

## TECHNICAL NOTES

### AUTOREGRESSIVE MODEL ANALYSIS AND DECAY RATIO

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**Abstract**—Although weights of some system poles of the AR model are asymptotically constant for model order change, they have transient model-order dependences. These transient order dependences cause the robustness of the decay ratio near the hard-model instability, when the value of it is determined by using an AR model. The relationship between the robustness of the decay ratio and transient order dependences of the AR weights are discussed qualitatively.

#### 1. INTRODUCTION

The decay ratio is used for the evaluation of reactor stability in boiling water reactors. Due to the simplicity, this value is mainly defined as the ratio of impulse responses, though there are other definitions of it. In order to calculate it, a parametric model is often used (Upadahyaya and Kitamura, 1981; Upadahyaya *et al.*, 1982; Kanemoto *et al.*, 1984; Suzuki *et al.*, 1986), that is, an AR (autoregressive) model. There is a well-established FORTRAN program, such as TIMSAC, to evaluate impulse responses from time series data, by using the recursion formula (Levinson, 1947; Durbin, 1960) and the information criterion (Akaike, 1974).

In the linear stability criterion, there are mainly two types of instabilities in a reactor (Kishida *et al.*, 1976, 1977). One is called the soft-mode instability in which one real system pole approaches the unit circle in the complex plane in the discrete time representation. The other is the hard-mode instability in which a pair of complex system poles approach the unit circle. In the post-hard-mode instability, there is a limit cycle region. This may be a generalized phase transition from a stable region to a limit cycle region. The decay ratio has an important role in the hard-mode instability, as well as the irreversible circulation of fluctuations which is an integral index indicating an information of forerunner phenomena for the instability. The decay ratio is mainly evaluated via the AR model analysis.

The method of the AR model is used for a black-box analysis, in which the relationship between physical processes and its model is not made clear. To give a physical foundation of it, we have already examined the properties of the AR model and can summarize them in the following two rules (Kishida, 1988; Suda *et al.*, 1989). Let a system be described by an AR-MA (autoregressive moving average) model.

1. The pole location rule (Kishida *et al.*, 1987a,b): poles of the AR model with large enough order are classified in three groups: (1) system (AR-MA) poles inside the convergence circle; (2) multiple circular or ring poles

which are equivalent to some of the AR-MA zeros; and (3) robust and/or non-robust singular poles outside the convergence circle. Here the convergence circle is defined as a circle centered at the origin with a radius of the absolute value of the AR-MA zero nearest to the unit circle in the complex plane.

2. The pole separation rule (Kishida and Yamada, 1988; Kishida *et al.*, 1989): (1) if the weight of an AR pole is constant for AR model order change, the AR pole is one of the system poles inside the convergence circle; and (2) if the weight of an AR pole is inversely proportional to the AR order, the AR pole is one of the ring (or circular) poles.

In practical numerical examples, some weights of the AR poles corresponding to system poles inside the convergence circle do not have constant behavior and have model order dependence, though their asymptotic values are constant by the weight rule. That is, weights of the AR poles have transient behaviors of model order dependence. In some examples of data analysis, more than 100% changes are found in weights of the AR poles corresponding to system poles. These transient behaviors of weights will be discussed in the next section. Since weights of system poles inside the convergence circle have transient dependencies for a change of AR model order as in the next section, the decay ratio will also be expected to have some transient behaviors of model-order dependence. From the safety viewpoint, this property of decay ratio must be examined in order to apply the AR model analysis for system identification and diagnosis. In the present paper, properties of the decay ratio by using the AR model will be discussed.

#### 2. DECAY RATIO AND WEIGHT OF AR POLE

The reactor core stability should be evaluated on the basis of the identified transfer function which is considered to include all dynamic properties of the reactor. So, the decay ratio should be obtained from impulse responses of the transfer function. Though there is a method in which transfer functions are determined approximately, there is no method where it will be given exactly from observed time series data within the author's knowledge. For the estimation of reactor

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stability, however, a univariate AR model is often used in the reactor noise analysis.

In this paper it is assumed that the stability of the reactor core is identified by a scalar AR model, since we want to present the essence of the relation between model order dependence of AR weight and decay ratio without loss of generality. That is, let an AR model of order  $m$  be:

$$\psi(m, z^{-1})y(n) = \varepsilon(m, n), \quad (1)$$

where  $y(n)$  is an observable variable at discrete time  $n$ ,  $\varepsilon(m, n)$  is an innovation,  $\psi$  is a polynomial of order  $m$  in  $z^{-1}$ ,

$$\psi(m, z^{-1}) = 1 + \sum_{j=1}^m a(m, j)z^{-j}$$

and  $z^{-1}$  is the time shift operator:  $z^{-1}y(n) = y(n-1)$ . The transfer function of the AR model,  $G_{AR}(z^{-1})$ , is given from equation (1) as:

$$G_{AR}(z^{-1}) = \frac{1}{\psi(m, z^{-1})} = \frac{1}{\prod_{j=1}^m (1 - \beta_j z^{-1})} = \sum_{j=1}^m \frac{w_j}{1 - \beta_j z^{-1}}, \quad (2)$$

where  $\beta_j^{-1}$  are poles of the AR model in the  $z^{-1}$  complex plane and  $w_j$  are weights corresponding to AR poles.

Since we have already asymptotically calculated weights of the AR poles as in previous papers (Kishida and Yamada, 1988; Kishida, 1988; Kishida *et al.*, 1989), we will rewrite it briefly: let  $d$  AR-MA poles be  $\alpha_j^{-1}$  ( $j = 1, 2, \dots, d$ ) and an AR-MA zero  $\lambda^{-1}$ . By setting a new variable  $\zeta_j = \alpha_j^{-1}/\lambda^{-1} = \lambda/\alpha_j$ , the weights of the AR poles are given as:

$$w_j = \frac{1 - \zeta_j}{(1 - \zeta_j^m) \prod_{\substack{k=1 \\ k \neq j}}^d (1 - \alpha_k \alpha_j^{-1})}, \quad (3)$$

[cf. equation (11) in Kishida *et al.* (1989)] for an AR pole inside the convergence circle, which is one of the system poles according to the pole location rule, and

$$w_j = \frac{1 - \omega^j}{m \prod_{k=1}^d (1 - \alpha_k v_j^{-1})}, \quad (4)$$

[cf. equation (12) in Kishida *et al.* (1989)] for an AR pole which is one of the ring poles;  $v_j = \lambda \omega^{-j}$  where  $\omega$  is the  $m$ th root of 1;  $\exp(2\pi i/m)$ . To obtain these results, we have used the following assumptions:

- (a) a system is described by an AR-MA( $d, 1$ ) model;
- (b) all AR-MA poles are inside the convergence circle and zero is single and real;
- (c) the bias between an AR model and a TAR model is neglected, where the TAR model is defined by truncating the Taylor expansion of the AR-MA model transfer function of  $z^{-1}$  at a finite order. Each pole-position of the TAR model is replaced by each limit position corresponding to that of the TAR model of infinite order.

The difference between weights of AR poles and those of AR-MA poles inside the convergence circle is expressed by the term:

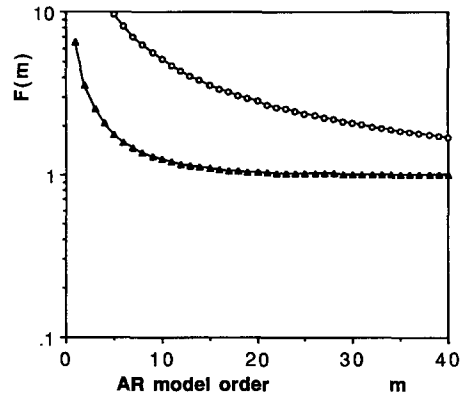


Fig. 1. AR model-order dependencies of  $F(m) = (1 - \zeta^m)^{-1}$  in the case of real  $\zeta$ , where  $\circ$ :  $\zeta = 1.272/1.3$  and  $\triangle$ :  $\zeta = 1.272/1.5$ .

$$\frac{1}{1 - \zeta_j^m}, \quad (5)$$

since the weight of corresponding the system pole is evaluated, as mentioned in the paper [cf. equation (14) in Kishida *et al.* (1989)] by:

$$w_j = \frac{1 - \zeta_j}{\prod_{\substack{k=1 \\ k \neq j}}^d (1 - \alpha_k \alpha_j^{-1})}$$

Since all system poles are inside the convergence circle, we have  $|\alpha_j^{-1}| < |\lambda^{-1}|$  and  $|\zeta_j| < 1$ . Therefore the difference between them disappears in a large enough model order, since  $\lim_{m \rightarrow \infty} \zeta_j^m = 0$ . However, the difference is still remaining in intermediate model orders. That is, the factor  $(1 - \zeta_j^m)^{-1}$  has model-order dependence. Its feature is mainly divided into two typical types: one is the case where  $\zeta_j$  is real, the other is the case where  $\zeta_j$  is complex. In the case of a real  $\zeta_j$ , the weight of the AR pole has a monotonic approach to that of a corresponding system pole as seen in Fig. 1. On the other hand, the weight of the AR pole has an oscillatory approach as in Fig. 2 in the case of complex  $\zeta_j$ . In both cases where  $|\zeta_j|$

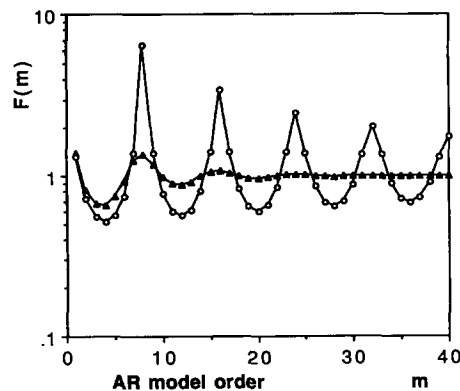


Fig. 2. AR model-order dependencies of  $F(m) = (1 - \zeta^m)^{-1}$  in the case of complex  $\zeta$ , where  $\circ$ :  $\zeta = (0.9 + 0.9i)/1.3$  and  $\triangle$ :  $\zeta = (0.9 + 0.9i)/1.5$ .

is nearly equal to 1, the weight of the AR pole seems not to converge to that of system pole during intermediate model orders. Moreover, in the complex case, the digit of weight of the AR pole is changeable to the model order as in Fig. 2. In the numerical analysis of real-time series data which were distributed in SMORN-IV, these two types of AR weights have been reported in Figs 2 and 4 by Kishida and Yamada (1988). One had monotonic behavior, however, the deviation was opposite to equation (5), and the other had oscillatory behavior, however, the system pole was real and the zeros were complex. It is considered that this property of weight affects the evaluation of decay ratio by using the AR model.

Next let us examine impulse responses of the AR model in order to evaluate a decay ratio. For simplicity we also use the discrete time representation, though the conventional definition of it is given in the continuous time representation. Impulse responses  $h_j$  are defined from:

$$y(n) = H(m, z^{-1})\epsilon(m, n), \tag{6}$$

where

$$H(m, z^{-1}) = \sum_{j=0}^{\infty} h_j z^{-j}.$$

From equation (2), we obtain:

$$h_j = \sum_{i=1}^m \beta_i^j w_i, \tag{7}$$

since

$$\begin{aligned} H(m, z^{-1}) &= \frac{1}{\psi(m, z^{-1})} = \sum_{i=1}^m \frac{w_i}{1 - \beta_i z^{-1}} \\ &= \sum_{i=1}^m w_i \left( \sum_{j=0}^{\infty} \beta_i^j z^{-j} \right) = \sum_{j=0}^{\infty} \left( \sum_{i=1}^m \beta_i^j w_i \right) z^{-j}. \end{aligned}$$

Finally, the decay ratio is calculated as a suitable ratio of impulse responses:

$$Dr = \frac{h_{L+j}}{h_j}, \tag{8}$$

where  $L$  is a length of one period from a peak to the next one. If we put  $j = 0$  in the above relation (8), then we have:

$$Dr(L) = \frac{\sum_{i=1}^m \beta_i^L w_i}{\sum_{j=1}^m w_j} = \sum_{j=1}^m \beta_j^L w_j. \tag{9}$$

Since a pair of system poles near the unit circle cause the hard-mode instability, the dominant terms in equation (9) are the ones containing the complex system poles  $\beta_s^{-1}$ , where the subscript  $s$  means the dominant system poles. Then, equation (9) is furthermore evaluated as:

$$Dr(L) \sim \beta_s^L w_s + \overline{\beta_s^L w_s}, \tag{10}$$

where  $\overline{\quad}$  denotes the complex conjugate.

By the way, we must pay attention to model order dependence of the decay ratio, since there are evidently statistical errors in practical observations. This means that the robustness of  $Dr(L)$  should be examined for the dependence of AR model order, since different AR models with various orders are obtained in the same system owing to statistical errors. When weights of the complex system poles are constant regardless of statistical errors, the decay ratio is mainly evaluated by the factor  $\beta_s^L$  from equation (10). This shows us that the estimation of decay ratio is related to the identification

of dominant system poles, and to the examination of reactor stability. On the other hand, we must examine factor  $w_j$  in addition to the dominant system poles  $\beta_s^{-1}$  for the robustness of decay ratio to statistical errors, if weights have the order dependence of the AR model. Much attention should be especially paid to the effect of model order dependence of weight when the system is near an unstable point, since  $|\beta_s^{-1}| \sim 1$ . In this case the decay ratio loses the robustness for the AR model-order dependence near the hard-mode instability, since the factor  $(1 - \zeta_j^m)^{-1}$  becomes large when  $|\zeta_j| \sim 1$  in the case where both the dominant poles and a real zero are also near the unit circle.

### 3. CONCLUDING REMARKS

In this paper we have qualitatively examined evaluation of the decay ratio through the method of the AR model. In practical applications, the following considerations are required:

1. In the evaluation of weight it has been assumed that an AR model is asymptotically equal to a TAR model, and that the system is described by the AR-MA( $d, 1$ ) model of which zero is single and real. In the intermediate region of the AR model order, however, differences between AR and TAR models should be taken into consideration. On the other hand, the evaluation of weight still holds in the wider class of AR-MA process as mentioned in the paper (Kishida *et al.*, 1989).
2. In the case where an AR-MA process describing a system has the complex zeros nearest to the unit circle, the phase angle of  $\zeta_j$  is determined from an angle between pole and zero, and gives a change of the period from peak to next peak for the AR model order from equation (5). This effect should also be examined carefully when the reactor system has the complex zeros nearest to the unit circle.
3. I believe that the decay ratio is evaluated on the basis of the transfer function which is considered to include all dynamic properties of the reactor. However, there is no method to obtain a transfer function exactly from time series data, though there are a lot of papers in which an approximate method is introduced to determine it by using the AR model. To establish an exact method will be a future subject for the evaluation of decay ratio.
4. The quantitative evaluation of the decay ratio is also needed, and the effect of discrete time representation must also be examined.

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