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The Cauchy Identity for $Sp(2n)$

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A bijection establishing the Cauchy identity for $Sp(2n)$

$$\prod_{1 \leqslant i < j \leqslant n} (1 - t_i t_j) \prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1} = \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t_1, ..., t_n)$$

is presented, using the insertion algorithm of Berele. A key element in the bijection is a new encoding of up–down tableaux. We present this as a correspondence proving the following enumerative formula for the number of up-down tableaux of length k and shape μ:

$$\tilde{f}_k^\mu = \sum_{\substack{\beta \vdash (k - |\mu|) \\ \beta' \text{ even}}} \binom{k}{|\mu|} f^\beta f^\mu.$$

1. INTRODUCTION

The main result of this paper is a bijection establishing the Cauchy identity for the symplectic group $Sp(2n)$, which is

$$\prod_{1 \leqslant i < j \leqslant n} (1 - t_i t_j) \prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1} = \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t_1, ..., t_n). \tag{1}$$

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In the above equation $sp_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1})$ is the character of the irreducible representation of $Sp(2n, \mathbf{C})$ indexed by λ (where the number of parts of $\lambda = l(\lambda) \leqslant n$), and may be viewed combinatorially as the generating function for all **symplectic** tableaux of shape λ, which we now proceed to define.

Fix n, a partition λ with at most n parts, and an alphabet $1 < \bar{1} < \cdots < i < \bar{i} < \cdots < n < \bar{n}$.

DEFINITION 1.1. A column-strict tableau (i.e., rows increase weakly left-to-right, columns increase strictly top-to-bottom) $\tilde{P}_\lambda$ of shape λ with entries in the above alphabet is symplectic if it satisfies the condition

(K) all entries in row i are larger than or equal to i.

We shall refer to (K) as the symplectic condition. The fact that, with the weighting scheme

$$i \to x_i, \qquad \bar{i} \to x_i^{-1},$$

these tableaux index the weights of the irredcible representation of $Sp(2n)$ corresponding to λ, is a result due to King [Ki].

The starting point for our bijection will be Berele's insertion scheme [Be], with which we shall assume familiarity. Berele insertion in fact proves the following character identity in the ring $\tilde{\Lambda}_n = \mathbf{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}]^{B_n}$, where B_n is the Weyl group of $Sp(2n)$ (i.e., the hyperoctahedral group),

$$(x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})^k = \sum_{\substack{\mu \\ l(\mu) \leqslant n}} \tilde{f}_k^\mu(n)\, sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}) \tag{2}$$

Here $\tilde{f}_k^\mu(n)$ is the number of n-**symplectic up–down tableaux** of shape μ and length k, that is, sequences of shapes $(\varnothing = \mu^0, \mu^1, ..., \mu^k = \mu)$ such that

(1) two consecutive shapes differ by exactly one box, i.e., for all $i = 1, ..., k$, either $\mu^i/(\mu^{(i-1)} = (1)$ or $\mu^{(i-1)}/\mu^i = (1)$.

(2) $l(\mu^i) \leqslant n$, for all $i = 1, ..., k$.

As a mapping, Berele insertion, which we denote by $\mathscr{B}$, maps words w of length k from $1, \bar{1}, ..., n, \bar{n}$ into pairs $(\tilde{P}_\mu, S_\mu^k(n))$, where $\tilde{P}_\mu$ is a symplectic tableau of shape μ and $S_\mu^k(n)$ is an n-symplectic up-down tableau of length k, shape μ. We write $(\varnothing \xleftarrow{\mathscr{B}} w) = (\tilde{P}_\mu, S_\mu^k(n))$.

Let $\tilde{f}_k^\mu$ denote the cardinality of the set F_k^μ of all up–down tableaux of length k, shape μ, i.e., all sequences of shapes $(\varnothing = \mu^0, \mu^1, ..., \mu^k = \mu)$ such that two consecutive shapes differ by exactly one box. Thus $\tilde{f}_k^\mu = \lim_{n \to \infty} \tilde{f}_k^\mu(n)$.

2. A FORMULA FOR $\mathcal{J}_k^\mu$

This section studies the combinatorial object which appears as the second component in the image of Berele's correspondence, playing the rôle of the standard Young tableau in the Knuth–Schensted algorithm. We write $[n]$ for the set of integers $\{1, ..., n\}$, for a positive integer n.

We begin with an example of an up-down tableau:

EXAMPLE 2.1.

$$S^5_{(2,1)} = (\varnothing, (1), (2), (1), (1, 1), (2, 1))$$

is an up–down tableau of length 5 and shape (2, 1).

Clearly a standard Young tableau is a particular case of an up–down tableau, since it may be viewed as a sequence of shapes such that each shape is one box larger than its predecessor.

We write f^μ for the number of standard Young tableau of shape μ.

LEMMA 2.2.

$$\mathcal{J}_k^\mu = \binom{k}{|\mu|} (2r-1)!!\, f^\mu, \qquad \mu \vdash (k-2r). \tag{3}$$

Proof. We set up a bijection between up–down tableaux S^k_μ and pairs (L, Q_μ), where Q_μ is a standard Young tableau of shape μ and L is a two-line array

$$L = \begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix}$$

with the j's in the top row written in increasing order; the i's in the bottom row are such that $j_k > i_k$ for each $k = 1, ..., r$, and the j's and i's are all distinct and $\{\text{entries in } Q_\mu\} \cup \{\text{entries in } L\} = [k]$. The latter observation will account for the binomial coefficient $\binom{k}{|\mu|}$.

Start with a sequence S^k_μ; the idea is to build up an associated sequence of tableaux, one for each shape of the sequence. As long as the sequence is increasing, we follow the usual labelling of a standard Young tableau, placing a j in the box of μ^j that was added to μ^{j-1}. (μ^1 is always a single box, so we can start off the process.) In general, at step j, given that we have the standard Young tableau T_{j-1} associated with μ^{j-1}, and μ^j is one box larger than μ^{j-1}, T_j is simply the standard Young tableau obtained by adding a j to T_{j-1} in the position of the added box (in the skew-shape μ^j/μ^{j-1}).

Now suppose μ^j is one box less that μ^{j-1}; let T_i be the standard Young tableau correspondig to μ^i. To get T_j, we do the following:

(1) Bump out the extra entry of T_{j-1} (the one in the unique square of μ^{j-1} which is not a square of μ^j) by columns (i.e., inverse Schensted column-insertion) to get a tableau T_j of shape μ^j, and a letter x. This means that by column-inserting x into T_j we would retrieve the previous bigger standard Young tableau T_{j-1}, and hence its shape μ^{j-1}. See the example in Fig. 1.

(2) Record the fact that a removal occurred at step j by putting the pair (j, x) into a two-line array L, with j on top. Note that since the x was bumped out at step j, it must have been inserted in an earlier step, so $x < j$.

We continue this process to the end of the sequence. Arranging the two-line array L so that the top row is in increasing order, we clearly end up with the requisite two-line array L and a standard Young tableau Q_μ of shape μ (the (final) shape of S_μ^k). The process clearly reverses: we work our way backwards from Q_μ, which is the kth step of the sequence, reconstructing the preceding tableaux and hence the sequence of shapes. If we have the standard Young tableau T_j for the jth step, and wish to get the tableau T_{j-1}, again two cases arise:

(1) j does not appear in the top row of the two-line array L, indicating that μ^j was not the result of removing a box from μ^{j-1}, but rather came about by adding a box labelled j to μ^{j-1}. Thus deleting the box labelled j from T_j will recover a standard Young tableau T_j of the correct shape (one box less that μ^j).

(2) j does appear in the top row of L, i.e., a pair (j, i) is in L. This says that T_j was obtained from T_{j-1} as a result of an inverse column-bumping which knocked out the i, or, equivalently, $T_{j-1} = (i \to T_j)$. (In the previous example, work backwards, i.e., right to left.) ∎

$$\mu^{j-1} \qquad\qquad \mu^j$$

$$T_{j-1} = \begin{matrix} 1\,2\,5 \\ 3\,4 \\ 6\,7 \end{matrix} \qquad \begin{matrix} & 1\,2 \\ 3 \leftarrow & 4\,5 \\ & 6\,7 \end{matrix} = T_j$$

$$\begin{Bmatrix} j \\ 3 \end{Bmatrix} \in L$$

FIG. 1. Proof of Lemma 2.2.

The example in Fig. 2 should make the bijection transparent:

THEOREM 2.3.

$$\mathcal{J}_k^{\mu} = \sum_{\substack{\beta \vdash (k - |\mu|) \\ \beta' \text{ even}}} \binom{k}{|\mu|} f^{\beta} f^{\mu}. \tag{4}$$

Proof. We need only refine the bijection of the previous lemma, which associates to the up–down k-tableau S_μ^k a pair (L, Q_μ), where L is a two-line array of distinct integers (j, i) such that the j's are written in increasing order the top row and each j is strictly greater than the corresponding i below it. L may consequently be viewed as a product of disjoint transpositions (i, j), and as such, Schensted row-insertion applied to the resulting permutation produces a pair of identical standard Young tableaux Q_β of shape β, where β has even columns [Kn]; conversely, since Schensted insertion is a bijection, the permutation, and thus the two-line array L, are uniquely recoverable from a standard Young tableau Q_β of shape β with even columns. For our purposes it will be more convenient to use the Burge correspondence [Bu] between two-line arrays and tableaux with even columns (in this case, restriction to the distinct entries in the two-line array produces the standard Young tableau). It can be shown that the tableaux produced by the two correspondences coincide, so this does not affect the output of our bijection. We conclude by noting that the binomial coefficient in (4) is accounted for by the fact that the construction in effect splits the integers $[k]$ into two disjoint sets, one contributing to the entries of Q_μ and the other to those of Q_β. ∎

The complete mapping appears in Fig. 3.

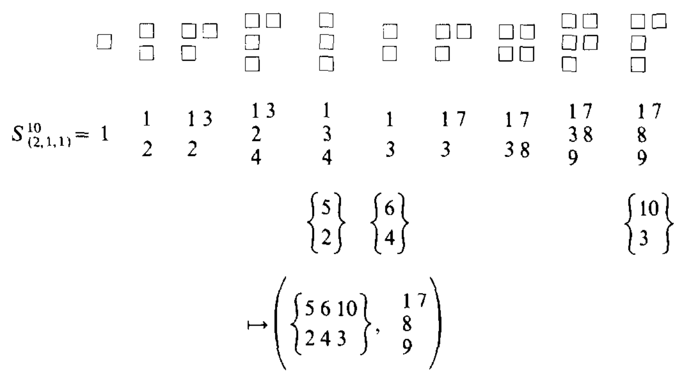

FIG. 2. The bijection of Lemma 2.2.

In the previous example,

$$L=\left\{\begin{matrix}5\ 6\ 10\\ 2\ 4\ 3\end{matrix}\right\}\longleftrightarrow (2\ 5)(4\ 6)(3\ 10)\longleftrightarrow\begin{pmatrix}2\ 3 & 3\ 5\ 6\ 10\\ 5\ 10\ 6\ 2\ 4\ 3\end{pmatrix}\overset{\mathcal{X}\mathcal{S}}{\longleftrightarrow}\begin{array}{ll}2 & 3\\ 4 & 6\\ 5 & \\ 10 & \end{array}$$

Thus

$$S^{10}_{(2,1^2)}\leftrightarrow\begin{pmatrix}\begin{array}{ll}2 & 3\\ 4 & 6\\ 5 & \\ 10 & \end{array} & \begin{array}{ll}1 & 7\\ 8 & \\ 9 & \end{array}\end{pmatrix}$$

FIG. 3. The bijection of Theorem 2.3.

3. TECHNICALITIES ON BERELE INSERTION

The first lemma is an obvious but nonetheless important observation about Berele insertion, which we state separately only because it is invoked so often:

LEMMA 3.1. *Suppose Berele insertion* $(\tilde{P}_\mu \overset{\mathcal{B}}{\longleftarrow} x)$ *of* x *(*x *in* $1, \bar{1}, ..., n, \bar{n}$*) into a symplectic tableau* $\tilde{P}_\mu$ *causes a cancellation. Then Knuth–Schented insertion* $(\tilde{P}_\mu \leftarrow x)$ *results in augmenting the shape* μ *of* $\tilde{P}_\mu$ *by a square in column* 1.

LEMMA 3.2. *Let* w *be a word of length* k *in* $1, \bar{1}, ..., n, \bar{n}$ *and let* x, x' *be consecutive letters in* w*, with* x *preceding* x' *in* w*, such that* $x \leqslant x'$*. Then in applying the Berele algorithm to the word* w*, if inserting* x' *(i.e.,* $((\tilde{P}_\mu \overset{\mathcal{B}}{\longleftarrow} x) \overset{\mathcal{B}}{\longleftarrow} x')$*) causes a cancellation, so did the (prior) insertion* $(\tilde{P}_\mu \overset{\mathcal{B}}{\longleftarrow} x)$ *of* x.

Proof. For suppose not; let the symplectic tableau built up prior to inserting x be $\tilde{P}$; then Berele insertion of x into $\tilde{P}$ is the same as Knuth–Schensted row insertion of x into $\tilde{P}$ and produces some larger tableau $\tilde{P}'$. On the other hand, Berele insertion of x' into $\tilde{P}'$ results in a cancellation, so that Schensted row insertion of x' into $\tilde{P}'$ must yield a tableau $\tilde{P}''$ whose shape differs from that of $\tilde{P}'$ by an extra square in column one (since the Schensted bumping path of $(\tilde{P}' \leftarrow x')$ must end in column one). But by an elementary property of Schensted insertion (see [Su] for details), $x \leqslant x'$ implies that the Schensted (row-insertion) bumping path of x' lies to the right and ends no lower than that of x, so we have a contradiction. ▮

LEMMA 3.3. *Let w be a k-word in $1, \bar{1}, \ldots n, \bar{n}$ and let x, x' be consecutive letters in w with x preceding x' and $x \leqslant x'$. Suppose Berele insertion performed on w causes cancellations at x and at x'; then the taquin path followed by the hole created upon inserting x ends in the same row as, or higher than, the taquin path of the hole created by the subsequent insertion of x'.*

Proof. We refer to Fig. 4 in the course of our argument. Suppose $\tilde{P}$ is the symplectic tableau built up prior to insertion of x, and let x cause a cancellation in row i of $\tilde{P}$, so that Berele insertion of x before the *jeu de taquin* (see [Schu]) step results in the punctured tableau of Fig. 4. Let r be the smallest index such that $b_r \leqslant a_r$; consider now the effect of *jeu de taquin* on $\tilde{P}$. In row i the hole slides out to the space above b_r, then swaps positions with b_r; again in row $i+1$ it slides rightwards until it stops above c_s, where s is least such that $c_s \leqslant b_{s+1}$ and $s \geqslant r$. This continues until it hits the boundary of $\tilde{P}$, leaving a tableau $\tilde{P}'$ of smaller shape.

Now consider the Berele insertion of x' into $\tilde{P}'$. Since $x' \geqslant x$, it is clear that the cancellation caused by x' occurs in a row no higher than row i, where the cancellation caused by x occurred (cf. Lemma 3.2). Thus prior to *jeu de taquin*, we have a punctured tableau as in Fig. 4, with the hole in row i or lower. To perform *jeu de taquin*, in each row we look for the smallest index r in the row below such that the rth element in it is less than or equal to the $(r+1)$th element in the current row, as before. It is easily seen that the new hole must migrate to a position which is to the LEFT of the old hole produced by x, if we are dealing with a row which was affected by the taquin path of insertion of x. For instance, in row $(i+1)$, the entry (b_{r+1}) in column $(r-1)$ is now less than or equal to the entry b_r in column r of row i, so the new hole slides down into row $(i+1)$ at worst in column $(r-1)$. The statement of the lemma follows. ▮

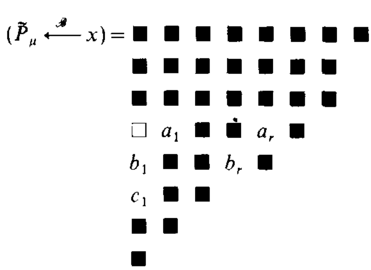

FIG. 4. Proof of Lemma 3.3.

The converse of the above lemma is also true:

LEMMA 3.4. *Let w be a k-word in* $1, \bar{1}, ..., n, \bar{n}$ *and let* x, x' *be consecutive letters in w with x preceding* x'. *Suppose Berele insertion performed on w causes cancellations at x and at* x', *such that the taquin path followed by the hole created upon inserting x ends in the same row as, or lower than, that of the hole created by the subsequent insertion of* x'. *Then* $x \leqslant x'$.

Proof. Suppose to the contrary that $x > x'$. Consider the results of ordinary Schensted insertion $(\tilde{P} \leftarrow x \leftarrow x')$, where as usual $\tilde{P}$ is the initial tableau. Then both bumping paths end in the first column, since x must do so, and that of x' must end below it. Also the path of x' lies above that of x until it reaches the first column, so the symplectic violation due to x' is in a row $r(x') < r(x)$, if $r(x)$ is the row of the hole produced upon (Berele) insertion of x. (See also the argument of Lemma 3.6 below.) But by reversing the arguments in the previous lemma (i.e., follow the taquin paths backwards), this contradicts the hypothesis about the culminating squares of the two taquin paths. ∎

LEMMA 3.5. *Let w be a k-word in* $1, \bar{1}, ..., n, \bar{n}$ *and let* x, x' *be consecutive letters in w with x preceding* x'. *Suppose Berele insertion performed on w causes a cancellation at x but not at* x'. *Then* $x \leqslant x'$.

Proof. Suppose to the contrary that $x' < x$. By considering the effect of ordinary Schensted insertion of first x, then x', into the symplectic tableau created up to the insertion of x, we see that the bumping path of x' lies strictly to the left of that of x until it hits column 1 (which must happen since it does for x by Lemma 3.1). Suppose inserting x caused a symplectic violation in row i; thus the (Schensted-insertion) path of x bumps an $\bar{i}$ out of row i. Consequently the path of x' bumps the entry a_i out of row i, where $a_i \leqslant \bar{i}$, contradicting the hypothesis that x does not cause a cancellation (the tableau is symplectic up to row i). ∎

LEMMA 3.6. *Let* $\tilde{P}$ *be a symplectic tableau in* $1, \bar{1}, ..., n, \bar{n}$, *and let* x, x', *be letters in* $1, \bar{1}, ..., n, \bar{n}$ *such that* $x > x'$. *If Berele insertion* $P \leftarrow x$ *causes a cancellation, then so does the subsequent Berele insertion of* x'.

Proof. If not, the ordinary Schensted bumping path p of x culminates in a new row, in column 1, so the (Berele and) Schensted bumping path p' of x', which must lie weakly above p (because $x > x'$), also ends in column 1, in a new row. Suppose inserting x caused a symplectic violation in row i, so that the new tableau has an i in row i, column 1, and an $\bar{i}$ in row $(i+1)$, column 1.

Consider the entry y in row $(i-1)$, column 1. We have $y \geqslant (i-1)$, since there are no symplectic violations higher than row $(i+1)$; on the other hand the $(i, 1)$ entry is i, so column-strictness says $y < i$. Since the path p' must hit column 1 in row j where $j \leqslant i$, this knocks y into row $i+1$, creating a symplectic violation, contradiction. ∎

LEMMA 3.7. *Let $\tilde{P}$ be as in the preceding lemma, with x, x' such that Berele insertion of x is the same as ordinary insertion, but the subsequent Berele insertion of x' causes a cancellation. Then $x > x'$.*

Proof. Suppose $x \leqslant x'$. A contradiction follows immediately from Lemma 3.2. ∎

4. A PIERI RULE FOR $Sp(2n, \mathbf{C})$

In this section we appeal to the technical properties of Berele insertion just developed to derive a rule for decomposing the product

$$sp_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1})\, sp_{(k)}(x_1^{\pm 1}, ..., x_n^{\pm 1})$$

as an integer combination of symplectic Schur functions.

THEOREM 4.1 (A Pieri rule for symplectic Schur functions). *Let λ be any partition of length at most n; k any nonnegative integer. Then*

$$\begin{aligned} &sp_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1})\, sp_{(k)}(x_1^{\pm 1}, ..., x_n^{\pm 1}) \\ &\quad = \sum_{\substack{\nu \\ l(\nu) \leqslant n}} sp_\nu(x_1^{\pm 1}, ..., x_n^{\pm 1}) \\ &\quad\quad \times \left(\sum_{r=0}^{k} \sum_{\substack{\mu \\ \mu \subseteq \lambda, \mu \subseteq \nu \\ |\lambda/\mu| = r, |\nu/\mu| = k-r}} c^{\lambda}_{\mu,(r)} c^{\nu}_{\mu,(k-r)} \right), \end{aligned} \tag{5}$$

where $c^{\lambda}_{\mu,\nu}$ denotes the Littlewood–Richardson coefficient, i.e., the multiplicity of the Schur function s_ν in the product $s_\lambda \cdot s_\mu$ (see [Macd]).

Proof. The left-hand side enumerates pairs $(\tilde{P}_\lambda, \omega)$, where $\tilde{P}_\lambda$ is a symplectic tableau of shape λ and $w = w_1 \leqslant \cdots \leqslant w_k$ is a weakly increasing k-word in $x_1^{\pm 1}, ..., x_n^{\pm 1}$. Recall that the Pieri rule for ordinary Schur functions is essentially a combinatorial statement about the effect of Schensted insertion of such a word into a column-strict tableau: the added squares form a horizontal strip. It is a tribute to the power of the Berele algorithm that the effect of Berele insertion of an increasing word into a symplectic tableau may be described just as elegantly.

So let us consider the situation $\tilde{P}_\lambda \xleftarrow{\mathscr{B}} w$, where $w_1 \leqslant \cdots \leqslant w_k$. Observe that

(1) The letters of w which cause cancellations must form an initial segment of w, so there is some $r \geqslant 0$ such that $w_1, ..., w_r$ cause cancellations, but w_i does not for $i > r$. This follows from Lemma 3.2.

(2) If for $i = 1, ..., r$, inserting w_i results in the loss of a square in row p_i, then $p_1 \leqslant \cdots \leqslant p_r$, by Lemma 3.3.

(3) If the letters whose insertion results in additions to the tableau are, as above, $w_{r+1}, ..., w_k$, with respective added squares in rows $q_1, ..., q_{k-r}$, then $q_1 \geqslant \cdots \geqslant q_{k-r}$. This is precisely the statement that the squares so added form a horizontal strip (i.e., no two in a column).

Thus Berele insertion of a row into a symplectic tableau $\tilde{P}_\lambda$ produces a pair $(\tilde{P}_\nu, S^k_{(\lambda \to \nu)})$, $l(\nu) \leqslant n$, where the second component is a k-sequence of shapes of the form

$$(\lambda = \mu^0 \supset \mu^1 \supset \cdots \supset \mu^r \subset \mu = \nu^1 \subset \nu^2 \subset \cdots \nu^{k-r} = \nu),$$

such that:

- the difference between two consecutive shapes is exactly one box;
- if we label, in order, the boxes lost in the shape λ by $1, ..., r$ then these filled boxes form a column-strict skew-plane partition (i.e., rows decrease right to left, columns decrease top-to-bottom) of shape λ/μ (since $i+1$ appears strictly above or to the left of i (by (2) above)), so in particular λ/μ is a horizontal strip;
- if we label the boxes added to μ in the remaining $k-r$ steps, these form a column-strict skew-tableau of shape ν/μ so, in particular, ν/μ is a horizontal strip.

Conversely, given a symplectic tableau of shape ν and a k-sequence of shapes starting at λ and ending in ν, both shapes having length at most n, with the properties described in the preceeding paragraph, it is clear that reversing the Berele insertion results in a word of length k which decreases weakly from left-to-right: this follows from Lemma 3.4 and the following observation:

By Lemma 3.5, if r is the cut off point as in (1) above, so that w_r is the last cancellation and w_{r+1} starts off a sequence of additions to the tableau, then $w_r \leqslant w_{r+1}$.

Thus the coefficient of sp_ν in $sp_\lambda sp_{(k)}$ is

$$|\{\mu : \mu \subseteq \nu, \mu \subseteq \lambda, \nu/\mu, \lambda/\mu \text{ are horizontal strips, and } |\nu/\mu| + |\lambda/\mu| = k\}|.$$

We now remind the reader that the Littlewood–Richardson coefficient $c^\lambda_{\mu,(r)}$ is nonzero only when λ/μ is a horizontal strip, in which case it is 1. ∎

EXAMPLE 4.2. We compute

$$sp_{\begin{smallmatrix}\square&\square\\\square\\\square\end{smallmatrix}}(x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1})\cdot\bar{h}_3,$$

by listing all the possible up–down sequences of shapes of length 3 which begin with the shape $\lambda = (2, 1, 1)$.

We need to find all sequences ending in some shape ν, characterized by $\mu \subseteq \lambda$, $\mu \subseteq \nu$, such that

$$\lambda/\mu \text{ and } \nu/\mu \text{ are horizontal strips, and } |\nu/\mu| + |\lambda/\mu| = 3.$$

The computation appears in Fig. 5.

For $|\lambda/\mu| = 0$:
The shapes ν are simply the ones which are 3-Pieri over λ, and these are

$$\begin{smallmatrix}\square&\square&1&1&1\\\square\\\square\end{smallmatrix},\quad \begin{smallmatrix}\square&\square&1&1\\\square&1\\\square\end{smallmatrix},\quad \begin{smallmatrix}\square&\square&1&1\\\square\\\square\\1\end{smallmatrix},\quad \begin{smallmatrix}\square&\square&1\\\square&1\\\square\\1\end{smallmatrix}.$$

For $|\lambda/\mu| = 1$:
For $\mu = (1^3)$, the shapes ν which are 2-Pieri over μ are:

$$\begin{smallmatrix}\square&1&1\\\square\\\square\end{smallmatrix},\quad \begin{smallmatrix}\square&1\\\square\\\square\\1\end{smallmatrix}.$$

For $\mu = (2, 1)$, the shapes ν which are 2-Pieri over μ are:

$$\begin{smallmatrix}\square&\square&1&1\\\square\end{smallmatrix},\quad \begin{smallmatrix}\square&\square&1\\\square&1\end{smallmatrix},\quad \begin{smallmatrix}\square&\square&1\\\square\\1\end{smallmatrix},\quad \begin{smallmatrix}\square&\square\\\square&1\\1\end{smallmatrix}.$$

For $|\lambda/\mu| = 2$: The only μ such that λ/μ is 2-Pieri is $\mu = (1^2)$, and is counted exactly once;

(note that a sequence such as $\begin{smallmatrix}\square&\square\\\square\\\square\end{smallmatrix},\ \begin{smallmatrix}\square\\\square\\\square\end{smallmatrix},\ \begin{smallmatrix}\square\\\square\end{smallmatrix}$

is not allowed, since it would violate observation (2) of Theorem 4.1.)
The shapes ν which are 1-Pieri over $\mu = (1^2)$ are

$$\begin{smallmatrix}\square&1\\\square\end{smallmatrix},\quad \begin{smallmatrix}\square\\\square\\1\end{smallmatrix}.$$

Finally, observe that there are no shapes μ such that λ/μ is 3-Pieri.
Thus

$$sp_{\begin{smallmatrix}\square&\square\\\square\\\square\end{smallmatrix}}(x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1})\cdot\bar{h}_3,$$

$$= sp_{(5,1^2)} + sp_{(4,2,1)} + sp_{(4,1^3)} + sp_{(3,2,1^2)} + sp_{(3,1^2)} + sp_{(4,1)} + sp_{(3,2)} + sp_{(3,1^2)} + sp_{(2^2,1)} + sp_{(2,1)} + sp_{(1^3)}.$$

FIG. 5. Computing $sp_{(2,2,1,1)}(x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, x_4^{\pm 1})\cdot\bar{h}_3$.

We can also deduce a result which traditionally follows from Lie algebra theory.

COROLLARY 4.3. *In the ring $\tilde{\Lambda}_n$, the coefficient of sp_λ in the expression of $\tilde{h}_2^k$ $(k \geq 0)$ as a linear combination of the basis elements $\{sp_\mu\}_{l(\mu) \leq n}$ is zero unless $\lambda \vdash 2m$ for some $m \geq 0$.*

(*In representation-theoretic terms, this says the following: Suppose the irreducible $Sp(2n, \mathbf{C})$-module $\tilde{N}^\lambda$ appears in the decomposition of the kth tensor power of the adjoint action of $Sp(2n, \mathbf{C})$. Then $|\lambda|$ is a multiple of 2.*)

Proof. Let $\tilde{P}_\mu$ be any symplectic tableau of shape μ. By the Pieri rule, it is clear that if the up–down sequence produced by

$$(\tilde{P}_\mu \xleftarrow{\mathscr{B}} w)$$

for an increasing word w of length 2, has shape λ, then $|\lambda|$ is equal to $|\mu|$ or $|\mu| \pm 2$. Thus if μ is even, so is $|\lambda|$. ∎

The next result gives a similar decomposition for $sp_\lambda \cdot e_k$, where e_k is the kth elementary symmetric function in the variables $x_1^{\pm 1}, ..., x_n^{\pm 1}$.

THEOREM 4.4. *For $k \leq 2n$, $l(\lambda) \leq n$, the coefficient of sp_ν in $sp_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}) \cdot e_k(x_1^{\pm 1}, ..., x_n^{\pm 1})$ is the number of shapes μ, $l(\mu) \leq n$, such that*

$$\mu \supseteq \nu,\ \mu \supseteq \lambda,\ \mu/\nu,\ \mu/\lambda \text{ are vertical strips, and } |\mu/\nu| + |\mu/\lambda| = k.$$

Proof. This time the left-hand side enumerates pairs $(\tilde{P}_\lambda, w)$, where $\tilde{P}_\lambda$ is a symplectic tableau of shape λ and $w = w_1 > \cdots > w_k$ is a strictly decreasing k-word in $x_1^{\pm 1}, ..., x_n^{\pm 1}$.

Examining the insertion $\tilde{P}_\lambda \xleftarrow{\mathscr{B}} w$, where $w = w_1 > \cdots > w_k$, we conclude that the output is a pair $(\tilde{P}_\nu, T^k_{(\lambda \to \nu)})$, $l(\nu) \leq n$, where the second component is a k-sequence of shapes of the form

$$(\lambda = \mu^0 \subset \mu^1 \subset \cdots \subset \mu^r \supset \mu = \nu^1 \supset \nu^2 \supset \cdots \nu^{k-r} = \nu)$$

(this follows from Lemma 3.6) such that:

- the difference between two consecutive shapes is exactly one box;
- if we label, in order, the boxes added to the shape λ by 1, ..., r then these filled boxes form a column-strict skew-tableau of shape λ/μ (since $i+1$ appears strictly below i, by ordinary Schensted effect of inserting a strictly decreasing sequence), so in particular μ/λ is a vertical strip;
- if we label the boxes removed from μ in the remaining $k-r$ steps, in the order of removal, then by Lemma 3.3, these form a column-strict

skew-plane partition (i.e., columns decrease from top to bottom) of shape ν/μ so, in particular, μ/ν is a vertical strip.

Conversely, given a symplectic tableau of shape ν and a k-sequence of shapes starting at λ and ending in ν, both shapes having length at most n, with the properties described in the preceding paragraph, it is clear that reversing the Berele insertion results in a strictly decreasing word of length k: this follows from Lemmas 3.4 and 3.7. ∎

It is well known that the character of the kth fundamental representation $(k \leqslant n)$ of $Sp(2n)$ satisfies

$$sp_{1^k} = e_k - e_{(k-2)}.$$

Hence the above result leads to a formula for the decomposition of the tensor product of the irreducible $Sp(2n)$-module corresponding to the partition λ, with the kth fundamental $Sp(2n)$-representation.

5. Facts about the Cauchy Identity for $GL(n)$

The ordinary Cauchy identity states that

$$\prod_{i,j} (1 - x_i y_j)^{-1} = \sum_{\lambda} s_\lambda(x)\, s_\lambda(y), \tag{6}$$

where, for a finite number of variables $x_1, ..., x_n$ and a partition λ with at most n parts, the Schur function $s_\lambda(x_1, ..., x_n)$ is the formal character of the general linear group $Gl(n, \mathbf{C})$ for the irreducible representation indexed by the partition λ and may be interpreted combinatorially as the generating function for all column-strict tableaux of shape λ in the alphabet $1 < 2 < \cdots < n$. A combinatorial proof of (6) is given by the well-known Knuth–Schensted–Robinson (KSR) insertion algorithm (see [Kn; S]). The bijection takes a word w in the x_i's and y_j's and associates to it a pair of column-strict tableaux (P_λ, Q_λ) of the same shape λ. We shall denote this by $(\varnothing \xleftrightarrow{\mathscr{KS}} w) = (P_\lambda, Q_\lambda)$. For our purposes we shall need Knuth's encoding of such a word into two-line arrays

$$\mathscr{T} = \begin{pmatrix} u_{i_1} \cdots u_{i_k} \\ v_{i_1} \cdots v_{i_k} \end{pmatrix},$$

where each u represents a letter x_u in the set $\{x_1, x_2, ...\}$, and each v represents a y_v; the letters are arranged such that the top row of u's is weakly increasing, with the property that $u_i = u_{i+1}$ implies $v_i \leqslant v_{i+1}$. We shall henceforth refer to such an array as a Knuth two-line array.

We shall be dealing extensively with a particular kind of Knuth two-line array, which we define next:

DEFINITION 5.1. Call a two-line array

$$L=\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix}$$

a **Burge** two-line array (after William Burge, cf. [Bu]) if

(1) $j_1 \leqslant j_2 \cdots \leqslant j_r$;

(2) $j_k = j_{k+1}$ implies $i_k \leqslant i_{k+1}$, for all $k = 1, ..., r$.

(3) $j_k > i_k$ for all $k = 1, ..., r$.

Notice that (1) and (2) are simply the conditions for L to be a Knuth two-line array.

Write λ' for the conjugate of λ, and call λ **even** if all the parts of λ have even length.

Applying the Knuth correspondence to symmetrised Burge two-line arrays (i.e., Knuth arrays in which each occurrence of a pair $\binom{j}{i}$ is accompanied by the pair $\binom{i}{j}$) establishes a bijection between symmetric non-negative integer matrices with trace zero and column-strict tableaux of shape with even columns (see [Kn]). This in turn is a bijective proof of the Schur function identity:

THEOREM 5.2 (Littlewood).

$$\prod_{i<j} (1-t_i t_j)^{-1} = \sum_{\beta' \text{ even}} s_\beta(t_1, t_2, ...). \tag{7}$$

This is one of six identities discovered by Littlewood, for which he supplies algebraic proofs in [Li, p. 235].

William Burge (see [Bu]) gives direct bijective proofs of four of these identities; we shall be especially interested in (7) above. We describe what we shall henceforth refer to as the Burge correspondence in

THEOREM 5.3. *The following procedure is a bijection between Burge two-line arrays and column-strict tableaux of even-columned shape: Given a Burge two-line array*

$$L=\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix},$$

construct a single column-strict tableaux T (instead of a pair), with columns of even length, as follows:

(1) *Set* $k=0$, $T_k=\varnothing$.

(2) *Set* $k \to k+1$. T_k *is obtained from* T_{k-1} *in two steps*:

(a) $S=(T_{k-1} \leftarrow i_k)$; *suppose the new square in* S *created by inserting* i_k *is in position* (s_k, t_k) (*i.e., row* s_k, *column* t_k).

(b) $T_k = (S$ *with* j_k *placed in position* $(s_k+1, t_k))$, *i.e.,* S *with* j_k *placed immediately below the new square by inserting* i_k *into* T_{k-1}.

(3) *If* $k=r$, *set* $T=T_r$ *and stop*; *otherwise, go back to step* (2).

Since we shall need to know how to recover the two-line array, given an even-columned tableau T, we describe the inverse of the above insertion process:

(1) Set $k=r$, $T_k=T$;

(2) Remove largest entry x in T_k; set $j_k=x$.

(3) Row-remove the entry $y<x$ immediately above the former position of x in T, bumping out an element z (note $z<y<x$) and leaving a tableau S. Set $i_k=z$, $k \leftarrow k-1$, $T_k=S$.

(4) If $k=0$, stop; otherwise, go back to step (2).

See Fig. 6 for an illustration of the correspondence.

We will have occasion to refer to another bijection in [Bu], which establishes the symmetric function identity

THEOREM 5.4 (The dual Burge correspondence).

$$\prod_{i \leqslant j} (1-t_i t_j)^{-1} = \sum_{\substack{\lambda \\ \lambda \text{ even}}} s_\lambda(t). \tag{8}$$

Proof. This time the left-hand side counts **dual Burge two-line arrays**

$$L* = \begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix},$$

The Burge array

$$L = \begin{pmatrix} 3 & 3 & 3 & 4 & 4 \\ 1 & 2 & 2 & 1 & 1 \end{pmatrix} \mapsto \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & \\ 4 & 4 & \end{array}$$

via the sequence of insertions

$$\begin{array}{l} 1 \\ {}_3 \end{array}, \quad \begin{array}{ll} 1 & 2 \\ 3 & {}_3 \end{array}, \quad \begin{array}{lll} 1 & 2 & 2 \\ 3 & 3 & {}_3 \end{array}, \quad \begin{array}{lll} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 3 & & \\ {}_4 & & \end{array}, \quad \begin{array}{lll} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 3 & 3 & \\ 4 & {}_4 & \end{array}.$$

FIG. 6. Burge's correspondence.

where

(1) $j_1 \leqslant \cdots \leqslant j_r$.

(2) $j_k \geqslant i_k$, all $k = 1, ..., r$.

(3) $j_k = j_{k+1}$ implies $i_k > i_{k+1}$.

The insertion process works essentially as in Theorem 5.3, except that it now produces a row-strict tableau with even columns, the conjugate of which then yields the contribution to the right-hand side of Eq. (8).

The algorithm proceeds exactly as before, with one change: we replace ordinary row-insertion, where i_k inserted into T_k bumps the left-most element larger than itself, by requiring that each element being inserted into a row bump the first element larger than or equal to itself. (This immediately guarantees row-strictness.) ∎

6. The Cauchy Identity for $Sp(2n, \mathbf{C})$

We begin by remarking that Eq. (1) may be obtained algebraically by using the following classical result [We; Li] from representation theory:

Theorem 6.1. *If* $l(\lambda) \leqslant n$ *then*

$$s_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}) = \sum_{\mu \subseteq \lambda} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1}) \Bigg(\sum_{\substack{\beta \dashv |\lambda/\mu| \\ \beta' \text{ even}}} c^\lambda_{\mu,\beta} \Bigg), \tag{9}$$

where $c^\lambda_{\mu,\nu}$ *denotes the Littlewood–Richardson coefficient, i.e., the multiplicity of the Schur function* s_ν *in the product* $s_\lambda \cdot s_\mu$ *(see* [Macd]*).*

A combinatorial proof of this result, and a generalisation to the case $l(\lambda) > n$, appears in [Su]. Substituting for $s_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1})$ in the Cauchy identity for ordinary Schur functions, writing $s_\lambda(t)$ for $s_\lambda(t_1, ..., t_n)$ and $sp_\mu(x)$ for $sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})$, we have

$$\prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1}$$

$$= \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} s_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}) \, s_\lambda(t)$$

$$= \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} \sum_{\mu \subseteq \lambda} sp_\mu(x) \Bigg(\sum_{\beta' \text{ even}} c^\lambda_{\mu,\beta} \Bigg) s_\lambda(t)$$

$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x) \sum_{\beta' \text{ even}} \left(\sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} c^\lambda_{\mu,\beta} s_\lambda(t) \right)$$

$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x) \sum_{\beta' \text{ even}} s_\mu(t)\, s_\beta(t)$$

$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x)\, s_\mu(t) \left(\sum_{\beta' \text{ even}} s_\beta(t) \right)$$

$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x)\, s_\mu(t) \prod_{\substack{i,j=1 \\ i<j}}^{n} (1 - t_i t_j)^{-1},$$

where the last equality follows by Littlowood's identity (8). In order to give a bijective proof of the identity (1), it clearly simplifies matters to re-write it as (see above proof)

$$\prod_{i,j=1}^{n} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1}$$
$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t) \left(\sum_{\beta' \text{ even}} s_\beta(t) \right).$$

We may enumerate the left-hand side by means of Knuth two-lines arrays

$$\mathscr{T} = \begin{pmatrix} t_{i_1} \cdots t_{i_k} \\ y_{i_1} \cdots y_{i_k} \end{pmatrix},$$

where the y_{i_j}'s are in the set $\{x_1^{\pm 1}, ..., x_n^{\pm 1}\}$; such a two-line array would correspond to the term $(t_{i_1} \cdots t_{i_k} y_{i_1} \cdots y_{i_k})$ in the expansion of the left side as a formal power series. We impose the usual lexicogaphic ordering on the arrays, viz, $t_{i_1} \leqslant \cdots \leqslant t_{i_k}$, and $t_{i_j} = t_{i_{j+1}}$ implies $y_{i_j} \leqslant y_{i_{j+1}}$.

The right-hand side clearly counts the set of all triples $(\tilde{P}_\mu(x), P_\mu(t), P_\beta(t))$, where $\tilde{P}_\mu(x)$ is a symplectic tableau of shape μ with entries $1, \bar{1}, ..., n, \bar{n}$, corresponding to the variables $\{x_1^{\pm 1}, ..., x_n^{\pm 1}\}$, $P_\mu(t)$ is an ordinary tableau with entries in $[n]$ of the same shape μ as P_μ, and $P_\beta(t)$ is an ordinary tableau of shape β with entries in $[n]$, where β has even columns.

We now demonstrate a correspondence between these two sets of objects, in the proof of

THEOREM 6.2. *There is a bijection establishing the identity*:

$$\prod_{i,j=1}^{n} (1-t_i x_j)^{-1} (1-t_i x_j^{-1})^{-1}$$
$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t_1, ..., t_n) \left(\sum_{\beta' \text{ even}} s_\beta(t_1, ..., t_n) \right). \quad (10)$$

Proof. The basic idea is to use the Berele algorithm on the bottom row of x's and x^{-1}'s in the two-line array $\mathscr{T}$, thereby obtaining a symplectic tableau $\tilde{P}_\mu$ which contributes to the symplectic Schur function $sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})$ and keeping track appropriately with the t's in the top row of the array. The remaining output of the original Berele algorithm was an up–down tableau $S_\mu^k(n)$. However, the reformulation in Lemma 2.2 encoded this as a pair (Q_μ, L), where Q_μ is a standard Young tableau of shape μ and L is a two-line array,

$$\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix},$$

with $j_1 < \cdots < j_r$ and $j_k > i_k$, all $k = 1, ..., r$. L was in turn converted into a standard Young tableau Q_β of shape β with even columns (Lemma 2.3).

By following exactly the same algorithm as in Lemma 2.2, we can go from the two-line array $\mathscr{T}$ to a triple $(\tilde{P}_\mu, P_\mu, \mathscr{L})$, where P_μ, which takes the place of Q_μ in Lemma 2.2, is no longer necessarily standard, but is an ordinary tableau of shape μ in $[n]$, and $\mathscr{L}$ is a two-line array with repeated entries, coming from the top row of t's,

$$\mathscr{L} = \begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix},$$

such that $j_1 \leqslant \cdots \leqslant j_r$. To be able to apply the general form of the Burge correspondence to $\mathscr{L}$, with the aim of converting it into a tableau of shape β with even columns, $\mathscr{L}$ must be a Burge two-line array, or equivalently, must have the properties:

(1) $j_k > i_k$, all $k = 1, ..., r$.

(2) $\mathscr{L}$ is in lexicographic order, i.e., $j_k = j_{k+1}$ implies $i_k \leqslant i_{k+1}$.

Before verifying that (1) and (2) hold, we review the procedure of Lemma 2.2 by describing it in the present context of repeated labels (the t's).

We may view the present situation as a generalisation of the setting of the Berele algorithm, whose input is a set of words $y_{i_1} \cdots y_{i_k}$, in $1, \bar{1}, ..., n, \bar{n}$, or equivalently, two-line arrays,

$$T = \begin{pmatrix} 1 \cdots k \\ y_{i_1} \cdots y_{i_k} \end{pmatrix},$$

where the top row consists of distinct, strictly increasing labels. Thus our input is now in the form of two-lines arrays,

$$\mathscr{T} = \begin{pmatrix} t_{i_1} \cdots t_{i_k} \\ y_{i_1} \cdots y_{i_k} \end{pmatrix},$$

where the bottom row is a work in $1, \bar{1}, ..., n, \bar{n}$ and the top row has labels which may be repeated, but are still in increasing order; in addition, the array is written so that $t_{i_j} = t_{i_{j+1}}$ implies $y_{i_j} \leqslant y_{i_{j+1}}$.

We begin by applying the Berele algorithm to the word $y_{i_1} \cdots y_{i_k}$. As long as there are no cancellations, this is the same as the Knuth–Schensted process; consequently we can mimic the Knuth correspondence [Kn] for the two-line array up to this point, so that if at the jth step we have built up a sequence of pairs of tableaux $(\tilde{P}_j, P_j)$, at step $j+1$ we set

$$\tilde{P}_{j+1} = \tilde{P}_j \xleftarrow{\mathscr{B}} y_{i_{j+1}} \qquad \text{(Berele insertion)}.$$

If Berele insertion does not result in a cancellation, set

$$P_{j+1} = P_j$$

with $t_{i_{j+1}}$ added in the unique position so as to force shape$(P_{j+1}) =$ shape$(\tilde{P}_{j+1})$; otherwise, shape$(\tilde{P}_{j+1}) = \mu^{j+1}$, say, has one box less than shape$(\tilde{P}_j) = \mu^j$, and we get P_{j+1} from P_j as follows:

- bump out the extra entry of P_j (the one in the unique square of μ^j which is not a square of μ^{j+1}) by columns (i.e., inverse Schensted column insertion) to get a tableau P_{j+1} of shape μ^{j+1}, and a letter x. This means that by column-inserting x into P_{j+1} we would retrieve the previous larger tableau P_j of shape μ^j.
- We record the fact that a removal has occurred at step j by putting the pair (t_{i_j}, x) into a two-line array $\mathscr{L}$, with t_{i_j} on top. Note that since x was already present in the tableau at step t_{i_j}, $x \leqslant t_{i_j}$.

We continue this process to the end of the word $y_{i_1} \cdots y_{i_k}$, arranging the two-line array $\mathscr{L}$ so that the top row is weakly increasing.

EXAMPLE 6.3. The word

$$t_1 x_1^{-1} t_1 x_2 t_2 x_1^{-1} t_2 x_2 (t_4 x_1)^2 \, t_4 x_1^{-1} t_5 x_1$$

corresponds to the two-line array

$$\mathcal{T} = \begin{pmatrix} 1 & 1 & 2 & 2 & 4 & 4 & 4 & 5 \\ \bar{1} & 2 & \bar{1} & 2 & 1 & 1 & \bar{1} & 1 \end{pmatrix}.$$

We proceed with Berele insertion of the word in the bottom row of $\mathcal{T}$, from left to right as usual. In the schematic that follows, the computation is arranged so that:

- the first row contains the successive symplectic tableaux, ending in the final tableau $\tilde{P}_\mu$,
- the second row encodes the up–down tableau resulting from the Berele insertion, ending in the final tableau P_μ,
- the third row encodes the removals in the form of pairs which, at the end of the process, may be put together into a Burge two-line array $\mathcal{L}$, so that ultimately the pair $(P_\mu, \mathcal{L})$ contains all the information to completely and uniquely specify the up–down tableau. See Fig. 7.

We have thus shown that a two-line array $\mathcal{T}$ above may be mapped to a triple $(\tilde{P}_\mu, P_\mu, \mathcal{L})$, where $\tilde{P}_\mu$ is a symplectic tableau of shape μ, P_μ is an ordinary tableau of the same shape, and $\mathcal{L}$ is a two-line array

$$\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix}$$

such that $j_1 \leqslant \cdots \leqslant j_r$ and $j_k \geqslant i_k$, all $k = 1, ..., r$, with entries from the top row of labels t. (We will see shortly that in fact $j_k > i_k$.)

If $\mathcal{L}$ has no repeated entries in its top row, we could apply the argument of Lemma 2.2 to conclude that this mapping is reversible, that is, that

$$\mathcal{T} \to \begin{pmatrix} \bar{1} & \bar{1}\,2 & \begin{array}{l}\bar{1}\,\bar{1}\\2\end{array} & \begin{array}{l}\bar{1}\,\bar{1}\,2\\2\end{array} & \begin{array}{l}\bar{1}\,2\\2\end{array} & 2\,2 & \begin{array}{l}\bar{1}\,2\\2\end{array} & 2\,2 & \tilde{P}_\mu \\ 1 & 1\,1 & \begin{array}{l}1\,1\\2\end{array} & \begin{array}{l}1\,1\,2\\2\end{array} & \begin{array}{l}1\,2\\2\end{array} & 1\,2 & \begin{array}{l}1\,2\\4\end{array} & 1\,2 & P_\mu \\ & & & & \begin{Bmatrix}4\\1\end{Bmatrix} & \begin{Bmatrix}4\\2\end{Bmatrix} & & \begin{Bmatrix}5\\4\end{Bmatrix} & \mathcal{L} \end{pmatrix}$$

$$\to \left(2\,2,\quad 1\,2,\quad \begin{Bmatrix}4\,4\,5\\1\,2\,4\end{Bmatrix}\right)$$

$$\to \left(2\,2,\quad 1\,2,\quad \begin{array}{l}1\,2\,4\\4\,4\,5\end{array}\right) = (\tilde{P}_{(2)}, P_{(2)}, P_{(3,3)}).$$

FIG. 7. The Cauchy identity bijection (a).

the pair $(P_\mu, \mathscr{L})$ contains all the information necessary to recover the up–down tableau arising from applying Berele to $y_{i_1} \cdots y_{i_k}$. Recall that the essential observation in this case is that $\mathscr{L}$ enables us to locate uniquely the labels (= steps of the Berele algorithm) at which the removals occurred.

If, on the other hand, there are repeated entries z among the top row of $\mathscr{L}$, and there are more z's in the original two-line array, then we need to know which of the labels z resulted in removals. This ambiguity is conveniently resolved by Lemma 3.2, which tells us the following:

If a letter y in the bottom row, corresponding to some repeated label z, in the top row of $\mathscr{T}$ caused a cancellation, so did all letters preceding y and having the same label z in the top row, since, in the chosen ordering for $\mathscr{T}$, all such letters are $\leqslant y$.

Thus a segment of $\mathscr{T}$ consisting of equal labels z in the top row, whose corresponding y's cause cancellations upon Berele insertion, is an initial segment of the sub-array of $\mathscr{T}$ consisting of all pairs with the label z, thereby enabling us to associate the removals with the correct labels uniquely, and consequently to retrieve the associated up–down tableau. See Fig. 8.

In general, to go from

$$\begin{pmatrix} t_{i+1} \\ T_{i+1} \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} t_i \\ T_i \end{pmatrix},$$

- if $T_{i+1} \supseteq T_i$, then T_i is the tableau obtained by deleting the rightmost entry t_{i+1} in T_{i+1},

Conversely, given $(\tilde{P}_{(2)}, P_{(2)}, P_{(3,3)})$ as above, to retrieve $\mathscr{T}$: First observe that the top row of $\mathscr{T}$ is distributed between the entries of P_μ and those of $\mathscr{L} \leftrightarrow P_\beta$. Also from the arguments in the text, we know where to place each removal pair (j, i) of $\mathscr{L}$; thus we have

$$\begin{array}{cccccccc} 1 & 1 & 2 & 2 & 4 & 4 & 4 & 5 \\ & & & & \left\{\begin{matrix}4\\1\end{matrix}\right\} & \left\{\begin{matrix}4\\2\end{matrix}\right\} & & \left\{\begin{matrix}5\\4\end{matrix}\right\} \end{array}$$

We can therefore reconstruct the up–down tableau, working backwards from P_μ and the removal pairs above:

$$\begin{array}{ccccccccc} 1 & 1 & 2 & 2 & 4 & 4 & 4 & 5 & \\ & & & & \left\{\begin{matrix}4\\1\end{matrix}\right\} & \left\{\begin{matrix}4\\2\end{matrix}\right\} & & \left\{\begin{matrix}5\\4\end{matrix}\right\} & \\ 1 & 1\,1 & \begin{matrix}1\,1\\2\end{matrix} & \begin{matrix}1\,1\,2\\2\end{matrix} & \begin{matrix}1\,2\\2\end{matrix} & 1\,2 & \begin{matrix}1\,2\\4\end{matrix} & 1\,2 & P_\mu \end{array}$$

FIG. 8. The Cauchy identity bijection (b): Reconstructing the up–down tableau.

- otherwise, there is a pair

$$\begin{Bmatrix} t_{i+1} \\ x \end{Bmatrix}$$

corresponding to the insertion at the step indexed by this occurrence of t_{i+1}, so $T_i = (x \rightarrow T_{i+1})$.

This gives the up–down tableau

$$S_\mu^8 = \left(\square \quad \square\square \quad \begin{smallmatrix}\square\square \\ \square\end{smallmatrix} \quad \begin{smallmatrix}\square\square\square \\ \square\end{smallmatrix} \quad \begin{smallmatrix}\square\square \\ \square\end{smallmatrix} \quad \square\square \quad \begin{smallmatrix}\square\square \\ \square\end{smallmatrix} \quad \square\square \right).$$

Finally the word forming the bottom row of $\mathscr{T}$,

$$\bar{1}2\bar{1}211\bar{1}1,$$

is recovered from the pair

$$(\tilde{P}_{(2)} = 22,\ S_{(2)}^8).$$

Lemma 3.2 also establishes property (1) for $\mathscr{T}$: no j_k can equal an i_k, since all labels equal to j_k which preceded it also caused cancellations and therefore are already in the top row of $\mathscr{L}$ before j_k.

To verify that property (2) holds, we invoke Lemma 3.3. Suppose x, x' are two consecutive letters in the bottom row of $\mathscr{T}$, both of which have the same label z, say, in the top row, and suppose $x \leqslant x'$ and both x, x' result in cancellations on Berele insertion, contributing labels i_p, i_{p+1} to the bottom row of the two-line array $\mathscr{L}$ (with the same top label z). Suppose the ordinary tableau prior to insertion of x was P_μ, and became P'_μ after inserting x, and P''_μ after inserting x'. (Thus $(i_{p+1} \rightarrow P''_\mu) = P'_\mu$; $i_p \rightarrow P'_\mu = P_\mu$.) By Lemma 3.3, the taquin path for x' ends in a weakly lower row than that of x, so that the bumping path of $(i_{p+1} \rightarrow P''_\mu)$ ends in a weakly lower row than that of $(i_p \rightarrow P'_\mu) = (i_p \rightarrow i_{p+1} \rightarrow P''_\mu)$, and thus a fundamental property of Schensted insertion immediately yields $i_{p+1} \geqslant i_p$.

We have shown how to construct, from the two-line array $\mathscr{T}$, a triple $(\tilde{P}_\mu, P_\mu, \mathscr{L})$. Because of properties (1) and (2) of $\mathscr{L}$, it is clear that $(P_\mu, \mathscr{L})$ encodes the up–down tableau resulting from the Berele algorithm applied to the lower row of $\mathscr{T}$; hence this up–down tableau is uniquely recoverable from $(P_\mu, \mathscr{L})$. Now it is simply a matter of reversing the Berele bijection, starting with the symplectic tableau P_μ, to retrieve the word forming the lower row of $\mathscr{T}$. The top row of $\mathscr{T}$, of course, is just the set of entries in P_μ and $\mathscr{L}$, arranged in increasing order. We note that the two-line array $\mathscr{T}$ recovered in this manner obeys the ordering defined above, as can be seen by reversing the preceding arguments.

The final step is to exhibit a bijection between two-line arrays $\mathscr{L}$ and ordinary tableau with even-columned shapes. But this is precisely what the Burge correspondence achieves: $\mathscr{L}$ is the right type of two-lines array because of (1) and (2). ∎

It is easy to see that the correspondence described above is a bijection between the stated objects as long as the number m of variables t is less than or equal to n. Hence we can state for $m \leqslant n$,

$$\prod_{1 \leqslant i < j \leqslant m} (1 - t_i t_j) \prod_{\substack{1 \leqslant i \leqslant m \\ 1 \leqslant j \leqslant n}} (1 - t_i x_j)^{-1} (1 - t_i x_j^{-1})^{-1}$$

$$= \sum_{\substack{\mu \\ l(\mu) \leqslant n}} sp_\mu(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t_1, ..., t_m). \tag{11}$$

Note that various specialisations in the formula (11) are possible. Equating coefficients of the square-free term $t_1 \cdots t_m$ in the t's we get, for $m \leqslant n$,

$$\sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k (x_1 + x_1^{-1} + \cdots + x_n + x_n^{-1})^{m-2k} \binom{m}{2k} \binom{2k}{k} \frac{k!}{2^k}$$

$$= \sum_{\substack{\lambda \vdash m \\ l(\lambda) \leqslant n}} f^\lambda sp_\lambda(x_1^{\pm 1}, ..., x_n^{\pm 1}), \tag{12}$$

and further substitution of $x_i = 1$, for all i, in this gives, for $m \leqslant n$,

$$\sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k (2n)^{m-2k} \binom{m}{2k} \binom{2k}{k} \frac{k!}{2^k} = \sum_{\substack{\lambda \vdash m \\ l(\lambda) \leqslant n}} f^\lambda \dim \tilde{N}^\lambda, \tag{13}$$

where $\tilde{N}^\lambda$ is the irreducible $Sp(2n, \mathbf{C})$-module for λ. Specialising to $x_i = 1$ in (11) gives, for $m \leqslant (n+1)$,

$$\prod_{1 \leqslant i < j \leqslant m} (1 - t_i t_j) \prod_{i=1}^{m} (1 - t_i)^{-2n} = \sum_{\substack{\lambda \\ l(\lambda) \leqslant n}} s_\lambda(t_1, ..., t_m) \dim \tilde{N}^\lambda, \tag{14}$$

another formula involving the dimensions of the irreducible representations of $Sp(2n, \mathbf{C})$.

7. A Dual Berele Algorithm

By using the formula (9) in the dual Cauchy identity

$$\prod_{i,j} (1 + x_i y_j) = \sum_\lambda s_\lambda(x)\, s_{\lambda'}(y),$$

we get an analogous dual Cauchy identity for $Sp(2n, \mathbf{C})$:

$$\prod_{1 \leqslant i, j \leqslant n} (1 + t_i x_j)(1 + t_i x_j^{-1}) = \sum_{\substack{\mu \\ l(\mu') \leqslant n}} sp_{\mu'}(x)\, s_\mu(t) \sum_{\gamma \text{ even}} s_\gamma(t). \tag{15}$$

Using the identity (8), we obtain

THEOREM 7.1 (The dual Cauchy identity for $Sp(2n, \mathbf{C})$).

$$\prod_{1 \leqslant i \leqslant j \leqslant n} (1 - t_i t_j) \prod_{1 \leqslant i, j \leqslant n} (1 + t_i x_j)(1 + t_i x_j^{-1})$$
$$= \sum_{\substack{\mu \\ l(\mu') \leqslant n}} sp_{\mu'}(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t_1, ..., t_n). \tag{16}$$

Equation (16) immediately suggests the existence of a dual to the Berele correspondence, in the spirit of the dual Knuth correspondence [Kn]. Before proceeding to present this, we adopt the following

Notation. In this section we denote the conjugate of a tableau P by P', and the conjugate of a shape μ by μ'.

First, a definition:

DEFINITION 7.2. Call a tableau P_λ of shape λ **row-symplectic** if

(1) P_λ is strictly increasing along rows, weakly increasing down columns

(2) every entry in column i of P_λ is larger than or equal to i.

Clearly P_λ is row-symplectic iff its conjugate P'_λ is symplectic.

THEOREM 7.3. *There is a bijection $\mathscr{B}*$ between words of length k in the alphabet $1, \bar{1}, ..., n, \bar{n}$ and pairs $(P_\mu *, S_\mu^k(n)*)$, where $P_\mu *$ is a row-symplectic tableau of shape μ, and $S_\mu^k(n)*$ is an up–down k-tableau of shape μ, $S_\mu^k(n)* = (\varnothing, \mu^1, ..., \mu^k = \mu)$, such that each μ^i has at most n columns.*

Proof. Let $w = w_1 \cdots w_k$ be a word of length k. The bijection is nothing but a column-insertion version of the Berele algorithm: Column-insert w $(w \to \varnothing)$ from the right end leftwards, with the bumping rule being as follows: an element x about to be inserted into column i bumps the first (highest) element in column i which is strictly larger than itself. (This ensures row-strictness.) If at some stage of the column-insertion, an i is about to displace an $\bar{\imath}$ from column i into column $i+1$,

- replace the first (highest) $\bar{\imath}$ in column i with an i; remove the first (highest) i in column i (which must be in row 1), thereby creating a punctured tableau with a hole in position $(1, i)$;

- slide the hole in row 1, column i out to the right boundary of the tableau via jeu de taquin, leaving a row-symplectic tableau whose shape has one box less than the previous shape.

It is clear that continuing in this manner produces a row-symplectic tableau of some shape μ, where μ has $\leqslant n$ columns, and a k-sequence of shapes culminating in the shape μ, such that two consecutive shapes differ by exactly one box, and each shape has at most n columns.

It is not difficult to see that the process reverses, in complete analogy with the Berele algorithm. ∎

We shall refer to this bijection as the **dual** Berele insertion algorithm, writing $(w \xrightarrow{\mathscr{B}*} \varnothing)$ to signify applying the insertion to the word w.

In order to apply this to obtain a bijective proof of (15), it is clear that we need an encoding of up–down tableaux similar in spirit to Theorem 2.3. The image set will be the same, but will be obtained differently:

LEMMA 7.4. *There is another bijection $S*$ between up–down k-tableaux S^k_μ of shape μ and the set of all pairs $(Q_\mu, L*)$ such that Q_μ is a standard Young tableau of shape μ and $L*$ is a two-line array with entries*

$$\begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix}$$

such that $j_1 < \cdots < j_r$ and $j_k > i_k$, all $k = 1, ..., r$, and $\{$entries in $Q_\mu\} \cup \{$entries in $L*\} = [k]$. The image of an up–down k-tableau $S^k_\mu = (\varnothing = \mu^0 \cdots \mu^k = \mu)$ under $S*$ is as follows:

Build up a sequence of standard tableaux, one for each shape in S^k_μ, as in Theorem 2.3: as long as the shapes are increasing, follow the usual labelling of a standard Young tableau. In general, at step $j+1$, given that we have the tableau T_j associated with μ^j, and μ^{j+1} is one box larger than μ^j, T_{j+1} is simply the tableau obtained by adding a $j+1$ to T_j in the position of the added box (in the skew-shape μ^{j+1}/μ^j).

Now suppose μ^{j+1} is one box less than μ^j; let T_j be the standard Young tableau corresponding to μ^j. To get T^{j+1}, we do the following (see Fig. 9):

(1) *bump out the extra entry of T_j (the one in the unique square of μ^j*

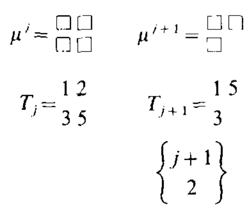

FIG. 9. A dual encoding of up–down tableaux.

which is not a square of μ^{j+1}*) by rows (i.e., inverse Schensted row insertion) to get a tableau* T_{j+1} *of shape* μ^{j+1} *and a letter* x*. This means that by row-inserting* x *into* T_{j+1} *we would retrieve the previous bigger SYT* $T_j (T_{j+1} = (T_j \leftarrow x))$*, and hence its shape* μ^j*.*

(2) *We record the fact that a removal occurred at step* j *by putting the pair* (j, i) *into a two-line array* $L*$*, with* j *on top. Note that since the* i *was bumped out at step* j*, it must have been inserted in an earlier step, so* $i < j$*.*

We continue this process to the end of the sequence. Arranging the two-line array $L*$ *so that the top row is in increasing order, we clearly end up with the requisite two-line array* $L*$ *and a SYT* Q_μ *of shape* μ *(the (final) shape of* S_μ^k*).*

Proof. That this is a bijection follows exactly as in Lemma 2.2. ∎

By using the dual Burge correspondence on the two-line array $L*$, we can convert it into an even-rowed standard Young tableau Q_γ of shape γ. Consequently we have

LEMMA 7.5. *There is a bijection between up–down tableaux* S_μ^k *of length* k *and shape* μ*, and pairs* (Q_μ, Q_γ) *of standard tableaux where* Q_μ *has shape* μ *and* Q_γ *has shape* γ*,* γ *with even rows, and* $k = |\mu| + |\gamma|$*.*

We now present the bijection which proves the dual Cauchy identity:

THEOREM 7.6 (The dual Cauchy identity for $Sp(2n, \mathbf{C})$). *There is a bijection establishing*

$$\prod_{1 \leqslant i, j \leqslant n} (1 + t_i x_j)(1 + t_i x_j^{-1})$$
$$= \sum_{\substack{\mu \\ l(\mu') \leqslant n}} sp_{\mu'}(x_1^{\pm 1}, ..., x_n^{\pm 1})\, s_\mu(t_1, ..., t_n) \left(\sum_{\gamma \text{ even}} s_\gamma(t_1, ..., t_n) \right). \quad (17)$$

Proof. The right side of the identity counts triples $(\tilde{P}_\mu*, P_\mu*, P_\gamma)$, where $\tilde{P}_\mu*$ is a row-symplectic tableau (so that its conjugate is a symplectic tableau of shape μ'), P_μ is a column-strict tableau of shape μ and P_γ is a column-strict tableau of shape γ where γ has even rows.

We may enumerate the left-hand side as two-line arrays $\mathscr{T}*$ consisting of vertically arranged pairs

$$\mathscr{T}* = \begin{pmatrix} t_{i_1} \cdots t_{i_k} \\ y_{i_1} \cdots y_{i_k} \end{pmatrix},$$

where the y's are in $1, \bar{1}, ..., n, \bar{n}$; clearly each pair occurs at most once. We choose to write $\mathscr{T}*$ by ordering the top row of t's in decreasing order from left to right, and then placing the corresponding y's so that if $t_{i_j} = t_{i_{j+1}}$ then $y_{i_j} < y_{i_{j+1}}$. For instance, the word $t_1 x_1^{-1} t_1 x_2^{-1} t_2 x_1 t_3 x_3 t_4 x_2^{-1} t_4 x_2 t_5 x_2^{-1}$ is represented by

$$\mathscr{T}* = \begin{pmatrix} 5 & 4 & 4 & 3 & 2 & 1 & 1 \\ \bar{2} & 2 & \bar{2} & 3 & 1 & \bar{1} & \bar{2} \end{pmatrix}.$$

Now we apply the dual Berele correspondence of Theorem 7.3 to the bottom row of x's and x^{-1}'s, replacing the resulting up–down tableau by a sequence of column-strict tableaux and a two-line array

$$L* = \begin{pmatrix} j_1 \cdots j_r \\ i_1 \cdots i_r \end{pmatrix},$$

constructed as in Lemma 7.4 and arranged so that the top row is increasing from left to right. Note that the only difference from the situation of Lemma 7.4 is that instead of using distinct labels we now have possibly repeated labels.

Consider the two-line array

$$\mathscr{T}* = \begin{matrix} 5 & 5 & 5 & 4 & 4 & 3 & 3 & 2 & 2 & 2 & 1 & 1 \\ 1 & \bar{1} & 2 & \bar{1} & 2 & \bar{1} & 3 & 1 & \bar{2} & \bar{4} & \bar{1} & \bar{2} \end{matrix}$$

Working from right-to-left, dual Berele-insertion on the bottom row gives:

5 1	5 $\bar{1}$	5 2	4 $\bar{1}$	4 2	3 $\bar{1}$	3 3	2 1	2 $\bar{2}$	2 $\bar{4}$	1 $\bar{1}$	1 $\bar{2}$
$\bar{1}\,3\,\bar{4}$ $\bar{1}$	$\bar{1}\,3\,\bar{4}$ $\bar{1}$ $\bar{1}$	$\bar{1}\,\bar{2}\,\bar{4}$ $\bar{1}\,3$ 2	$\bar{1}\,\bar{2}\,\bar{4}$ $\bar{1}$ 3	$\bar{1}\,\bar{2}\,\bar{4}$ $2\,\bar{2}$ 3	$\bar{1}\,\bar{2}\,\bar{4}$ $\bar{2}$ 3	$\bar{2}\,\bar{4}$ $\bar{2}$ 3	$\bar{2}\,\bar{4}$ $\bar{2}$	$\bar{1}\,\bar{2}$ $\bar{2}\,\bar{4}$	$\bar{1}\,\bar{2}$ $\bar{4}$	$\bar{1}\,\bar{2}$	$\bar{2}$
2 2 5 3	1 2 5 2 3	1 2 4 2 5 3	1 2 4 2 3	1 2 3 2 4 3	1 2 3 2 3	1 2 2 3	1 2 2	1 1 2 2	1 1 2	1 1	1
$\begin{Bmatrix} 5 \\ 1 \end{Bmatrix}$	$\begin{Bmatrix} 5 \\ 4 \end{Bmatrix}$		$\begin{Bmatrix} 4 \\ 3 \end{Bmatrix}$				$\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$				

$$\mapsto \left(\begin{matrix} \bar{1}\,3\,\bar{4} \\ \bar{1} \end{matrix}, \begin{matrix} 2\,2\,5 \\ 3 \end{matrix}, \begin{Bmatrix} 2\,4\,5\,5 \\ 1\,3\,4\,1 \end{Bmatrix} \right)$$

$$\mapsto \left(\begin{matrix} \bar{1}\,3\,\bar{4} \\ \bar{1} \end{matrix}, \begin{matrix} 2\,2\,5 \\ 3 \end{matrix}, \begin{pmatrix} 1\,3\,4 \\ 1\,4\,5 \\ 2 \\ 5 \end{pmatrix}^t \right)$$

$$\mapsto \left(\begin{matrix} \bar{1}\,3\,\bar{4} \\ \bar{1} \end{matrix}, \begin{matrix} 2\,2\,5 \\ 3 \end{matrix}, \begin{matrix} 1\,1\,2\,5 \\ 3\,4 \\ 4\,5 \end{matrix} \right)$$

FIG. 10. The bijection for the dual Cauchy identity.

Let us first observe that the procedure of Lemma 7.4 will indeed replace the sequence of shapes with column-strict tableaux: since the possibility of equal elements one above the other can only arise if, in the course of building up the row-symplectic tableau, we inserted first a y, then a $y' \geq y$ (take conjugates in Lemma 3.2). But then the associated labels t, t' are unequal, by our ordering of $\mathcal{T}*$.

We now claim that the two-line array $L*$ (whose entries are a subset of the top row of $\mathcal{T}*$) is such that

(1) if an entry y with label t causes a cancellation upon dual Berele insertion, so do all subsequent y's with the same label t. (Observe that "subsequent" now means "to the left" in $\mathcal{T}*$).

(2) $L*$ is in "dual Burge" lexicographic order: if $j_p = j_{p+1}$ then $i_p \geq i_{p+1}$; note that $j_p \leq j_{p+1}$ by construction.

As in the proof of Theorem 6.2, to show (1) we need:

LEMMA 7.7. *Let $\tilde{P}_\mu*$ be a row-symplectic tableau of shape μ, let x, x' be letters in $1, \bar{1}, ..., n, \bar{n}$ such that $x < x'$. Suppose dual Berele insertion of x' into $\tilde{P}_\mu*$ causes a cancellation. Then so does the subsequent insertion of x.*

Proof. Suppose not. Since dual Berele insertion of x' into $\tilde{P}_\mu*$ causes a cancellation, this means that when replaced by ordinary Schensted dual column-insertion, the bumping path of x' ends in row 1. But subsequent insertion of the strictly smaller x must produce a bumping path that ends strictly above that of x', contradiction. ∎

Now (1) follows easily, since a y' following a y with the same label t is, by our ordering of $\mathcal{T}*$, strictly less than y. To show (2) we need:

LEMMA 7.8. *Let $\tilde{P}_\mu*$ be a row-symplectic tableau of shape μ, and let x, x' be such that $x < x'$ and dual Berele insertion of x' into $\tilde{P}_\mu*$ causes a cancellation, as does the subsequent insertion of x. Then the taquin path of x ends in the same row as, or lower than, that of x'.*

Proof. Simply take conjugates in the proof of Lemma 3.3. ∎

To see why this gives us (2), observe that if $j_p = j_{p+1}$, and the label j_p was for an element x', and j_{p+1} was for x, then since insertion of x' preceded insertion of x, we must have $x < x'$ according to the ordering chosen for $\mathcal{T}*$. Also both x and x' caused cancellations (j_p, j_{p+1} both appear in the top row of $L*$), so Lemma (7.8) above applies. But this implies the following:

Suppose that j_p resulted in row-removing an i_p from the column-strict tableau $P_\mu*$ constructed up to this point, and then j_{p+1} resulted in row-removing i_{p+1}. Then re-inserting i_{p+1} row-wise recovers the end of the taquin path of x, and subsequently re-inserting i_p reproduces the end of the taquin path of x'.

Consequently the bumping path of i_{p+1} ends below or in the same row as that of i_p, which implies $i_{p+1} \geqslant i_p$.

It remains to observe that properties (1) and (2) of $L*$ ensure that the labels corresponding to cancellations can be uniquely identified from $L*$, and hence, using P_μ, the up–down tableau is recoverable as before, thereby leading to the retrieval of the word forming the bottom row of $\mathscr{T}*$. The top row consists of the entries in P_μ and $L*$, written in the pre-determined order. (Note that reversing the above arguments shows that the recovered two-line array $\mathscr{T}*$ will obey the correct ordering relations.)

Finally, because of (1) and (2), $L*$ is precisely the type of two-line array to which the dual Burge correspondence applies, producing an even-columned row-strict tableau, and thus (by taking conjugates) an even-rowed column-strict tableau. ∎

Figure 11 shows how the bijection reverses.

Continuing with the example of Fig. 10, given the triple

$$\left(\begin{matrix}\bar{1}\,3\,\bar{4}\\ \bar{1}\end{matrix},\ \begin{matrix}2\,2\,5\\ 3\end{matrix},\ \begin{matrix}1\,1\,2\,5\\ 3\,4\\ 4\,5\end{matrix}\right),$$

we know the top row of $\mathscr{T}$ and we know precisely at which of these labels the removal pairs corresponding to the even-rowed tableau are located. Thus, working backwards (i.e., **left-to-right**) from the tableau

$$\begin{matrix}2\,2\,5\\ 3\end{matrix},$$

we get the up–down tableau whose shapes are those of the sequence below:

5	5	5	4	4	3	3	2	2	2	1	1
$\begin{Bmatrix}5\\1\end{Bmatrix}$	$\begin{Bmatrix}5\\4\end{Bmatrix}$		$\begin{Bmatrix}4\\3\end{Bmatrix}$				$\begin{Bmatrix}2\\1\end{Bmatrix}$				
2 2 5 3	1 2 5 2 3	1 2 4 2 5 3	1 2 4 2 3	1 2 3 2 4 3	1 2 3 2 3	1 2 2 3	1 2 2	1 1 2 2	1 1 2	1 1	1
(←1)	(←4)	remove right –most 5	(←3)	remove right –most 4	remove right –most 3	remove right –most 3	(←1)	remove right –most 2	remove right –most 2	remove right –most 1	

FIG. 11. The dual bijection in reverse.

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