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ASYMPTOTIC EXPANSIONS IN MULTIVARIATE RENEWAL THEORY

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Let P be a probability measure on $\mathbb{R}^{m+1}$, and $R=\sum_{n=0}^{\infty} P^{*n}$ the associated renewal measure. A two term asymptotic expansion for R is derived under moment and smoothness conditions. The smoothness conditions imposed allow P to be arithmetic is some coordinates and absolutely continuous in the other coordinates.

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random walks * smoothing * Fourier analysis

1. Introduction

Let P be a probability measure on $\mathbb{R}^{m+1}$, and $R=\sum_{n=0}^{\infty} P^{*n}$ the associated renewal measure, where the $*$ denotes convolution. The main results of this paper concern approximations for the multivariate renewal measure R. In one dimension ($m=0$), the renewal theorem shows that far from the origin R may be approximated by Lebesgue measure over the mean of P (or a suitable multiple of counting measure in the arithmetic case). Extensions of this result to higher dimensions have been pursued by Bickel and Yahav (1965), Doney (1966), Stam (1968, 1969, 1971), Carlsson (1982) and Höglund (1988). The basic limit theory in the multivariate case is more interesting: now R is approximated by the product of Lebesgue measure over the length of the mean of P in the direction of drift with a normal measure in the orthogonal direction.

Asymptotic expansions for renewal measures in the plane are given by Keener (1988), and expansions for multivariate renewal measures are given by Carlsson and Wainger (1982). The results here extend the results of Keener (1988) to higher dimensions. Although the method of proof used here is similar, there are important technical differences discussed in the concluding remarks section. For simplicity, one less term is included in the expansions presented here. The results of Carlsson

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and Wainger (1982) approximate $R(s+B)$ as the projection of s in the direction of the mean of P approaches infinity, where B is an arbitrary parallelepiped. By contrast, the results here are designed to approximate the renewal measure of larger sets (see Corollary 2 below). Using linearity of R, their result has sufficient uniformity to approximate certain large sets (such as those that are disjoint unions of parallelepipeds), but the class of sets allowed cannot be as large as that considered in Corollary 2. Also, the lattice case is not covered by their result, and the moment conditions assumed here are less stringent.

This research is largely motivated by applications to boundary crossing problems arising in sequential analysis—this is pursued in Keener (1987, 1989). Suppose $\{S_n\}_{n\geqslant 0}$ is a random walk with $S_1 \sim P$, and $t = t_a = \inf\{n \geqslant 0\colon v\cdot S_n > a\}$ where v is a fixed vector. If $v\cdot S_1 \geqslant 0$ a.s., then a simple calculation shows that the distribution of S_t can be expressed as an integral against the renewal measure R. This integral can be approximated using Theorem 1 or 3 below, and this leads to an expansion for the distribution of S_t as $a\to\infty$. To relax the restriction $v\cdot S_1 > 0$ and to approximate the joint distribution of t and S_t, ladder variables are introduced. In this extension, the renewal measure of interest has P the joint distribution of t_0 and S_{t_0}. Hence it is crucial in these applications that we allow one coordinate of the random walk to have an arithmetic distribution, while the other coordinates are continuous. To describe the smoothness condition used for the nonarithmetic coordinates, call a random vector y *arithmetic* on $\mathbb{Z}^m$ if $P(Y\in\mathbb{Z}^m)=1$ but $P(Y\in B)<1$ for B any proper subgroup of $\mathbb{Z}^m$. In renewal theory, a random vector $X\in\mathbb{R}^j$ is called *strongly nonlattice* if $\liminf_{|p|\to\infty}|1-E\,\mathrm{e}^{ip\cdot X}|>0$, which is equivalent to Cramér's condition, $\limsup_{|p|\to\infty}|E\,\mathrm{e}^{ip\cdot X}|<1$. For the mixed case, a random vector $X\in\mathbb{R}^j$ is *strongly nonlattice* with Y if Y is arithmetic on $\mathbb{Z}^m$ and

$$\liminf_{|p|\to\infty}\ \inf_{q\in[-\pi,\pi]^m}|1-E\,\mathrm{e}^{ip\cdot X+iq\cdot Y}|>0.$$

This condition plays the same role here that Cramér's condition plays for expansions in the central limit theorem. An easy consequence of this condition is

$$\inf_{\mathbb{R}^j\times[-\pi,\pi]^m-N_0}|1-E\,\mathrm{e}^{ip\cdot X+iq\cdot Y}|>0$$

for any neighborhood N_0 of the origin. This in turn implies that

$$\sup_{\mathbb{R}^j\times[-\pi,\pi]^m-N_0}|1+E\,\mathrm{e}^{ip\cdot X+iq\cdot Y}|<2.$$

Let $(X,Y)\sim P$ with $X\in\mathbb{R}$ and $Y\in\mathbb{R}^m$. Also, let $\nu=EX$, $\gamma=EY/\nu$, $\Sigma=\operatorname{cov}(Y-\gamma X)$ and $Z=\Sigma^{-1/2}(Y-\gamma X)$. The following conditions are assumed in the sequel:

(A1) $Y=(Y_1,Y_2)$ where $Y_1\in\mathbb{R}^{m_1}$, Y_2 is arithmetic on $\mathbb{Z}^{m_2}$ (so $m_1+m_2=m$) and (X,Y_1) is strongly nonlattice with Y_2.

(A2) $\nu>0$ and $0<\Sigma<\infty$.

(A3) $E|Z|^{3+\delta}<\infty$ and $E|X|^{(3+\delta)/2}<\infty$ for some $\delta\in[0,1)$.

The cases $m_1=0$ and $m_2=0$ are allowed. When $m_2=0$, the interpretation for (A1) is that (X, Y) is strongly nonlattice, and when $m_1=0$, the interpretation is that X is strongly nonlattice with Y.

Let λ_1 be Lebesgue measure on $\mathbb{R}^{1+m_1}$, λ_2 counting measure on $\mathbb{Z}^{m_2}$ and $\lambda=\lambda_1\times\lambda_2$. Define the polynomials M_1 and M_2 as

$$M_1(q)=EXq\cdot Z \quad \text{and} \quad M_2(q)=\tfrac{1}{6}E(q\cdot Z)^3.$$

The polynomial g is

$$g(q)=-M_1(q)/\nu+M_2(q)-q^2M_1(q)/(2\nu),$$

and the associated multidimensional Hermite polynomial $\boldsymbol{H}_g$ is defined by

$$\boldsymbol{H}_g(\hat{z})\,\mathrm{e}^{-\hat{z}^2/2}=g(-\partial/\partial\hat{z})\,\mathrm{e}^{-\hat{z}^2/2}.$$

The measure $\hat{R}$ approximating R has λ-density given by

$$\hat{r}(x,y)=\frac{1}{\nu}\,\frac{\mathrm{e}^{-\hat{z}^2/2}}{\sqrt{|\Sigma|}(2\pi x/\nu)^{m/2}}\left\{1+\sqrt{\frac{\nu}{x}}\,\boldsymbol{H}_g(\hat{z})\right\},$$

where $\hat{z}=\Sigma^{-1/2}(y-\gamma x)/\sqrt{x/\nu}$ and $|\Sigma|$ is the determinant of Σ. Define the *oscillation function* $\omega_f(\cdot\,;\varepsilon)$ as

$$\omega_f(x,y;\varepsilon)=\sup\{|f(x,y)-f(x_1,y_1)|:(x-x_1)^2+(y-y_1)^2\le\varepsilon^2\}.$$

Let $\mathcal{F}_a$ be the set of all positive Borel measurable functions $f:\mathbb{R}^{m+1}\to[0,1]$ such that $f(x,y)=0$ whenever $x\notin[a,a+1]$.

Theorem 1. *For some* $\eta>0$,

$$\int f\,\mathrm{d}R=\int f\,\mathrm{d}\hat{R}+\mathrm{o}\{a^{(-1-\delta)/2}\}+\mathrm{o}\{a^{(-1-\delta-m)/2}\}\int[f+\omega_f(\cdot\,;\mathrm{e}^{-\eta a})]\,\mathrm{d}\lambda$$
$$+\mathrm{O}(1)\int\omega_f(\cdot\,;\mathrm{e}^{-\eta a})\,\mathrm{d}\hat{R}$$

as $a\to\infty$, *uniformly for* $f\in\mathcal{F}_a$.

With f an indicator function, Theorem 1 can be applied to give approximations for the renewal measure of sets with well behaved boundaries. One such result is the following corollary. Let $(\partial B)_\varepsilon$ denote all points within ε of the boundary of B.

Corollary 2. *If B is a bounded Borel set in $\mathbb{R}^m$ and if the Lebesgue measure of $(\partial B)_\varepsilon$ approaches zero at an algebraic rate as $\varepsilon\downarrow 0$, then*

$$R([a,a+1]\times\sqrt{a}[B+\gamma a])=\hat{R}([a,a+1]\times\sqrt{a}[B+\gamma a])+\mathrm{o}\{a^{(-1-\delta)/2}\}$$

as $a\to\infty$. □

The main weakness of Theorem 1 is that the answer is typically vacuous unless f vanishes as $|y|\to\infty$. In the setting of Corollary 2, this precludes approximations when B is an unbounded set. This deficiency is removed in Theorem 3 at the expense of a slightly larger error rate. Since both theorems give global approximations, they are uninteresting for those $f \in \mathscr{F}_a$ where $\int f \,\mathrm{d}\hat{R} = \mathrm{o}\{a^{(-1-\delta)/2}\}$, which can happen if $f \to 0$ too quickly as $|y - \gamma a| \to \infty$. For many applications, the main concern is accurate approximation for sequences of functions $f = f_a$ with form (or approximate form) $g(x-a, (y-\gamma a)/\sqrt{a})$, where $g \in \mathscr{F}_0$. Then $\int f_a \,\mathrm{d}\hat{R}$ has a positive limit (unless $g \equiv 0$) and the approximations here have small relative error.

Theorem 3. *For some* $\eta > 0$,

$$\int f \,\mathrm{d}R = \int f \,\mathrm{d}\hat{R} + \mathrm{O}(1) \int \omega_f(\cdot\,; \mathrm{e}^{-\eta a}) \,\mathrm{d}\hat{R} + \mathrm{o}\{a^{(-1-\delta)/2} (\log a)^{m/2}\}$$

as $a \to \infty$, *uniformly for* $f \in \mathscr{F}_a$.

2. Proofs of Theorem 1 and Theorem 3

The proof of Theorem 1 is similar in many respects to proofs for expansions in the central limit theorem. The spirit of the argument is as follows. Suppose H is a probability measure dominated by λ. Then $H * R < \lambda$ and it is possible to find the density of $H * R$ under regularity conditions for H by Fourier inversion. From the inversion formula it is possible to derive an asymptotic expansion for $\mathrm{d}H * R/\mathrm{d}\lambda$ and integrals against this density. This expansion is applied with H almost a point mass at the origin to obtain the expansion for R.

This argument needs two modifications. First, it is convenient to work with the symmetrized renewal measure $W = R + \bar{R}$ where $\bar{P}$ is the distribution of $-(X, Y)$ and $\bar{R} = \sum_{n=0}^{\infty} \bar{P}^{*n}$. This helps with integrability problems approximating the integrand in the inversion formula near the origin. Second, to make effective use of (A1) (the replacement for Cramer's condition), it is useful to convolve W with P_0^{*n} where $P_0 = \frac{1}{2}(\delta_0 + P)$ and δ_0 is a point mass at the origin. With this device, contributions to the inversion integral outside a neighborhood of the origin become small at an exponential rate, which allows H to approach δ_0 extremely quickly. The effect of convolving W with P_0^{*n} is similar to removing initial terms from the sum defining R, and the error induced by this convolution is handled by probabilistic arguments.

Let $\phi(p, q) = E\, \mathrm{e}^{\mathrm{i}pX + \mathrm{i}q\cdot Z}$, the joint characteristic function of X and Z. Let $\tilde{S} = \mathbb{R}^{1+m_1} \times [-\pi, \pi]^{m_2}$ and $S = \mathbb{R} \times \Sigma^{1/2} (\mathbb{R}^{m_1} \times [-\pi, \pi]^{m_2})$. Suppose H is a probability measure on $\mathbb{R}^{1+m_1} \times \mathbb{Z}^{m_2}$ with characteristic function $\tilde{h}$. If $\int_{\tilde{S}} |\tilde{h}| < \infty$, then $H < \lambda$ with density given by the inversion formula

$$\begin{aligned}
\frac{\mathrm{d}H}{\mathrm{d}\lambda}(x, y) &= \frac{1}{(2\pi)^{m+1}} \int_{\tilde{S}} \tilde{h}(p, q)\, \mathrm{e}^{-\mathrm{i}px - \mathrm{i}q\cdot y} \,\mathrm{d}p \,\mathrm{d}q \\
&= \frac{1}{(2\pi)^{m+1}\sqrt{|\Sigma|}} \int_S \tilde{h}(p - q\cdot \Sigma^{-1/2}\gamma, \Sigma^{-1/2} q)\, \mathrm{e}^{-\mathrm{i}px - \mathrm{i}q\cdot \Sigma^{-1/2}(y - \gamma x)} \,\mathrm{d}p \,\mathrm{d}q \\
&= \frac{1}{(2\pi)^{m+1}\sqrt{|\Sigma|}} \int_S h(p, q)\, \mathrm{e}^{-\mathrm{i}px - \mathrm{i}q\cdot z} \,\mathrm{d}p \,\mathrm{d}q,
\end{aligned}$$

where $h(p, q) = \tilde{h}(p - q \cdot \Sigma^{-1/2}\gamma, \Sigma^{-1/2}q)$ and z is related to x and y by $z = \Sigma^{-1/2}(y - \gamma x)$. In the sequel, z will always be related in this fashion to x and y. The function h is called the *transformed characteristic function* of H. The following lemma is the inversion formula that serves as the starting point for this work.

Lemma 4. *If* $\int_{\tilde{S}} |\tilde{h}| < \infty$ *then* $H * W < \lambda$ *and*

$$\frac{\mathrm{d}H * W}{\mathrm{d}\lambda}(x, y) = \frac{1}{\pi(2\pi)^m\sqrt{|\Sigma|}} \int_S h\psi\, \mathrm{e}^{-\mathrm{i}px - \mathrm{i}q\cdot z}\, \mathrm{d}p\, \mathrm{d}q,$$

where $\psi = \mathscr{R}\{1/(1-\phi)\}$.

Proof. This result is similar to Lemma 2.3 of Keener (1988). For $0 < r < 1$, the finite measure $H * \sum_{n=0}^{\infty} r^n(P^{*n} + \bar{P}^{*n})$ has characteristic function $2\tilde{h}\mathscr{R}\{1/(1 - r\tilde{\phi})\}$, where $\tilde{\phi}$ is the characteristic function of P. By monotone convergence and the inversion formula,

$$\frac{\mathrm{d}H * W}{\mathrm{d}\lambda}(x, y) = \lim_{r \uparrow 1} \frac{1}{\pi(2\pi)^m\sqrt{|\Sigma|}} \int_S h\mathscr{R}\left[\frac{1}{1 - r\phi}\right] \mathrm{e}^{-\mathrm{i}px - \mathrm{i}q\cdot z}\, \mathrm{d}p\, \mathrm{d}q.$$

Since $\mathscr{R}(1 - \phi) \geq 0$,

$$|1 - r\phi| = r\left\{\frac{(1-r)^2}{r^2} + |1 - \phi|^2 + 2\frac{1-r}{r}\mathscr{R}(1 - \phi)\right\}^{1/2} \geq r|1 - \phi|,$$

so the integrand is dominated by $2|h|/|1 - \phi|$ for $r > \frac{1}{2}$. By Taylor expansion of ϕ near the origin (see Lemma 8), $|1 - \phi| \sim \sqrt{p^2\nu^2 + \frac{1}{4}q^4}$ as $(p, q) \to 0$. By (A1), $1/|1 - \phi|$ is bounded on $S - N_0$ for any neighborhood N_0 of the origin. Hence $|h|/|1 - \phi|$ is integrable over S, and the lemma follows by dominated convergence. □

By the same proof, if ψ is replaced by $\frac{1}{2}(1 - \phi)^{-1}$, the inversion integral gives the density for $H * R$. Although it may be possible to prove Theorem 1 from this inversion formula for $H * R$, our attempts have failed. The problem is that the rational approximation $\hat{\psi}$ for ψ (presented below) has slightly better behavior than the corresponding rational approximation for $(1 - \phi)^{-1}$ near the origin. For more details, see the comments following the proof of Lemma 5 at the end of Section 3.

Let $U_n = H * (P_0^{*n}) * W$ and $u_n = \mathrm{d}U_n/\mathrm{d}\lambda$. By Lemma 4, if $\int_{\tilde{S}} |\tilde{h}| < \infty$,

$$u_n(x, y) = \frac{1}{\pi(2\pi)^m\sqrt{|\Sigma|}} \int_S h\phi_0^n\psi\, \mathrm{e}^{-\mathrm{i}px - \mathrm{i}q\cdot z}\, \mathrm{d}p\, \mathrm{d}q$$

where $\phi_0 = \frac{1}{2}(1 + \phi)$, the transformed characteristic function of P_0. The main contribution to this integral comes from a small neighborhood for the origin. Accordingly, an important technical task will be approximating ψ and ϕ_0 near the origin. Since a painful amount of delicate algebra is necessary, this derivation will be deferred to Section 3. The resulting approximations are

$$\hat{\psi}=\frac{\frac{1}{2}}{-\mathrm{i}p\nu+\frac{1}{2}q^2}+\frac{\frac{1}{2}}{\mathrm{i}p\nu+\frac{1}{2}q^2}-\frac{\frac{1}{2}pM_1++\frac{1}{2}\mathrm{i}M_2}{(-\mathrm{i}p\nu+\frac{1}{2}q^2)^2}-\frac{\frac{1}{2}pM_1-\frac{1}{2}\mathrm{i}M_2}{(\mathrm{i}p\nu+\frac{1}{2}q^2)^2}$$

for ψ, and

$$\hat{\chi}_n=\mathrm{e}^{-nq^2/4-2nKp^2}(1+\tfrac{1}{2}npM_1-\tfrac{1}{2}\mathrm{i}nM_2)$$

for $\chi_n=\phi_0^n\,\mathrm{e}^{-\mathrm{i}n\nu p/2}$, where K is a constant that will be set later and M_1 and M_2 are the polynomials in q given in the introduction. The following result from Section 3 will be used to prove Theorem 1. Primes denote $\partial/\partial p$, and $N_0=[-\rho,\rho]^{m+1}$ is a small neighborhood of the origin with $\rho>0$ to be chosen later.

Lemma 5.

$$\int_{N_0}|(\chi_n\psi)'-(\hat{\chi}_n\hat{\psi})'|=\mathrm{o}\{n^{(1-\delta-m)/2}\}$$

as $n\to\infty$.

The next lemma approximates u_n.

Lemma 6. *For some* $\alpha>0$,

$$u_n=\hat{r}+\mathrm{O}(1)\left\{\mathrm{e}^{-\alpha x}\int_S|h|+\sup_{N_0}|1-h|\right\}+\mathrm{o}\{x^{(-1-\delta-m)/2}\}$$

as $x\to\infty$ *and* $n\to\infty$ *with* $n/x\to c\in(0,1/\nu)$, *uniformly in* H.

Proof. Since

$$\left|\int_S h\phi_0^n\psi-\int_{N_0}\phi_0^n\psi\right|\le\sup_{N_0}|1-h|\int_{N_0}|\psi|+\left[\sup_{S-N_0}|\phi_0|\right]^n\sup_{S-N_0}|\psi|\int_S|h|,$$

it is sufficient to show that

$$\frac{1}{\pi(2\pi)^m\sqrt{|\Sigma|}}\int_{N_0}\phi_0^n\psi\,\mathrm{e}^{-\mathrm{i}px-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q=\hat{r}+\mathrm{o}\{x^{(-1-\delta-m)/2}\}$$

as $x\to\infty$. Now

$$\begin{aligned}\int_{N_0}\phi_0^n\psi\,\mathrm{e}^{-\mathrm{i}px-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q&=\int_{\mathbb{R}^{m+1}}\hat{\chi}_n\hat{\psi}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q\\&\quad-\int_{\mathbb{R}^{1+m}-N_0}\hat{\chi}_n\hat{\psi}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q\\&\quad+\int_{N_0}\{\chi_n\psi-\hat{\chi}_n\hat{\psi}\}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q.\end{aligned}\qquad(1)$$

The second integral is exponentially small. The third integral after an integration by parts equals

$$\frac{-\mathrm{i}}{x-\frac{1}{2}n\nu}\int_{N_0}\{(\chi_n\psi)'-(\hat{\chi}_n\hat{\psi})'\}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q$$
$$+\frac{\mathrm{i}}{x-\frac{1}{2}n\nu}\int_{[-\rho,\rho]^m}\{\chi_n\psi-\hat{\chi}_n\hat{\psi}\}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}\Big|_{p=-\rho}^{p=\rho}\,\mathrm{d}q.$$

The second integral here is exponentially small. The first is $o\{x^{(-1-\delta-m)/2}\}$ by Lemma 5. To study the first integral in (1), let $\mathscr{C}$ be the path in the complex plane where $\mathscr{R}\{p\}$ goes from $-\infty$ to ∞ with $\mathscr{I}\{p\}=-\frac{1}{2}|\mathscr{R}\{p\}|$. By residue theory,

$$I=\int_{\mathbb{R}^{m+1}}\hat{\chi}_n\hat{\psi}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}\,\mathrm{d}p\,\mathrm{d}q=\int_{\mathbb{R}^m}\mathrm{d}q\int_{\mathscr{C}}\mathrm{d}p\,\hat{\chi}_n\hat{\psi}\,\mathrm{e}^{-\mathrm{i}p(x-n\nu/2)-\mathrm{i}q\cdot z}.$$

The advantage of introducing $\mathscr{C}$ is that $\mathrm{e}^{-\mathrm{i}p(x-n\nu 2)}$ decreases at an exponential rate in the tails of the contour but $|\mathrm{e}^{-2nKp^2}|<1$ on $\mathscr{C}$, so integrability is never an issue. Rescaling p and q,

$$I=n^{-m/2}\int_{\mathbb{R}^m}\mathrm{d}q\int_{\mathscr{C}}\mathrm{d}p\{A_0+A_1/\sqrt{n}+A_2/n\}$$
$$\times\exp\{-\tfrac{1}{4}q^2-2Kp^2/n-\mathrm{i}p(x/n-\tfrac{1}{2}\nu)-\mathrm{i}q\cdot z/\sqrt{n}\},$$

where

$$A_0=\frac{\frac{1}{2}}{-\mathrm{i}p\nu+\frac{1}{2}q^2}+\frac{\frac{1}{2}}{\mathrm{i}p\nu+\frac{1}{2}q^2},$$
$$A_1=-\tfrac{1}{2}A_0(pM_1+\mathrm{i}M_2)-\frac{\frac{1}{2}pM_1+\frac{1}{2}\mathrm{i}M_2}{(-\mathrm{i}p\nu+\frac{1}{2}q^2)^2}-\frac{\frac{1}{2}pM_1-\frac{1}{2}\mathrm{i}M_2}{(\mathrm{i}p\nu+\frac{1}{2}q^2)^2}$$

and A_2 is a rational function. The contribution to I from the integral against A_2 is $O(n^{-1-m/2})$. Since $n|\mathrm{e}^{-2Kp^2/n}-1|\leq 2K|p|^2$ on $\mathscr{C}$, dominated convergence gives

$$I=O(n^{1-m/2})+n^{-m/2}\int_{\mathbb{R}^m}\mathrm{d}q\int_{\mathscr{C}}\mathrm{d}p\{A_0+A_1/\sqrt{n}\}$$
$$\times\exp\{-\tfrac{1}{4}q^2-\mathrm{i}p(x/n-\tfrac{1}{2}\nu)-\mathrm{i}q\cdot z/\sqrt{n}\}.$$

At this stage the integration over p is easily accomplished by completing the contour and using residue theory. The relevant identities are

$$\int_{\mathscr{C}}\frac{f(p)}{-\mathrm{i}p\nu+\frac{1}{2}q^2}\exp\{-\tfrac{1}{4}q^2-\mathrm{i}p(x/n-\tfrac{1}{2}\nu)\}\,\mathrm{d}p=\frac{2\pi}{\nu}f\left(\frac{-\mathrm{i}q^2}{2\nu}\right)\exp\left\{-\frac{xq^2}{2n\nu}\right\}$$

and

$$\int_{\mathscr{C}}\frac{f(p)}{(-\mathrm{i}p\nu+\frac{1}{2}q^2)^2}\exp\{-\tfrac{1}{4}q^2-\mathrm{i}p(x/n-\tfrac{1}{2}\nu)\}\,\mathrm{d}p$$
$$=\frac{2\pi}{\nu}\left\{\frac{\mathrm{i}}{\nu}f'\left(\frac{-\mathrm{i}q^2}{2\nu}\right)+\left(\frac{x}{n\nu}-\frac{1}{2}\right)f\left(\frac{-\mathrm{i}q^2}{2\nu}\right)\right\}\exp\left\{-\frac{xq^2}{2n\nu}\right\},$$

where f is any polynomial in p. Similar integrals where the denominator is $(ip\nu+\frac{1}{2}q^2)$ or $(ip\nu+\frac{1}{2}q^2)^2$ vanish because the integrand has no singularities in the lower half plane. Integrating over p,

$$I=n^{-m/2}\frac{\pi}{\nu}\int\left\{1-\frac{i}{\sqrt{n}}\left[\frac{M_1}{\nu}+\frac{x}{n\nu}\left(M_2-\frac{q^2M_1}{2\nu}\right)\right]\right\}\exp\left\{-\frac{xq^2}{2n\nu}-\frac{iq\cdot z}{\sqrt{n}}\right\}dq$$
$$+O(n^{1-m/2})$$
$$=\frac{\pi}{\nu(x/\nu)^{m/2}}\int\left\{1-i\sqrt{\frac{\nu}{x}}\left[\frac{M_1}{\nu}+\left(M_2-\frac{q^2M_1}{2\nu}\right)\right]\right\}e^{-q^2/2-iq\cdot\hat{z}}\,dq$$
$$+O(n^{1-m/2})$$

where $\hat{z}=z/\sqrt{x/\nu}$. To evaluate the integral over q, start with the identity

$$e^{-\hat{z}^2/2}=\frac{1}{(2\pi)^{m/2}}\int e^{-q^2/2-iq\cdot\hat{z}}\,dq. \tag{2}$$

Using the differential operator $g(-\partial/\partial\hat{z})$ on (2),

$$H_g(\hat{z})\,e^{-\hat{z}^2/2}=\frac{1}{(2\pi)^{m/2}}\int g(iq)\,e^{-q^2/2-iq\cdot z}\,dq.$$

Hence

$$I=O(n^{1-m/2})+\frac{\pi(2\pi)^{m/2}}{\nu(x/\nu)^{m/2}}\left\{1+\sqrt{\frac{\nu}{x}}\,H_g(\hat{z})\right\}e^{-\hat{z}^2/2}$$

which proves the lemma. □

Proof of Theorem 1. Let $H^{(1)}$ be a probability measure with the following properties: $H^{(1)}$ is degenerate in the last m_2 coordinates, i.e., $H^{(1)}(\mathbb{R}^{1+m_1}\times\{0\}^{m_2})=1$; the support of $H^{(1)}$ is a subset of the unit ball; and $\int_S|h^{(1)}|<\infty$. The measure $H^{(\varepsilon)}$ will be $H^{(1)}$ scaled by ε, i.e., $H^{(\varepsilon)}(B)=H^{(1)}(B/\varepsilon)$ for any Borel set B. Note that $H^{(\varepsilon)}$ has support contained in the ε-ball and this implies the smoothing bound

$$\left|\int f\{d(W*P_0^{*n})-d(H^{(\varepsilon)}*W*P_0^{*n})\}\right|\leq\int\omega_f(\cdot\,;\varepsilon)\,d(H^{(\varepsilon)}*W*P_0^{*n}). \tag{3}$$

Now let $\varepsilon=\varepsilon_a=e^{-\eta a}$ and $n=n_a$ where $n/a\to c\in(0,1/\nu)$ as $a\to\infty$. Since the transformed characteristic function of $H^{(\varepsilon)}$ is $h^{(\varepsilon)}(p,q)=h^{(1)}(\varepsilon p,\varepsilon q)$, if $\eta>0$ is sufficiently small,

$$e^{-a\alpha}\int_S|h^{(\varepsilon)}|+\sup_{N_0}|1-h^{(\varepsilon)}|=o\{a^{(-1-\delta-m)/2}\}$$

as $a\to\infty$. Let $U_n^{(\varepsilon)}=H^{(\varepsilon)}*(P_0^{*n})*W$ and $u_n^{(\varepsilon)}=dU_n^{(\varepsilon)}/d\lambda$. Then by Lemma 6,

$$u_n^{(\varepsilon)}=\hat{r}+o\{a^{(-1-\delta-m)/2}\}$$

as $a\to\infty$, uniformly for $x\in[a,a+1]$. Hence

$$\int f\,\mathrm{d}(H^{(\varepsilon)}*W*P_0^{*n})=\int f\hat{r}\,\mathrm{d}\lambda+\mathrm{o}\{a^{(-1-\delta-m)/2}\}\int f\,\mathrm{d}\lambda \tag{4}$$

as $a\to\infty$, uniformly for $f\in\mathscr{F}_a$. Also,

$$\int \omega_f(\cdot\,;\varepsilon)\,\mathrm{d}(H^{(\varepsilon)}*W*P_0^{*n})=\int \omega_f(\cdot\,;\varepsilon)[\hat{r}+\mathrm{o}\{a^{(-1-\delta-m)/2}\}]\,\mathrm{d}\lambda$$

as $a\to\infty$, uniformly for $f\in\mathscr{F}_a$. Using (3) and (4) the proof will be completed showing that integrals against R are close to integrals against $W*P_0^{*n}$. Using Brillinger's (1962) rate for convergence in the weak law of large numbers, since the x-marginal of P_0 has mean $\frac{1}{2}\nu$ and a finite moment of order $\frac{1}{2}(3+\delta)$,

$$P_0^{*n}\{(-\infty,0]\times\mathbb{R}^m\}=\mathrm{o}\{n^{(-1-\delta)/2}\}=\mathrm{o}\{a^{(-1-\delta)/2}\} \tag{5}$$

as $a\to\infty$. Let $S_n=\sum_{j=1}^n X_j$ where $X_1,X_2,\dots,$ are i.i.d. with $X_j\sim X$. Let

$$\Lambda=\sup_{t\in\mathbb{R}} R([t,t+1]\times\mathbb{R}^m)$$

which is finite by the renewal theorem in one dimension. Let $M=\inf\{S_n: n\geqslant 0\}$. Since $E|X|^{(3+\delta)/2}<\infty$, $E|M|^{(1+\delta)/2}<\infty$. Conditioning on the first time $S_j\leqslant -a$,

$$\begin{aligned}\bar{R}([a,a+1]\times\mathbb{R}^m)&=R([-a-1,-a]\times\mathbb{R}^m)\\ &\leqslant \Lambda P(M\leqslant -a)=\mathrm{o}\{a^{(-1-\delta)/2}\}\end{aligned} \tag{6}$$

as $a\to\infty$. Similarly, if $M=\inf\{\frac{3}{2}j\nu-S_j: j\geqslant 0\}$, then

$$\begin{aligned}P(S_j\geqslant a,\exists j\leqslant n)&=P(\tfrac{3}{2}j\nu-S_j\leqslant\tfrac{3}{2}j\nu-a,\exists j\leqslant n)\\ &\leqslant P(\tfrac{3}{2}j\nu-S_j\leqslant\tfrac{3}{2}n\nu-a,\exists j\geqslant 0)=\mathrm{o}\{a^{(-1-\delta)/2}\}\end{aligned}$$

as $a\to\infty$, because $\frac{3}{2}n\nu-a\sim-\frac{a}{4}$. Conditioning on the first time S_j exceeds a,

$$\sum_{j=0}^n P^{*j}([a,a+1]\times\mathbb{R}^m)\leqslant \Lambda P(S_j\geqslant a,\exists j\leqslant n)=\mathrm{o}\{a^{(-1-\delta)/2}\}$$

as $a\to\infty$. Now

$$P_0^{*n}=\sum_{k=0}^n\binom{n}{k}P^{*k}/2^k,$$

and $R*P^{*k}=\sum_{l=0}^\infty P^{*(l+k)}=R-\sum_{j<k}P^{*j}\geqslant R-\sum_{j\leqslant n}P^{*j}$ for all $k\leqslant n$. It follows that

$$R-\sum_{j\leqslant n}P^{*j}\leqslant R*P_0^{*n}\leqslant R,$$

and hence for $f\in\mathscr{F}_a$,

$$\left|\int f\,\mathrm{d}R-\int f\,\mathrm{d}(R*P_0^{*n})\right|\leqslant\sum_{j\leqslant n}P^{*j}([a,a+1]\times\mathbb{R}^m)=\mathrm{o}\{a^{(-1-\delta)/2}\} \tag{7}$$

as $a \to \infty$. Using (5) and (6),

$$\int f \, \mathrm{d}(\bar{R} * P_0^{*n}) \leqslant \bar{R} * P_0^{*n}([a, a+1] \times \mathbb{R}^m)$$

$$= \int \bar{R}([a-x, a+1-x] \times \mathbb{R}^m) \, \mathrm{d}P_0^{(*n)}(x, y)$$

$$\leqslant A P_0^{*n}((-\infty, 0] \times \mathbb{R}^m) + \sup_{t \geqslant a} \bar{R}([t, t+1] \times \mathbb{R}^m)$$

$$= \mathrm{o}\{a^{(-1-\delta)/2}\} \tag{8}$$

as $a \to \infty$. The theorem now follows from (7) and (8), because $W * P_0^{*n} = R * P_0^{*n} + \bar{R} * P_0^{*n}$. $\square$

The proof of Theorem 3 will use the following lemma, a rate of convergence result in one dimensional renewal theory that appears as Theorem 3.5 of Kalma (1972). Under the stronger smoothness condition that the distribution of S_n has an absolutely continuous component for some n, the lemma follows from results of Stone and Wainger (1967). They also study the lattice case as does Frenk (1987).

Lemma 7. *As* $a \to \infty$,

$$R([a, a+1] \times \mathbb{R}^m) = 1/\nu + \mathrm{o}\{a^{(-1-\delta)/2}\}. \quad \square \tag{9}$$

Proof of Theorem 3. Let us begin by noting that $\hat{R}([a, a+1] \times \mathbb{R}^m) = 1/\nu + \mathrm{o}\{a^{(-1-\delta)/2}\}$ as $a \to \infty$. This would hold with equality if λ were Lebesgue measure. The integrations against the counting measures are Riemann approximations to the corresponding Lebesgue integral, and arguments from numerical analysis can be used to show that the difference is sufficiently small (the trapezoid rule, keeping careful track of the error, is good enough). Alternatively, one could use Euler–MacLaurin summation formulas (see Bhattacharya and Rao, 1976, Appendix 4) or Fourier methods to establish this result. Let $S_a = \{y: |y - \gamma a| \leqslant K_0 \sqrt{a \log a}\}$. If K_0 is large enough, then

$$\hat{R}([a, a+1] \times (\mathbb{R}^m - S_a)) = \mathrm{o}\{a^{(-1-\delta)/2}\} \tag{10}$$

as $a \to \infty$. By Theorem 1 with f the indicator of $[a, a+1] \times S_a$,

$$R([a, a+1] \times S_a) = \hat{R}([a, a+1] \times S_a) + \mathrm{o}\{a^{(-1-\delta)/2} (\log a)^{m/2}\}$$

$$= 1/\nu + \mathrm{o}\{a^{(-1-\delta)/2} (\log a)^{m/2}\} \tag{11}$$

as $a \to \infty$. From Lemma 7, it follows that as $a \to \infty$,

$$R([a, a+1] \times (\mathbb{R}^m - S_a)) = \mathrm{o}\{a^{(-1-\delta)/2} (\log a)^{m/2}\}. \tag{12}$$

Now write $f = f_1 + f_2$ where $f_1(x, y) = f(x, y) I\{y \in S_a\}$. Using (12), $\int f_2 \, \mathrm{d}R = \mathrm{o}\{a^{(-1-\delta)/2} (\log a)^{m/2}\}$ and the theorem follows by applying Theorem 1 to f_1. $\square$

3. Approximations for ψ and χ_n

When proving expansions in the central limit theorem, Taylor series arguments are used to approximate the characteristic function of normalized partial sums. The corresponding task in renewal theory is approximating ψ and χ_n, but there is one important difference: after the appropriate rescaling, p^2 and q will be of the same magnitude. In the Taylor expansions to be developed, the highest power for q will be twice the highest power for p, and the distance from (p, q) to the origin will be measured by $\tau = \sqrt{|p|} + |q|$.

Lemma 8. *Let*

$$\hat{\phi} = 1 + \mathrm{i}p\nu - \tfrac{1}{2}q^2 - pM_1 - \mathrm{i}M_2.$$

Then

$$\phi = \hat{\phi} + \mathrm{o}(\tau^{3+\delta}) \tag{13}$$

and

$$\phi' = \hat{\phi}' + \mathrm{o}(\tau^{1+\delta}) \tag{14}$$

as $\tau \to 0$ (i.e., as $(p, q) \to 0$).

Proof. Let $K_1 = \sup_{x \neq 0} |\mathrm{e}^{\mathrm{i}x} - 1 - \mathrm{i}x| / |x|^{(3+\delta)/2} < \infty$. Then

$$\left| \frac{\mathrm{e}^{\mathrm{i}pX} - 1 - \mathrm{i}pX}{|p|^{(3+\delta)/2}} \mathrm{e}^{\mathrm{i}q\cdot Z} \right| \leq K_1 |X|^{(3+\delta)/2},$$

so by dominated convergence,

$$\phi(p, q) = E\ \mathrm{e}^{\mathrm{i}q\cdot Z} + \mathrm{i}pEX\ \mathrm{e}^{\mathrm{i}q\cdot Z} + \mathrm{o}(|p|^{(3+\delta)/2})$$

as $p \to 0$, uniformly in q. Similarly, if $K_2 = \sup_{x\neq 0} |\mathrm{e}^{\mathrm{i}x} - 1 - \mathrm{i}x + \frac{1}{2}x^2 + \frac{1}{6}\mathrm{i}x^3| / |x|^{3+\delta}$, then

$$\frac{|\mathrm{e}^{\mathrm{i}q\cdot Z} - 1 - \mathrm{i}q\cdot Z + \frac{1}{2}(q\cdot Z)^2 + \frac{1}{6}\mathrm{i}(q\cdot Z)^3|}{|q|^{3+\delta}} \leq K_2 |Z|^{3+\delta}$$

and dominated convergence gives

$$E\ \mathrm{e}^{\mathrm{i}q\cdot Z} = 1 - \tfrac{1}{2}q^2 - \mathrm{i}M_2 + \mathrm{o}(|q|^{3+\delta})$$

as $q \to 0$. Finally, let $K_3 = \sup_{x\neq 0} |\mathrm{e}^{\mathrm{i}x} - 1 - \mathrm{i}x| / |x|^{1+\delta}$. Then

$$\frac{|X(\mathrm{e}^{\mathrm{i}q\cdot Z} - 1 - \mathrm{i}q\cdot Z)|}{|q|^{1+\delta}} \leq |X||Z|^{1+\delta} \leq |X|^{(3+\delta)/2} + |Z|^{3+\delta},$$

and

$$EX\ \mathrm{e}^{\mathrm{i}q\cdot Z} = \nu + \mathrm{i}M_1 + \mathrm{o}(|q|^{1+\delta}) \tag{15}$$

as $q \to 0$ by dominated convergence and (13) follows. To establish (14), let $K_4 = \sup_{x \neq 0} |e^{ix} - 1| / |x|^{(1+\delta)/2}$. Then

$$\frac{|iX(e^{ipX} - 1)\, e^{iq \cdot Z}|}{|p|^{(1+\delta)/2}} \leq K_4 |X|^{(3+\delta)/2}$$

as $p \to 0$, uniformly in q. Hence

$$\phi'(p, q) = E iX\, e^{ipX + iq \cdot Z} = iEX\, e^{iq \cdot Z} + o\{|p|^{(1+\delta)/2}\}$$

as $p \to 0$, uniformly in q, and (14) then follows from (15). $\square$

From this lemma it is a simple matter to approximate ψ and ψ'. Using the Taylor expansion

$$\frac{1}{b+a} = \frac{1}{b} - \frac{a}{b^2} + O\left(\frac{|a|^2}{|b|^3}\right)$$

as $a/b \to 0$,

$$\frac{1}{1-\phi} = \frac{1}{-ip\nu + \frac{1}{2}q^2} - \frac{pM_1 + iM_2}{(-ip\nu + \frac{1}{2}q^2)^2} + o(\tau^{\delta - 1})$$

as $\tau \to 0$. Taking real parts,

$$\psi = \hat{\psi} + o(\tau^{\delta - 1}) \tag{16}$$

as $\tau \to 0$. Using the approximation (14) for ϕ',

$$\psi' = \hat{\psi}' + o(\tau^{\delta - 3}) \tag{17}$$

as $\tau \to 0$.

The next step is to approximate χ_n and χ'_n. This is rather delicate because bounds are needed that hold as $n \to \infty$ uniformly in a neighborhood of the origin, and separate bounds are needed as $n \to \infty$ with $n\tau^3 \to 0$. Using the Taylor expansion $\log(1+a) = a + O(a^2)$ as $a \to 0$,

$$\log \chi_n = n\{\log \tfrac{1}{2}(1+\phi) - \tfrac{1}{2}i\nu\} = -\tfrac{1}{4}nq^2 - \tfrac{1}{2}npM_1 - \tfrac{1}{2}inM_2 + o(n\tau^{3+\delta}) \tag{18}$$

as $\tau \to 0$. Using $e^a = 1 + a + O(a^2)$ as $a \to 0$,

$$\chi_n = \hat{\chi}_n + n\, e^{-nq^2/4} o(\tau^{3+\delta}) \tag{19}$$

as $n \to \infty$ with $n\tau^3 \to 0$, where

$$\hat{\chi}_n = e^{-nq^2/4 - 2nKp^2}(1 - \tfrac{1}{2}npM_1 - \tfrac{1}{2}inM_2). \tag{20}$$

The factor e^{-2nKp^2} is one to sufficiently high order in this limit to be negligible, but will play an important role when bounds are sought as $n \to \infty$ holding uniformly for (p, q) in a neighborhood of the origin. The constant K will be set later.

For uniform bounds in some neighborhood of the origin, the following lemma is necessary. This lemma would follow from Taylor expansion if $EX^2 < \infty$, but is curious in that moments for X and Z are not used. Let $\xi = (p, q)$.

Lemma 9. *Let*

$$1-\alpha=\sup_{\varepsilon\leq|\xi|\leq 2\varepsilon}|\chi_1|.$$

Then for ε sufficiently small, $\alpha>0$ and

$$|\chi_1(\xi)|\leq 1-\alpha\xi^2/\varepsilon^2\leq \mathrm{e}^{-\alpha\xi^2/\varepsilon^2}$$

for all $|\xi|\leq\varepsilon$.

Proof. The bound appears as Lemma 1, Chapter IV of Cramér (1970). That $\alpha>0$ for ε sufficiently small follows easily from assumption (A1). It also follows from the weaker assumption that P_0 (or distribution with characteristic function χ_1) is fully $(m+1)$-dimensional in the sense that its support is not contained in any m-dimensional hyperplane in $\mathbb{R}^{m+1}$. □

Using this lemma, fix a neighborhood $N_0=[-\rho,\rho]^{m+1}$ of the origin and a constant $K\in(0,\frac{1}{8})$ so that $\tau^2\psi$ is bounded on N_0 and

$$|\chi_1|\leq \mathrm{e}^{-2K(p^2+q^2)}$$

on N_0. Then

$$|\chi_n|\leq \mathrm{e}^{-2nK(p^2+q^2)} \tag{21}$$

on N_0. It is worth noting that if f is bounded on N_0 and $f=\mathrm{o}(1)$ as $\tau\to 0$, then

$$\chi_n f=\mathrm{o}(1)\,\mathrm{e}^{-nK(p^2+q^2)}$$

as $n\to\infty$ uniformly for $(p,q)\in N_0$. (This explains the spurious 2 in 21.) Using this, (16) and (17),

$$\chi_n(\psi'-\hat{\psi}')=\mathrm{e}^{-nK(p^2+q^2)}\mathrm{o}(\tau^{\delta-3}), \tag{22}$$

and

$$\chi_n'(\psi-\hat{\psi})=n\chi_n(\log\chi_1)'(\psi-\hat{\psi})=n\,\mathrm{e}^{-nK(p^2+q^2)}(|q|+\tau^{\delta+1})\mathrm{o}(\tau^{\delta-1}) \tag{23}$$

as $n\to\infty$, uniformly on N_0. Also, by (14),

$$(\log\chi_1)'=\frac{\phi'-\mathrm{i}\nu}{1+\phi}=-\tfrac{1}{2}M_1+\mathrm{o}(\tau^{1+\delta})$$

as $\tau\to 0$, so

$$\chi_n\{(\log\chi_1)'+\tfrac{1}{2}M_1\}=\mathrm{e}^{-nK(p^2+q^2)}\mathrm{o}(\tau^{1+\delta}) \tag{24}$$

as $n\to\infty$, uniformly on N_0.

Approximating χ_n and χ_n' is harder—small o bounds may not hold uniformly on N_0.

Lemma 10. *As* $n \to \infty$,

$$\chi_n - \hat{\chi}_n = n\, \mathrm{e}^{-nK(p^2+q^2)} \mathrm{O}(\tau^{3+\delta}), \tag{25}$$

uniformly on N_0.

Proof. The lemma follows from (19) if $n \to \infty$ with $n\tau^3 \to 0$. Suppose $n \to \infty$ with $n\tau^{3+\delta} > b > 0$. Then by (23),

$$|\chi_n| \leq \mathrm{e}^{-nK(p^2+q^2)} = \mathrm{e}^{-nK(p^2+q^2)} \mathrm{O}(n\tau^{3+\delta}).$$

The same bound holds for $|\hat{\chi}_n|$, because

$$\begin{aligned} \frac{\mathrm{e}^{nK(p^2+q^2)}}{n\tau^{3+\delta}} \hat{\chi}_n &= \mathrm{O}(1)\, \mathrm{e}^{-nK(p^2+q^2)} \left\{ \frac{n|p||q| + n|q|^3}{n\tau^{3+\delta}} \right\} \\ &= \mathrm{O}(1) \exp\left\{ -Kb \frac{p^2+q^2}{\tau^{3+\delta}} \right\} \left\{ \frac{p^2+q^2}{\tau^{3+\delta}} + \rho \frac{q^2}{\tau^{3+\delta}} \right\} \\ &\leq \mathrm{O}(1) \frac{1+\rho}{Kb\, \mathrm{e}}, \end{aligned}$$

the last line since $\sup_{t>0} t\, \mathrm{e}^{-ct} = 1/(c\, \mathrm{e})$. The proof will be finished by showing that (25) holds as $n \to \infty$ with $n\tau^3 > a$ and $n\tau^{3+\delta} < b$. In this limit, $np^2 \leq n\tau^4 \to 0$. Using this and (18),

$$\chi_n = \mathrm{e}^{-nq^2/4 - npM_1/2 - \mathrm{i}nM_2/2} \{1 + \mathrm{O}(n\tau^{3+\delta})\}. \tag{26}$$

Now with K_5 chosen so that $|M_1| \leq K_5 |q|$,

$$\begin{aligned} \mathrm{e}^{nK(p^2+q^2)}\, \mathrm{e}^{-q^2/4 - npM_1/2} &\leq \exp\{-n(\tfrac{1}{4} - K)q^2 + \tfrac{1}{2} nK_5 |p||q| + nKp^2\} \\ &\leq \exp\left\{ np^2 \left(\frac{K_5^2}{4(1-4K)} \right) + nKp^2 \right\} \to 1. \end{aligned} \tag{27}$$

Hence in (26), the large O term is $n\mathrm{O}(\tau^{3+\delta})\, \mathrm{e}^{-nK(p^2+q^2)}$, and it is sufficient to show that

$$\frac{\mathrm{e}^{nK(p^2+q^2)}}{n\tau^{3+\delta}} \{\mathrm{e}^{-nq^2/4 - npM_1/2 - \mathrm{i}nM_2/2} - \mathrm{e}^{-nq^2/4 - 2Knp^2}(1 - \tfrac{1}{2}npM_1 - \tfrac{1}{2}\mathrm{i}nM_2)\}$$

is bounded. Since $\mathrm{e}^{nKp^2} - 1 = \mathrm{O}(n\tau^4)$, it is sufficient to show that

$$\frac{\mathrm{e}^{-nkq^2}}{n\tau^{3+\delta}} \{\mathrm{e}^{-npM_1/2 - \mathrm{i}nM_2/2} - 1 - \tfrac{1}{2}npM_1 + \tfrac{1}{2}\mathrm{i}nM_2\} \tag{28}$$

is bounded. To continue, note that for any complex ξ,

$$|\mathrm{e}^{\xi} - 1 - \xi| = \left| \int_0^1 (1-t)\xi^2\, \mathrm{e}^{t\xi}\, \mathrm{d}t \right| \leq \tfrac{1}{2} |\xi|^2\, \mathrm{e}^{|\Re \xi|}.$$

Hence (28) is bounded by a multiple of

$$\frac{e^{-nKq^2}}{n\tau^{3+\delta}}(n^2p^2q^2+n^2q^6)\,e^{n|p||q|K_5/2}$$

$$\leqslant\left\{\frac{p^2+q^4}{\tau^{3+\delta}}\right\}\{nq^2\,e^{-nKq^2/2}\}\{e^{-nKq^2/2+nK_5|p||q|/2}\}.$$

The first factor is bounded by $2\tau^{1-\delta}\to 0$, the second factor is bounded by $1/e=\sup_{t>0} t\,e^{-t}$, and the last factor is bounded by algebra similar to that in (27). This proves the lemma. □

Approximating χ_n' is now relatively easy. By (24) and Lemma 10,

$$\begin{aligned}\chi_n' &= n(\log\chi_1)'\chi_n\\ &= -\tfrac{1}{2}nM_1\chi_n+o(n\tau^{1+\delta})\,e^{-K(p^2+q^2)}\\ &= -\tfrac{1}{2}nM_1\hat{\chi}_n+o(n\tau^{1+\delta})\,e^{-nK(p^2+q^2)}\end{aligned}$$

as $n\to\infty$ uniformly on N_0. Since

$$\begin{aligned}\hat{\chi}_n'+\tfrac{1}{2}nM_1\hat{\chi}_n &= \{-4nKp(1-\tfrac{1}{2}npM_1-\tfrac{1}{2}\mathrm{i}nM_2)\\ &\qquad -\tfrac{1}{4}nM_1(npM_1+\mathrm{i}nM_2)\}\,e^{-nq^2/4-2nKp^2}\\ &= e^{-nq^2/4-2nKp^2}O\{n\tau^2+n^2\tau^2|p||q|+n^2\tau^2q^2\}\\ &= e^{-nK(p^2+q^2)}O(n\tau^2),\end{aligned}$$

we have

$$\chi_n'-\hat{\chi}_n'=e^{-nK(p^2+q^2)}o(n\tau^{1+\delta}) \tag{29}$$

as $n\to\infty$, uniformly on N_0.

The final preliminary estimates needed are

$$\hat{\psi}=O(q^2/\tau^4+|q|/\tau^2) \tag{30}$$

and

$$\hat{\psi}'=O(q^2/\tau^6+|q|/\tau^4) \tag{31}$$

which follow from the definition of $\hat{\psi}$ after some algebra.

Proof of Lemma 5. Let $\Delta=(\chi_n\psi)'-(\hat{\chi}_n\hat{\psi})'$. Then $\Delta=\Delta_1+\Delta_2$ where

$$\Delta_1=\chi_n\psi'-\hat{\chi}_n\hat{\psi}'=\chi_n(\psi'-\hat{\psi}')+\hat{\psi}'(\chi_n-\hat{\chi}_n)$$

and

$$\Delta_2=\chi_n'\psi-\hat{\chi}_n'\hat{\psi}=\chi_n'(\psi-\hat{\psi})+\hat{\psi}(\chi_n'-\hat{\chi}_n').$$

Using (22), (31), (25) and (23), (30), (29),

$$\Delta_1 = e^{-nK(p^2+q^2)}[o(\tau^{\delta-3}) + (q^2/\tau^6 + |q|/\tau^4)O(n\tau^{3+\delta})]$$

and

$$\Delta_2 = e^{-nK(p^2+q^2)}[o(n\tau^{2\delta}) + (q^2/\tau^4 + |q|/\tau^2)o(n\tau^{1+\delta})]$$

as $n \to \infty$ uniformly on N_0. Adding these,

$$\Delta(p/n, q/\sqrt{n}) = O\{n^{(3-\delta)/2}\}(1+q^2)\tau^{-3+\delta}\, e^{-Kq^2}$$
$$+ e^{-Kq^2 - Kp^2/n}[O(n^{(2-\delta)/2})|q|\tau^{\delta-1} + o(n^{1-\delta}\tau^{2\delta})]$$

as $n \to \infty$ uniformly for $(p/n, q/\sqrt{n}) \in N_0$. A similar but easier calculation, using (19) instead of (25), shows that $\Delta(p/n, q/\sqrt{n}) = o\{n^{(3-\delta)/2}\}$ as $n \to \infty$ pointwise in (p, q). Since

$$\int |q|\tau^{\delta-1}\, e^{(-Kq^2 - Kp^2/n)} \leq n^{(1+\delta)/4} \int \frac{|q|\, e^{-K(p^2+q^2)}}{|p|^{(1-\delta)/2}}\, dp\, dq$$
$$= O\{n^{(1+\delta)/4}\} = o(\sqrt{n})$$

and

$$\int \tau^{2\delta}\, e^{-Kq^2 - Kp^2/n}\, dp\, dq = n^{(1+\delta)/2} \int (\sqrt{|p|} + |q|/n^{1/4})^{2\delta}\, e^{-K(p^2+q^2)}\, dp\, dq$$
$$= o\{n^{(1+\delta)/2}\}$$

as $n \to \infty$, and since $\int (1+q^2)\tau^{-3+\delta}\, e^{-Kq^2}\, dp\, dq < \infty$,

$$\int_{N_0} |\Delta| = n^{-1-m/2} \int_{(p/n, q/\sqrt{n}) \in N_0} |\Delta(p/n, q/\sqrt{n})|\, dp\, dq = o\{n^{(1-\delta-m)/2}\}$$

as $n \to \infty$ by dominated convergence. This proves the lemma. $\square$

Careful inspection of the proof of Lemma 5 reveals the value of symmetrization. Without symmetrization ψ is replaced by $\frac{1}{2}(1-\phi)^{-1}$, and the rational approximation for $(1-\phi)^{-1}$ is $O(1/\tau^2)$ on N_0. Although this is only slightly larger than the bound in (30), it adds a new error term to Δ_2, namely

$$e^{-nK(p^2+q^2)} o(n\tau^{\delta-1}).$$

By dominated convergence,

$$n \int_{N_0} \tau^{\delta-1}\, e^{-nK(p^2+q^2)}\, dp\, dq \sim n^{(3-2\delta-2m)/4} \int |p|^{(\delta-1)/2}\, e^{-K(p^2+q^2)}\, dp\, dq$$

as $n \to \infty$, which is too large for the desired result.

4. Concluding remarks

Although this research is similar in approach to Keener (1988), there are a few important differences. One is convolving W with P_0^{*n}. This has two effects: contributions to the inversion integral over $S - N_0$ are exponentially small, and integrability problems in the q integration are eliminated. Without this device, integrability over q depends heavily on the dimension m.

The approach used to obtain Theorem 3 from Theorem 1 is slightly crude. A natural conjecture is that Theorem 3 remains valid without the log a that appears in the error rate. A similar problem in encountered in expansions in the central limit theorem, and two ideas are particularly useful in that setting. One is to use integration by parts to improve the error rate in the tails of the distribution. The other idea is to work with truncated variables. (See Bhattacharya and Rao, 1976, Theorems 19.2 and 19.5.) Although integration by parts is used successfully in Keener (1988), my attempts to use this technique in this multivariate case have failed. The trouble is that differentiation with respect to p generally improves the p-integrability of the error terms approximating $\chi_n\psi$, and to differentiate with respect to q, derivatives with respect to p must be sacrificed. The use of truncation in multivariate renewal theory seems difficult, but may be the next major step towards a general theory.

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