} = T^{-2} \frac{\int_0^1 \psi \left[\psi'' + \frac{n-1}{r} \psi' \right] s^{n-1}\, ds}{\int_0^1 \psi(s)^2\, s^{n-1}\, ds}$$

$$\leqslant T^{-2}\mu_1, \tag{4.5}$$

where μ_1 is as defined above. Next,

$$\int_0^T \left(f'(u)+\frac{\lambda_N}{r^2}\right)\phi^2 r^{n-1}\,dr$$
$$= T^{n-2}\left[\int_0^1 T^2 f'(u)\,\psi^2 s^{n-1}\,ds + \lambda_N \int_0^1 \psi^2 s^{n-3}\,ds\right]$$
$$= T^{n-2}\left[\int_0^1 T^2 f'(u)\,\psi^2 s^{n-2}\,ds + \lambda_N \int_0^1 \psi^2 s^{n-1}\,ds\right]. \tag{4.6}$$

Choose $\varepsilon > 0$ so small that $f'(0) > \varepsilon$ and $-\lambda_{N_0} > \mu_1 + (f'(0)+\varepsilon)(f'(0)-\varepsilon)^{-1}(c^{\mathrm{D}}_{n,k})^2$. For this ε, take p_1 so close to 0 that $f'(0)-\varepsilon < f'(u(r,p)) < f'(0)+\varepsilon$, $0<p<p_1$, $0 \leqslant r \leqslant T(p)$. On this range, $f(u)/u = f'(\xi) > f(0)-\varepsilon$, $u=u(r,p)$. Setting $g(u)=(f'(0)-\varepsilon)u$ gives $f(u)/u \geqslant g(w)/w$, $0<w$, $u<p$. It follows from [7, Lemma 4.4], that $T(p) \leqslant \sigma_{n,k}$, where $\sigma_{n,k} = \rho^{1,0}_{k,0}(f'(0)-\varepsilon)$ and where we have used Proposition 2.4, with $A = f'(0)-\varepsilon$. Using Lemma 2.5, we find $\sigma_{n,k} = c^{\mathrm{D}}_{k,n}/\sqrt{f'(0)-\varepsilon}$, so that for $0<p<p_1$ and $u=u(\cdot,p)$,

$$\int_0^1 T^2 f'(u)\,\psi^2 s^{n-1}\,ds \leqslant (c^{\mathrm{D}}_{k,n})^2 \frac{f'(0)+\varepsilon}{f'(0)-\varepsilon}\int_0^1 \psi(s)^2 s^{n-1}\,ds.$$

Combining this with (4.4)–(4.6) gives

$$\lambda \leqslant T^{-2}\left[\mu_1 + \frac{f'(0)+\varepsilon}{f'(0)-\varepsilon}(c^{\mathrm{D}}_{k,n})^2 + \lambda_N\right]$$
$$\leqslant T^{-2}\left[\mu_1 + \frac{f'(0)+\varepsilon}{f'(0)-\varepsilon}(c^{\mathrm{D}}_{k,n})^2 + \lambda_{N_0}\right] < 0,$$

as desired. ∎

If $-\pi/2 \leqslant \theta_0 \leqslant 0$ (which includes both Dirichlet and Neumann boundary conditions), we can give an easier proof of a yet stronger result. To this end, let

$$N_2 = \min\{N \in \mathbb{Z}_+ : -\lambda_N > (c^{\mathrm{D}}_{k,n})^2\}. \tag{4.7}$$

PROPOSITION 4.4. *Suppose $-\pi/2 \leqslant \theta_0 \leqslant 0$, and let N_2 be defined by* (4.7). *Then for p near* 0 *and $N \geqslant N_2$, L^p_N has only negative spectrum.*

Proof. For p near 0, $T_D(p)$ is near $\rho^{1,0}_{k,n}$; thus there is a $\delta>0$ and $p_2>0$ such that $-\lambda_{N_0} > T_D(p)^2(f'(0)+\delta)$, $0<p\leqslant p_2$. We may assume that for $0<p<p_2$ and $0\leqslant r \leqslant T_D(p)$, $0<f'(u(r,p))<f'(0)+\delta$. Then if $N \geqslant N_2$, $0<p\leqslant p_2$, and $L^p_N\phi = \lambda\phi$, $\|\phi\|=1$, we have, upon integrating by parts,

$$\lambda = \lambda \|\phi\|^2 = \langle L_N^p \phi, \phi \rangle$$
$$= \int_0^{T(p)} \{\phi(r^{n-1}\phi')' + q\phi^2 r^{n-1}\}\, dr$$
$$= T(p)^{n-1}\,\phi(T(p))\,\phi'(T(p)) - \int_0^{T(p)} r^{n-1}\phi'^2\, dr + \int_0^{T(p)} q\phi^2 r^{n-1}\, dr$$
$$\leqslant \int_0^{T(p)} q\phi^2 r^{n-1}\, dr < 0,$$

since $\phi(T(p))\,\phi'(T(p)) \leqslant 0$. ∎

Proof of Theorem 4.1. Let $N \in \mathbb{Z}_+$, $N \geqslant N_0$. From Proposition 3.16, there is a point p near γ for which $\phi_N(T(p), p) < \theta_0 - k\pi$, and Proposition 4.3 implies the existence of a point q near 0 for which $0 > \phi_N(T(q), q) > \theta_0 - \pi$; ($\phi_N$ is defined in $(3.3)_N$, $\mu = 0$; i.e., $\phi_N(r, p) = \tan^{-1}(a'_N(r, p)/a_N(r, p))$). Since dom($T$) is connected (Proposition 2.3), the intermediate value theorem shows that there are points p_j $(j = 1, ..., k)$, $0 < p_j < p_{j+1} < \gamma$, for which $\phi_N(T(p_j), p_j) = \theta_0 - j\pi$. That is, 0 is in the spectrum of each operator $L_N^{p_j}$, $j = 1, ..., k$. This completes the proof. ∎

In order to prove Theorem 4.2, we need the following analogue of Proposition 4.3.

PROPOSITION 4.5. *Let f satisfy* (1.3), *and let N_1 be defined by* (4.2). *Let $q \in (\bar{p}, \gamma)$ satisfy $N_1(q) = N_1$. Then if $N \geqslant N_1$, L_N^q has negative spectrum.*

Proof If $N \geqslant N_1$, $u = u(\cdot, q)$, $T = T(q)$, and $L_N^q \phi = \lambda\phi$, then

$$\int_0^T \left(f'(u) + \frac{\lambda_N}{r^2}\right)\phi^2 r^{n-1}\, dr$$
$$\leqslant \int_0^T \left(M + \frac{\lambda_N}{r^2}\right)\phi^2 r^{n-1}\, dr$$
$$\leqslant T^{-2}(MT^2 + \lambda_N)\,\|\phi\|^2.$$

Hence if in the proof of Proposition 4.3 we replace (4.6) by this inequality we find

$$\lambda \leqslant T^{-2}[\mu_1 + MT + \lambda_N]$$
$$\leqslant T^{-2}[\mu_1 + MT^2 + \lambda_{N_1}]$$
$$< 0,$$

and the result follows. ∎

As above, if $-\pi/2 \leqslant \theta_0 \leqslant 0$, we may replace N_2 by N_3, where

$$N_3 = \min\{\bar{N}_1(p) \colon \bar{p} < p < \gamma\},$$

where

$$N_3(p) = \min\{N \in \mathbb{Z}_+ : -\lambda_N > MT(p)^2\}.$$

Finally, the proof of Theorem 4.2 follows easily from Propositions 3.15 and 4.5.

APPENDIX: Proof of Lemma 3.12

Let B satisfy

$$b < B < A < \gamma,$$

where we recall that b and γ are defined in hypotheses (1.3), and A is defined in (3.6). Now choose p such that $A < p < \gamma$; then if p is near γ, the orbit $(u(\cdot, p), v(\cdot, p))$ meets the lines $u = A$ and $u = B$ (this is similar to Proposition 2.2), so that there is a function $T^B(p)$ satisfying

$$u(T^B(p), p) = B. \tag{A$_1$}$$

We assume that v does not change sign along the orbit segment between A and B; i.e., T^B is minimal with respect to (A$_1$). As before, let $H(r, p) \equiv H(u(r, p), v(r, p))$ be defined by

$$H(r, p) = \frac{v^2(r, p)}{2} + F(u(r, p)),$$

where $F' = f$ and $F(0) = 0$. Then as we have seen earlier, $H' = -(n-1)\, v^2/r$ along orbits of (2.5), so H decreases along such orbits. Let $b < u_1 < u_2 < \gamma$, and for p near γ, let $T^i(p)$ be defined by $u(T^i(p), p) = u_i$, and set $H_i(p) = H(u(T^i(p), p), v(T^i(p), p))$, $i = 1, 2$.

Lemma A$_1$. *Given any $\varepsilon > 0$, we can choose p so close to γ $(p < \gamma)$, such that along the orbit segment between u_1 and u_2, if v is of constant sign,*

$$0 < H_2(p) - H_1(p) < \varepsilon.$$

Proof. Suppose that $v \leqslant 0$ along the orbit segment from u_2 to u_1 (the proof for $v \geqslant 0$, whereby the orbit goes from u_1 to u_2 is similar). We have

$$\begin{aligned} 0 < H_2(p) - H_1(p) &= \int_{T^1(p)}^{T^2(p)} H'(r, p)\, dr \\ &= -(n-1) \int_{T^1}^{T^2} \frac{v^2}{r}\, dr \\ &= (n-1) \int_{T^1}^{T^2} \frac{(-v)(-v)}{r}\, dr \\ &\leqslant \frac{(n-1)}{T^2(p)} \max(-v(r\ p)) \int_{T^2}^{T^1} -v\, dr, \end{aligned}$$

where the max is taken over $0 \leqslant r \leqslant T(p)$ (and is thus bounded independently of p by virtue of Lemma 2.1). It follows that

$$\begin{aligned} 0 < H_2(p) - H_1(p) &\leqslant \frac{(n-1)\max(-v)}{T^2(p)} \int_{u_2}^{u_1} -du \\ &= \frac{(n-1)\max(-v)}{T^2(p)} (u_2 - u_1). \end{aligned}$$

On the other hand, it is easy to show (cf. Proposition 2.2) that $T^2(p) \to \infty$ as $p \to \gamma$, whence the lemma follows.

LEMMA A_2. *There is a closed level curve $H(u, v) = \delta$ such that if p is near γ $(p < \gamma)$, the orbit segment $(u(r, p), v(r, p))$, $0 \leqslant r \leqslant T(p)$, stays outside of this level curve.*

Proof. This is a consequence of Proposition 2.2.

Now choose $B < 0$ such that

$$\min\{u: H(u, v) = \delta\} < B < 0. \qquad (A_2)$$

Notice that for p close to γ $(p < \gamma)$, all orbit segments $(u(r, p), v(r, p))$ meet the lines $u = A$ and $u = B$ in k points (cf. Proposition 2.2).

LEMMA A_3. *There is a constant $\bar{v} > 0$ and a point $q < \gamma$ such that if $q < p < \gamma$ and v is of one sign along the orbit segment from A to B, then $v(r, p)^2 \geqslant \bar{v}$ along this orbit segment.*

Proof. Again we will only consider the case where $v(r, p) < 0$ along the orbit segment in question. Choose $q > A$ so close to γ that the conclusions of the previous two lemmas hold if $q < p < \gamma$, and $F(q) > F(u)$ if $B \leqslant u \leqslant A$. Let $p > q$ and define $T^q(p)$ by $u(T^q(p), p) = q$. Set

$$2\sigma = F(q) - \sup_{B \leqslant u \leqslant A} F(u) > 0,$$

and note that σ is independent of p. Then for $p > q$, p near γ, Lemma A_1 implies that

$$\sigma > H(u(T^q(p), p), v(T^q(p), p)) - H(u(T^B(p), p), v(T^B(p), p))$$
$$\geqslant H(u(T^q(p), p), v(T^q(p), p)) - H(u(r, p), v(r, p)),$$

if $T^A(p) \leqslant r \leqslant T^B(p)$. Thus

$$\sigma > F(q) - F(u(r, p)) - \frac{v^2(r, p)}{2} \geqslant 2\sigma - \frac{v^2(r, p)}{2},$$

so that $v^2(r, p) \geqslant 2\sigma$, if $T^A(p) \leqslant r \leqslant T^B(p)$, and the proof is complete. ∎

We can now complete the proof of Lemma 3.12. Let δ be a small positive number as described in Lemma A_2, and take p so close to γ as to make valid the conclusions of the previous lemmas. The orbit $(u(r, p), v(r, p))$, $r \geqslant 0$, meets the lines $u = A$ in two points P_A^1, P_A^2, where $P_A^i = (A, v(r_i, p))$, $r_1 < r_2$, $r_1 = T_1^A(p)$, $r_2 = T_2^A(p)$; it meets the line $u = B$ in two points P_B^1, P_B^2, where $P_B^i = (B, v(s_i, p))$, $s_1 < s_2$, $s_1 = T_1^B(p)$, $s_2 = T_2^B(p)$, and it meets the line $v = 0$ at $Q = (u(t, p), 0)$, $t = t(p)$, where $T_1^A(p) < T_1^B(p) < t(p) < T_2^B(p) < T_2^A(p)$; see Fig. 12.

Now we may write

$$T_2^A(p) - T_1^A(p) = (T_2^A(p) - T_2^B(p)) + (T_2^B(p) - t(p))$$
$$+ (t(p) - T_1^B(p)) + (T_1^B(p) - T_1^A(p)).$$

We shall show that each of the four terms on the right side of this last equation is bounded independently of p, and this will complete the proof. Thus

$$T_1^B(p) - T_1^A(p) = \int_A^B \frac{du}{v} = \int_B^A \frac{du}{-v}.$$

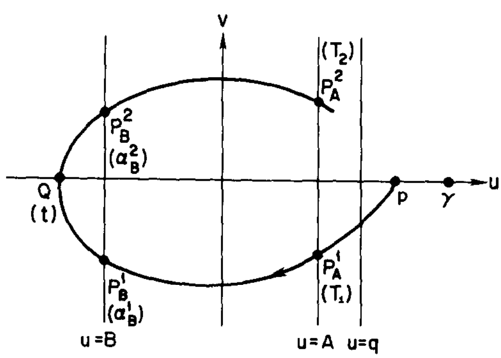

FIGURE 12

But from Lemma A_3, $-v \geqslant \bar{v} > 0$, so

$$T_1^B(p) - T_1^A(p) \leqslant (A-B)/\bar{v}. \tag{i}$$

Also,

$$t(p) - T_1^B(p) = \int_{\tilde{v}}^0 \frac{du}{v'} = \int_{\tilde{v}}^0 \frac{dv}{-\dfrac{n-1}{r} v - f(u)},$$

for some $\tilde{v} = \tilde{v}(p)$. Also, there is a constant $k > 0$ for which $f(u) \leqslant -k$, if $b \leqslant u \leqslant B$. Thus,

$$\begin{aligned} t(p - T_1^B(p) &\leqslant \int_{\tilde{v}}^0 dv \Big/ \left(-\frac{n-1}{r} v + \sigma \right) \\ &\leqslant \int_{\tilde{v}}^0 dv \Big/ \left(-\frac{(n-1)}{t(p)} + \sigma \right) \\ &= \frac{t}{1-n} \log \frac{\sigma t}{\sigma t - (n-1)\tilde{v}} = \frac{t}{n-1} \log \left(1 - \frac{(n-1)\tilde{v}}{\sigma t} \right) \\ &\leqslant \frac{t}{n-1} \left(\frac{-(n-1)\tilde{v}}{\sigma t} \right) = -\frac{\tilde{v}}{\sigma}, \end{aligned}$$

as $\log(1+x) < x$ if $x > 0$. Thus using Lemma 2.1,

$$t(p) - T_1^B(p) \leqslant 2M_0/\sigma. \tag{ii}$$

Next,

$$T_2^B(p) - t(p) = \int_0^{\tilde{w}} \frac{dv}{v'} = \int_0^{\tilde{w}} \frac{dv}{-\dfrac{n-1}{r} v - f(u)},$$

for some $\tilde{w} = \tilde{w}(p)$. Since $f(u) \leqslant -\eta$, $\eta > 0$, on $b \leqslant u \leqslant B$, we have

$$\begin{aligned} T_2^B(p) - t(p) &\leqslant \frac{t(p)}{1-n} \log \left(1 - \frac{(n-1)\tilde{w}}{\sigma t(p)} \right) \\ &\leqslant \tilde{w}/\sigma \leqslant 2M_0/\sigma. \end{aligned} \tag{iii}$$

Finally,

$$T_2^A(p) - T_2^B(p) = \int_B^A \frac{du}{v} \leqslant (A-B)/\bar{v},$$

in view of Lemma A_3. This inequality, together with inequalities (i)–(iii) above completes the proof of Lemma 3.12.

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