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Products of Polynomials in Many Variables

BERNARD BEAUZAMY

*Institut de Calcul Mathématique, Université de Paris 7, Paris, France**

ENRICO BOMBIERI

Institute for Advanced Study, Princeton, New Jersey 08540†

PER ENFLO

Kent State University, Kent, Ohio 44242†

AND

HUGH L. MONTGOMERY

University of Michigan, Ann Arbor, Michigan 48109†

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We study the product of two polynomials in many variables, in several norms, and show that under suitable assumptions this product can be bounded from below independently of the number of variables. © 1990 Academic Press, Inc.

INTRODUCTION

In this paper, we let

$$P(x_1, ..., x_N) = \sum_{\alpha} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \qquad Q(x_1, ..., x_N) = \sum_{\beta} b_\beta x_1^{\beta_1} \cdots x_N^{\beta_N},$$

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with $\alpha=(\alpha_1, ..., \alpha_N)$, $\beta=(\beta_1, ..., \beta_N)$, be polynomials in N variables, $x_1, ..., x_N$, and with complex coefficients. We are interested in estimates from below for the product PQ, that is in estimates of the form

$$\|PQ\| \geqslant \lambda \|P\| \cdot \|Q\|, \tag{1}$$

for some norm $\|\cdot\|$ on the space of polynomials.

The norms we use are first related to the coefficients of P,

$$|P|_p = \left(\sum_\alpha |a_\alpha|^p\right)^{1/p},$$

for $1 \leqslant p < \infty$, and

$$|P|_\infty = \max_\alpha |a_\alpha|.$$

Other norms, also related to the coefficients of P, are

$$[P]_p = \left(\sum_\alpha \left(\frac{\alpha!}{m!}\right)^{p-1} |a_\alpha|^p\right)^{1/p},$$

where m is the total degree of P, and $\alpha! = \alpha_1! \cdots \alpha_N!$.

Comparison between these norms is given by the following inequalities:

$$\left(\frac{1}{m!}\right)^{1-1/p} |P|_p \leqslant [P]_p \leqslant |P|_p. \tag{2}$$

We can also consider P as a function on the polycircle, and introduce the L_p norms

$$\|P\|_p = \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |P(e^{i\theta_1}, ..., e^{i\theta_N})|^p \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi}\right)^{1/p},$$

for $1 \leqslant p < \infty$, and

$$\|P\|_\infty = \max_{\theta_1, ..., \theta_N} |P(e^{i\theta_1}, ..., e^{i\theta_N})|.$$

We observe the following relations between the norms already introduced:

$$|P|_\infty \leqslant \|P\|_1 \leqslant \|P\|_2 = |P|_2 \leqslant \|P\|_\infty \leqslant |P|_1.$$

Of course, for $1 \leqslant p \leqslant q \leqslant \infty$,

$$\|P\|_p \leqslant \|P\|_q.$$

Moreover, as we see in Section 1, there is a constant $C_{m,p,q} > 0$ such that if P is a polynomial with (total) degree at most m,

$$\|P\|_p \geqslant C_{m,p,q} \|P\|_q.$$

For the $|\cdot|_p$-norms, we have, for $1 \leqslant p \leqslant q \leqslant \infty$,

$$|P|_q \leqslant |P|_p,$$

but there is no constant depending only on the degree (and independent of the number of variables) such that the converse inequality holds: this is clear from the consideration of the example $P = (x_1 + \cdots + x_N)/N$.

In the case of polynomials in one variable, estimates of the type (1) were obtained, under specific assumptions, by B. Beauzamy and P. Enflo [1]. They of course extend to polynomials in several variables, but lead to estimates which depend on the number of variables.

We are interested here in estimates independent of the number of variables. The first result of this nature was obtained by P. Enflo [2], in the frame of *concentration at low degrees*. To define it, let $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Then we say that P has concentration d $(0 < d \leqslant 1)$ at degree k if

$$\sum_{|\alpha| \leqslant k} |a_\alpha| \geqslant d \sum |a_\alpha|.$$

Then the theorem in [2] is:

THEOREM. *There is a constant $\lambda(d, d'; k, k') > 0$ such that for any polynomials P and Q, with concentration d at degree k, concentration d' at degree k', respectively, one has*

$$|PQ|_1 \geqslant \lambda |P|_1 \cdot |Q|_1.$$

The important point is that the constant λ does not depend on the degrees of the polynomials, nor on the number of variables.

We study here similar problems, in various norms, as was done in [1]; moreover the proof given here in the case of the $|\cdot|_1$-norm is simpler than the original proof of [2]. For all these reasons, the present paper may be regarded as a continuation of [1, 2].

It is convenient for us to start with homogeneous polynomials.

1. Homogeneous Polynomials

In this section, P and Q are homogeneous polynomials of degrees m and n, respectively:

$$P(x_1, ..., x_N) = \sum_{|\alpha| = m} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}, \qquad Q(x_1, ..., x_N) = \sum_{|\beta| = n} b_\beta x_1^{\beta_1} \cdots x_N^{\beta_N}. \quad (1)$$

Then we have:

THEOREM 1.1. *There exists a constant $C_1(m, n) > 0$ such that, for any polynomials P and Q, homogeneous of degrees m and n, respectively, one has, for any p, $1 \leqslant p \leqslant \infty$,*

$$C_1(m, n)\,|P|_p \cdot |Q|_p \leqslant |PQ|_p \leqslant 2^{(m+n)(1-1/p)}\,|P|_p \cdot |Q|_p.$$

No information is given about the value of $C_1(m, n)$ in general. But in the case $p = 2$, precise bounds are obtained for the norm $[\cdot]_2$:

THEOREM 1.2. *Let P, Q be homogeneous polynomials of degrees m, n, respectively. Then*

$$[PQ]_2 \geqslant \sqrt{\frac{m!\,n!}{(m+n)!}}\,[P]_2\,[Q]_2,$$

and this estimate is best possible.

For the $\|\cdot\|_p$-norms, we obtain:

THEOREM 1.3. *There exist constants $C_2(p, m, n) > 0$ and $C_2'(p, m, n) > 0$ such that, for any polynomials P and Q, homogeneous of degrees m and n, respectively, one has, for any p, $1 \leqslant p < \infty$,*

$$C_2(p, m, n)\,\|P\|_p \cdot \|Q\|_p \leqslant \|PQ\|_p \leqslant C_2'(p, m, n)\,\|P\|_p \cdot \|Q\|_p,$$

and in the case $p = \infty$,

$$C_2(\infty, m, n)\,\|P\|_\infty\,\|Q\|_\infty \leqslant \|PQ\|_\infty \leqslant \|P\|_\infty\,\|Q\|_\infty,$$

with

$$C_2(\infty, m, n) = \prod_{k=1}^{\inf\{m,n\}} \tan^2 \frac{(2k-1)\pi}{4(m+n)}.$$

In order to prove Theorem 1.1, we start with some basic inequalities for the l_p norms. The first one gives the upper estimate in Theorem 1.1.

A. *Relations between Norms*

LEMMA 1.A.1. *For $1 \leqslant p \leqslant \infty$,*

$$|PQ|_p \leqslant 2^{(m+n)(1-1/p)}\,|P|_p \cdot |Q|_p.$$

Proof of Lemma 1.A.1. We write

$$PQ = \sum c_\gamma x_1^{\gamma_1} \cdots x_N^{\gamma_N}.$$

A coefficient c_γ is a sum

$$c_\gamma = \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta.$$

Of course, $\alpha_1 \leqslant \gamma_1, ..., \alpha_N \leqslant \gamma_N$, so the number of terms in this sum does not exceed

$$(\gamma_1+1)\cdots(\gamma_N+1) \leqslant 2^{\gamma_1+\cdots+\gamma_N} = 2^{m+n}.$$

So, by Hölder's inequality, for $1 \leqslant p < \infty$,

$$\begin{aligned} |PQ|_p^p &= \sum_\gamma \left| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right|^p \\ &\leqslant 2^{(m+n)(p-1)} \sum_\gamma \sum_{\alpha+\beta=\gamma} |a_\alpha b_\beta|^p \\ &\leqslant 2^{(m+n)(p-1)} |P|_p^p \, |Q|_p^p, \end{aligned}$$

and for $p = \infty$,

$$|PQ|_\infty = \max_\gamma \left| \sum_{\alpha+\beta=\gamma} a_\alpha b_\beta \right| \leqslant 2^{m+n} \max |a_\alpha| \cdot \max |b_\beta|,$$

which proves our lemma.

LEMMA 1.A.2. *For* $1 \leqslant p < \infty$,

$$|P+Q|_p^p \leqslant 2^{p-1}(|P|_p^p + |Q|_p^p).$$

Proof of Lemma 1.A.2. This inequality follows immediately from the fact that, for two complex numbers a and b,

$$|a+b|^p \leqslant 2^{p-1}(|a|^p + |b|^p).$$

LEMMA 1.A.3. *Let* $P_i = \partial P / \partial x_i$. *Then, for* $1 \leqslant p \leqslant \infty$,

$$m^{1/p} |P|_p \leqslant \left(\sum_{i=1}^N |P_i|_p^p \right)^{1/p} \leqslant m \, |P|_p.$$

Proof of Lemma 1.A.3. We have, for $1 \leqslant p < \infty$,

$$\sum_{i=1}^N |P_i|_p^p = \sum_\alpha \left(|a_\alpha|^p \sum_i \alpha_i^p \right)$$

and

$$m = \sum_i \alpha_i \leqslant \sum_i \alpha_i^p \leqslant \left(\sum_i \alpha_i \right)^p = m^p,$$

from which the lemma follows. Modifications in the case $p=\infty$ are obvious.

We now turn to the proof of Theorem 1.1. Let $k, m, n \in \mathbb{N}$, and let $C_k(m, n) = C_k(p, m, n)$ be the largest real number such that the inequality

$$C_k(m, n)\, |P|_p^k\, |Q|_p \leqslant |P^k Q|_p$$

holds for all homogeneous polynomials P, Q of degrees m, n, respectively.

Clearly, $0 \leqslant C_k(m, n) \leqslant 1$. Our object is to show that $C_k(m, n) > 0$ for every $k, m, n \in \mathbb{N}$: taking $k=1$, we get the theorem. This is accomplished by an inductive argument, the basic steps of which are fulfilled in the following two lemmas:

LEMMA 1.A.4. *For* $1 \leqslant p \leqslant \infty$,

$$C_{k+1}(m, 0) \geqslant m^{1/p-1} C_1(m-1, mk) \cdot C_k(m, 0).$$

Proof of Lemma 1.A.4. We assume $p<\infty$, the case $p=\infty$ being left to the reader. For any homogeneous polynomial P of degree m, with the notation of Lemma 1.A.3, we have, for every i,

$$\begin{aligned} |(P^{k+1})_i|_p &= (k+1)\, |P_i P^k|_p \\ &\geqslant (k+1)\, C_1(m-1, mk)\, |P_i|_p\, |P^k|_p \\ &\geqslant (k+1)\, C_1(m-1, mk)\, C_k(m, 0)\, |P_i|_p\, |P|_p^k \end{aligned}$$

which implies

$$\left(\sum_{i=1}^{N} |(P^{k+1})_i|_p^p\right)^{1/p} \geqslant (k+1)\, C_1(m-1, mk)\, C_k(m, 0)\, |P|_p^k \left(\sum_i |P_i|_p^p\right)^{1/p}.$$

By Lemma 1.A.3,

$$(k+1)\, m\, |P^{k+1}|_p \geqslant (k+1)\, C_1(m-1, mk)\, C_k(m, 0)\, |P|_p^{k+1}\, m^{1/p},$$

which gives the desired result.

LEMMA 1.A.5. *For* $1 \leqslant p \leqslant \infty$,

$$C_k(m, n) \geqslant C_{k+1}(m, n-1)\, n^{1/p} [(km+n)^p + (km)^p]^{-1/p}\, 2^{-(1-1/p)(km+m+n)}.$$

Proof of Lemma 1.A.5. Again, we treat only the case $p<\infty$. We have, for every i,

$$\begin{aligned}
C_{k+1}(m, n-1)^p \, |P|_p^{(k+1)p} \, |Q_i|_p^p \leqslant |P^{k+1}Q_i|_p^p \\
&= |P(P^kQ_i + kP^{k-1}P_iQ) - kP^kP_iQ|_p^p \\
&\leqslant 2^{p-1}(|P(P^kQ_i + kP^{k-1}P_iQ)|_p^p \\
&\quad + k^p \, |P^kP_iQ|_p^p), \qquad \text{by Lemma 1.A.2} \\
&\leqslant 2^{(p-1)(km+m+n)}(|P|_p^p \, |P^kQ_i + kP^{k-1}P_iQ|_p^p \\
&\quad + k^p \, |P^kQ|_p^p \, |P_i|_p^p), \qquad \text{by Lemma 1.A.1.}
\end{aligned}$$

But $P^kQ_i + kP^{k-1}P_iQ = (P^kQ)_i$. We sum both sides over i and apply Lemma 1.A.3. We find

$$\begin{aligned}
&nC_{k+1}(m, n-1)^p \, |P|_p^{(k+1)p} \, |Q|_p^p \\
&\quad \leqslant 2^{(p-1)(km+m+n)}((km+n)^p \, |P|_p^p \, |P^kQ|_p^p + k^p m^p \, |P|_p^p \, |P^kQ|_p^p),
\end{aligned}$$

which implies

$$\begin{aligned}
|P^kQ|_p \geqslant C_{k+1}(m, n-1) \, n^{1/p}[(km+n)^p + (km)^p]^{-1/p} \\
\times 2^{-(1-1/p)(km+m+n)} \, |P|_p^k \, |Q|_p,
\end{aligned}$$

and shows the result.

We now complete the proof of Theorem 1.1. By induction on m, we show that, for every $n \geqslant 0$,

$$C_1(m, n) > 0. \tag{1}$$

Clearly, $C_1(0, n) = 1$. Assume we know that

$$C_1(m-1, n) > 0, \qquad \text{for all} \quad n \geqslant 0. \tag{2}$$

First, we show that, for every $k \geqslant 1$,

$$C_k(m, 0) > 0, \qquad \text{for all} \quad m \geqslant 0. \tag{3}$$

This is done by an induction on k: $C_1(m, 0) = 1$, and the inductive step on k is made by Lemma 1.A.4, using (2). So (3) is proved.

Next, we show that

$$C_k(m, n) > 0, \qquad \text{for all} \quad k, m, n \geqslant 0. \tag{4}$$

This will be done by an induction on n. For $n = 0$, this is (3). Assume that $C_k(m, n-1) > 0$ for all k, m. Using Lemma 1.A.5, we find that $C_k(m, n) > 0$ for all k, m. This proves (4), which implies (1).

The constants occurring in all lemmas are uniformly bounded in p, $1 \leqslant p \leqslant \infty$, so in the final result we may give a constant independent of p.

Remark 1. Similar results can be obtained in the case $0 < p < 1$ (where $|\cdot|_p$ is only a quasi-norm); the constants then depend on p and become worse and worse when $p \to 0$.

Remark 2. If we consider the case $p = 1$, $m = n = k = 1$, Lemma 1.A.4 gives $C_2(1, 0) = 1$, and Lemma 1.A.5 gives $C_1(1, 1) \geqslant \frac{1}{3}$. Thus

$$|PQ|_1 \geqslant \tfrac{1}{3} |P|_1 \, |Q|_1,$$

for homogeneous polynomials P, Q of degree 1. But (at least for real coefficients), a better estimate can be found:

LEMMA 1.A.6. *Let P, Q be homogeneous polynomials of degree* 1, *with real coefficients. Then*

$$|PQ|_1 \geqslant \tfrac{1}{2} |P|_1 \, |Q|_1,$$

and this estimate is best possible.

Proof. We write $P = \sum_1^N a_i x_i$, $Q = \sum_1^N b_i x_i$, with $\sum |a_i| = \sum |b_i| = 1$. We can assume that all the b_i's are $\geqslant 0$, and that for some n $(1 \leqslant n < N)$, $a_i \geqslant 0$ if $i \leqslant n$, and $a_i \leqslant 0$ if $i > n$. We set $\alpha = \sum_1^n a_i$, $\beta = \sum_1^n b_i$, and

$$\tilde{P}(x, y) = \alpha x - (1 - \alpha) y,$$

$$\tilde{Q}(x, y) = \beta x + (1 - \beta) y.$$

Then one checks easily that

$$|PQ|_1 \geqslant |\tilde{P}\tilde{Q}|_1 \geqslant \tfrac{1}{2},$$

and this value is attained when $\alpha = \beta = \frac{1}{2}$.

The above remark and this result show that the method of proof of Theorem 1.1 does not produce constants which are best possible.

We now turn to the proof of Theorem 1.2.

B. *The Norms Deduced from Taylor's Formula*

We have defined the $[\cdot]_p$-norms in the Introduction. In the case of homogeneous polynomials, they are related to Taylor's formula as follows.

For a polynomial $P(x_1, ..., x_N)$, homogeneous of degree m, we can write

$$P(x_1, ..., x_N) = \frac{1}{m!} \sum_{i_1, ..., i_m = 1}^{N} \frac{\partial^m P}{\partial x_{i_1} \cdots \partial x_{i_m}} x_{i_1} \cdots x_{i_m}. \tag{1}$$

If P is written as before,

$$P(x_1, ..., x_N) = \sum_{|\alpha| = m} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N},$$

the norm $[P]_p$ defined as

$$[P]_p = \left(\sum_{|\alpha| = m} \left(\frac{\alpha!}{m!} \right)^{p-1} |a_\alpha|^p \right)^{1/p},$$

with $\alpha! = \alpha_1! \cdots \alpha_N!$, satisfies clearly

$$[P]_p = \frac{1}{m!} \left(\sum_{i_1, ..., i_m = 1}^{N} \left| \frac{\partial^m P}{\partial x_{i_1} \cdots \partial x_{i_m}} \right|^p \right)^{1/p}.$$

We define $k_p(m, n)$ as the largest constant $\lambda \geqslant 0$ such that the inequality

$$[PQ]_p \geqslant \lambda [P]_p \, [Q]_p$$

holds for all N, all polynomials P, Q in N variables, homogeneous of degrees m, n, respectively.

Theorem 1.2 asserts that $k_2(m, n) = \sqrt{m! \, n!/(m+n)!}$. It is derived from a statement about functions, which we now describe.

Let $u = (u_1, ..., u_m)$, $v = (v_1, ..., v_n)$, and let $F(u)$, $G(v)$ be functions on the unit cubes I^m, I^n, with complex values, and invariant under the symmetric groups $\mathscr{S}_m$, $\mathscr{S}_n$, respectively. Let also $w = (w_1, ..., w_{m+n})$, and define a *shuffle of type* (m, n) to be

$$(\mathscr{I}, \mathscr{J}) = (i_1, ..., i_m; j_1, ..., j_n),$$

where $i_1 < i_2 < \cdots < i_m$, $j_1 < j_2 < \cdots < j_n$, and $(\mathscr{I}, \mathscr{J})$ is a permutation of the set $\{1, 2, ..., m+n\}$.

The set of shuffles of type (m, n) is denoted by $\operatorname{sh}(m, n)$, and its cardinality is

$$|\operatorname{sh}(m, n)| = \frac{|\mathscr{S}_{m+n}|}{|\mathscr{S}_m| \, |\mathscr{S}_n|} = \frac{(m+n)!}{m! \, n!}.$$

We write $x_{\mathscr{I}}$ for $(x_{i_1}, ..., x_{i_m})$.

We now define $c_p(m, n)$ as the largest constant $\mu \geqslant 0$ for which the inequality

$$\left\| \sum_{(\mathscr{I}, \mathscr{J}) \in \operatorname{sh}(m, n)} F(x_{\mathscr{I}}) \, G(x_{\mathscr{J}}) \right\|_{L_p(I^{m+n})} \geqslant \mu \, \|F\|_{L_p(I^m)} \, \|G\|_{L_p(I^n)}$$

holds for all symmetric functions $F \in L_p(I^m)$, $G \in L_p(I^n)$.

The two constants $k_p(m, n)$ and $c_p(m, n)$ are related by:

LEMMA 1.B.1. *For all p, $1 \leqslant p \leqslant \infty$, all $m, n \in \mathbb{N}$,*

$$k_p(m, n) = \frac{m! \, n!}{(m+n)!} \, c_p(m, n).$$

Proof of Lemma 1.B.1. (a) We first show that

$$k_p(m, n) \geqslant \frac{m!\, n!}{(m+n)!}\, c_p(m, n).$$

Let F, G be continuous symmetric functions on I^m, I^n, respectively. We define (with variables in P as capital letters)

$$P(X_1, ..., X_N) = N^{-m/p} \sum_{i_1, ..., i_m = 1}^{N} F\left(\frac{i_1}{N}, ..., \frac{i_m}{N}\right) X_{i_1} \cdots X_{i_m}, \tag{2}$$

$$Q(X_1, ..., X_N) = N^{-n/p} \sum_{i_{m+1}, ..., i_{m+n} = 1}^{N} G\left(\frac{i_{m+1}}{N}, ..., \frac{i_{m+n}}{N}\right) X_{i_{m+1}} \cdots X_{i_{m+n}}. \tag{3}$$

These polynomials are of degree m, n, respectively, with N variables. So

$$\frac{\partial^m P}{\partial X_{i_1} \cdots \partial X_{i_m}} = N^{-m/p} m!\, F\left(\frac{i_1}{N}, ..., \frac{i_m}{N}\right),$$

and

$$[P]_p = \left(N^{-m} \sum_{i_1, ..., i_m = 1}^{N} \left| F\left(\frac{i_1}{N}, ..., \frac{i_m}{N}\right)\right|^p\right)^{1/p}.$$

This last quantity tends to $\|P\|_{L_p(I^m)}$ when $N \to \infty$, by approximation of the integral by means of Riemann sums.

We also have

$$PQ = N^{-(m+n)/p} \sum_{i_1, ..., i_{m+n} = 1}^{N} F\left(\frac{i_1}{N}, ..., \frac{i_m}{N}\right) G\left(\frac{i_{m+1}}{N}, ..., \frac{i_{m+n}}{N}\right) X_{i_1} \cdots X_{i_{m+n}},$$

$$[PQ]_p = \frac{N^{-(m+n)/p}}{(m+n)!} \left(\sum_{j_1, ..., j_{m+n} = 1}^{N} \left| \sum_{\{i_1, ..., i_{m+n}\} = \{j_1, ..., j_{m+n}\}} F\left(\frac{i_1}{N}, ..., \frac{i_m}{N}\right) \right.\right.$$

$$\left.\left. \times G\left(\frac{i_{m+1}}{N}, ..., \frac{i_{m+n}}{N}\right)\right|^p\right)^{1/p}$$

$$\to \frac{m!\, n!}{(m+n)!} \left\| \sum_{(\mathscr{I}, \mathscr{J}) \in \mathrm{sh}(m,n)} F(x_{\mathscr{I}})\, G(x_{\mathscr{J}}) \right\|_{L_p(I^{m+n})},$$

and this proves our first claim, first for continuous functions, then for all functions in L_p, $p < \infty$, and finally for $p = \infty$.

(b) We now show the converse inequality. If P, Q are homogeneous polynomials as before, we define

$$F(x_1, ..., x_m) = \frac{1}{m!} \frac{\partial^m P}{\partial X_{i_1} \cdots \partial X_{i_m}},$$

where $i_1 = [Nx_1] + 1, ..., i_m = [Nx_m] + 1$.

So F is constant in each cube, and therefore

$$\|F\|_{L_p(I^m)} = \frac{N^{-m/p}}{m!} \left(\sum_{i_1, ..., i_m = 1}^{N} \left| \frac{\partial^m P}{\partial X_{i_1} \cdots \partial X_{i_m}} \right|^p \right)^{1/p}$$
$$= N^{-m/p} [P]_p.$$

G being defined the same way from Q, we get

$$\|G\|_{L_p(I^n)} = N^{-n/p} [Q]_p$$

and

$$\left\| \sum_{(\mathscr{I}, \mathscr{J}) \in \operatorname{sh}(m,n)} F(x_{\mathscr{I}})\, G(x_{\mathscr{J}}) \right\|_{L_p(I^{m+n})}$$
$$= \left(\frac{N^{-m-n}}{(m!\, n!)^p} \sum_{(\mathscr{I}, \mathscr{J}) \in \operatorname{sh}(m,n)} \left| \frac{\partial^m P}{\partial X_{i_1} \cdots \partial X_{i_m}} \frac{\partial^n Q}{\partial X_{j_1} \cdots \partial X_{j_n}} \right|^p \right)^{1/p}$$
$$= N^{-(m+n)/p} \frac{(m+n)!}{m!\, n!} [PQ]_p,$$

and the lemma is proved. It will allow us to compute precisely $k_2(m, n)$. First we show:

THEOREM 1.B.2. *For all F, G which are $\mathscr{S}_m$, $\mathscr{S}_n$ invariant,*

$$\left\| \sum_{(\mathscr{I}, \mathscr{J}) \in \operatorname{sh}(m,n)} F(x_{\mathscr{I}})\, G(x_{\mathscr{J}}) \right\|_{L_2(I^{m+n})} \geqslant \sqrt{\frac{(m+n)!}{m!\, n!}} \|F\|_{L_2(I^m)} \|G\|_{L_2(I^n)}.$$

Proof. We have

$$\left\| \sum_{(\mathscr{I}, \mathscr{J}) \in \operatorname{sh}(m,n)} F(x_{\mathscr{I}})\, G(x_{\mathscr{J}}) \right\|^2_{L_2(I^{m+n})}$$
$$= \sum_{(\mathscr{I}, \mathscr{J}) \in \operatorname{sh}(m,n)} \sum_{(\mathscr{I}', \mathscr{J}') \in \operatorname{sh}(m,n)} \int F(x_{\mathscr{I}})\, G(x_{\mathscr{J}})\, \overline{F(x_{\mathscr{I}'})}\, \overline{G(x_{\mathscr{J}'})}\, dx_{\mathscr{I}}\, dx_{\mathscr{J}}$$
$$= \frac{(m+n)!}{m!\, n!} \|F\|^2_{L_2(I^m)} \|G\|^2_{L_2(I^n)}$$
$$+ \sum_{(\mathscr{I}, \mathscr{J}) \neq (\mathscr{I}', \mathscr{J}')} \int F(x_{\mathscr{I}})\, G(x_{\mathscr{J}})\, \overline{F(x_{\mathscr{I}'})}\, \overline{G(x_{\mathscr{J}'})}\, dx_{\mathscr{I}}\, dx_{\mathscr{J}}.$$

But

$$\int F(x_{\mathscr{I}})\,\overline{G(x_{\mathscr{I}'})}\,\overline{F(x_{\mathscr{I}'})}\,G(x_{\mathscr{I}})\,dx_{\mathscr{I}}\,dx_{\mathscr{I}}$$

$$= \int F(x_{\mathscr{I}\cap\mathscr{I}'}, x_{\mathscr{I}\cap\mathscr{I}'})\,\overline{G(x_{\mathscr{I}\cap\mathscr{I}'}, x_{\mathscr{I}\cap\mathscr{I}'})}\,\overline{F(x_{\mathscr{I}\cap\mathscr{I}'}, x_{\mathscr{I}\cap\mathscr{I}'})}$$

$$\times G(x_{\mathscr{I}\cap\mathscr{I}'}, x_{\mathscr{I}\cap\mathscr{I}'})\,dx_{\mathscr{I}\cap\mathscr{I}'}\,dx_{\mathscr{I}\cap\mathscr{I}'}\,dx_{\mathscr{I}\cap\mathscr{I}'}\,dx_{\mathscr{I}\cap\mathscr{I}'}$$

$$= \int \left| \int F(x_{\mathscr{I}\cap\mathscr{I}'}, z)\,\overline{G(z, x_{\mathscr{I}\cap\mathscr{I}'})}\,dz \right|^2 dx_{\mathscr{I}\cap\mathscr{I}'}\,dx_{\mathscr{I}\cap\mathscr{I}'} \geqslant 0,$$

and this proves Theorem 1.B.2. Theorem 1.2 now follows from Theorem 1.B.2 and Lemma 1.B.1.

The estimate $\sqrt{(m+n)!/m!\,n!}$ in Theorem 1.B.2 (and therefore the estimate of Theorem 1.2) is best possible. Indeed, take

$$F(x_1, ..., x_m) = 1, \qquad G(x_1, ..., x_n) = e^{2i\pi(x_1 + \cdots + x_n)}.$$

Then $\|F\|_{L_2(I^m)} = 1$, $\|G\|_{L_2(I^n)} = 1$, and

$$\left\| \sum_{(\mathscr{I},\mathscr{I}) \in \mathrm{sh}(m,n)} F(x_{\mathscr{I}})\,G(x_{\mathscr{I}}) \right\|^2_{L_2(I^{m+n})} = \frac{(m+n)!}{m!\,n!}.$$

Remark. From Theorem 1.1 and the comparison estimates

$$\left(\frac{1}{m!}\right)^{1-1/p} |P|_p \leqslant [P]_p \leqslant |P|_p, \tag{4}$$

it follows that $k_p(m,n) > 0$ for all p, $1 \leqslant p \leqslant \infty$, and therefore that $c_p(m,n) > 0$. But for $p \neq 2$, we do not know the precise value of these constants.

These theorems have some consequences:

First, we observe that Lemma 1.A.6 and Lemma 1.B.1 show that if f, g are real-valued functions,

$$\int_0^1 \int_0^1 |f(x)\,g(y) + f(y)\,g(x)|\,dx\,dy \geqslant \left(\int_0^1 |f(x)|\,dx\right)\left(\int_0^1 |g(y)|\,dy\right).$$

We also deduce from (4):

Proposition 1.B.3. *If P, Q are homogeneous polynomials of degrees m, n,*

$$|PQ|_2 \geqslant \frac{1}{\sqrt{(m+n)!}}\,|P|_2 \cdot |Q|_2.$$

As we already said, this estimate is independent of the number of variables, and we do not know how precise it is. However, in the case of two variables, it can be greatly improved:

PROPOSITION 1.B.4. *Let P, Q be homogeneous polynomials of degrees m, n, with two variables. Then*

$$|PQ|_2 \geqslant \binom{m+n}{m}^{-1/2} \binom{m}{[m/2]}^{-1/2} \binom{n}{[n/2]}^{-1/2} |P|_2 \, |Q|_2.$$

Proof of Proposition 1.B.4. From Theorem 1.B.1, we get

$$[PQ]_2 \geqslant \binom{m+n}{m}^{-1/2} [P]_2 \, [Q]_2. \tag{5}$$

But $[\cdot]_2 \leqslant |\cdot|_2$. Moreover, in the case of two variables, we have:

LEMMA 1.B.5. *Let P be a homogeneous polynomial of degree m, with two variables. Then*

$$[P]_2 \geqslant \binom{m}{[m/2]}^{-1/2} |P|_2.$$

Proof of Lemma 1.B.5. One uses the fact that, if $\alpha_1 + \alpha_2 = m$,

$$\alpha_1 ! \, \alpha_2 ! \geqslant [m/2]! \, (m - [m/2])!.$$

Proposition 1.B.3 follows immediately from the lemma.

Remark. A previously known bound, in the case of two-variable polynomials, was

$$|PQ|_2 \geqslant \binom{2m}{m}^{-1/2} \binom{2n}{n}^{-1/2} |P|_2 \, |Q|_2$$

$$\geqslant \frac{1}{2^{m+n}} |P|_2 \, |Q|_2.$$

Indeed, set $f(z) = P(z, 1)$. The above formula is easily deduced from the comparison between Mahler's measure,

$$M(f) = \exp \int_0^{2\pi} \log |f(e^{i\theta})| \frac{d\theta}{2\pi},$$

and the L_2-norm of f, $\|f\|_{L_2} = |P|_2$: see Mahler [4]. The bound given in Proposition 1.B.4 is slightly better.

We now turn to the estimates related to the L_p-norms.

C. *Estimates in L_p-Norms*

We now prove Theorem 1.3, first in the case $p<\infty$. Since $\|P\|_2=|P|_2$, the result is known in the case $p=2$. It follows for other values of p by comparison arguments between the various norms. The constants involved, of course, have to be independent of the number of variables.

LEMMA 1.C.1. *Let P be a homogeneous polynomial of degree m. Then*

$$\|P\|_2 \leqslant \|P\|_p \leqslant 2^{([\log_2 p]+1)m/2}\,\|P\|_2, \qquad \textit{if} \quad 2\leqslant p<\infty,$$
$$2^{-m}\,\|P\|_2 \leqslant \|P\|_p \leqslant \|P\|_2, \qquad \textit{if} \quad 1\leqslant p\leqslant 2.$$

Proof of Lemma 1.C.1. By Parseval's identity,

$$\|P\|_4^4=\|P^2\|_2^2=|P^2|_2^2.$$

By Lemma 1.A.1, with $P=Q$,

$$|P^2|_2^2\leqslant 2^{2m}\,|P|_2^4,$$

and thus

$$\|P\|_4\leqslant 2^{m/2}\,\|P\|_2. \tag{1}$$

Let now $l\geqslant 1$. We have

$$\begin{aligned}\|P\|_{2^{l+1}} &= \|P^{2^{l-1}}\|_4^{1/2^{l-1}}\\ &\leqslant (2^{2^{l-1}m/2}\,\|P^{2^{l-1}}\|_2)^{1/2^{l-1}}, \qquad \text{by (1)}\\ &\leqslant 2^{m/2}\,\|P^{2^{l-1}}\|_2^{1/2^{l-1}}\\ &= 2^{m/2}\,\|P\|_{2^l},\end{aligned}$$

and therefore, for $l\geqslant 1$,

$$\|P\|_{2^l}\leqslant 2^{lm/2}\,\|P\|_2.$$

Now, for $p\geqslant 2$, one chooses $l=[\log_2 p]+1$ and obtains the estimate.

We now consider the case $p\leqslant 2$. The inequality $\|P\|_p\leqslant\|P\|_2$ is obvious. The other inequality needs to be proved only in the case $p=1$. But then we have

$$\begin{aligned}\|P\|_2 &\leqslant \|P\|_1^{1/3}\cdot\|P\|_4^{2/3}\\ &\leqslant 2^{m/3}\,\|P\|_1^{1/3}\cdot\|P\|_2^{2/3}, \qquad \text{by (1)}.\end{aligned}$$

So

$$\|P\|_1\geqslant 2^{-m}\,\|P\|_2,$$

and our lemma is proved. Theorem 1.3, in the case $p<\infty$, follows immediately.

We now consider the case $p=\infty$. Then Lemma 1.C.1 is not true anymore, as the example of $P=x_1+\cdots+x_N$ shows. The right hand side inequality is obvious. We now prove the other one.

Let $\xi=(\xi_1,\dots,\xi_N)$, $\eta=(\eta_1,\dots,\eta_N)$ be points on the polycircle $|x_i|=1$ such that

$$|P(\xi)|=\max_{|x_i|=1}|P(x_1,\dots,x_N)|,$$

$$|Q(\eta)|=\max_{|x_i|=1}|Q(x_1,\dots,x_N)|.$$

We now consider, for $0\leqslant t\leqslant 1$,

$$f(t)=P(t\xi+(1-t)\,\eta),\qquad g(t)=Q(t\xi+(1-t)\,\eta).$$

Since $\max_{|x_i|=1}|P(x_1,\dots,x_N)\,Q(x_1,\dots,x_N)|$ is attained on the Shilov boundary $|x_i|=1$, we see that

$$\|PQ\|_\infty\geqslant\max_{0\leqslant t\leqslant 1}|f(t)\,g(t)|.$$

Our result now follows immediately from the following theorem of Kneser [3]:

THEOREM (H. Kneser). *Let $f(z)$, $g(z)$ be polynomials of degrees m, n and let E be a bounded continuum in $\mathbb{C}$. Then*

$$\max_E|f(z)\,g(z)|\geqslant c(m,n)\max_E|f(z)|\max_E|g(z)|,$$

where

$$c(m,n)=\prod_{k=1}^{m}\tan^2\frac{(2k-1)\,\pi}{4(m+n)}.$$

Equality is attained if, setting $z=\cos\phi$, $f(z)\,g(z)=\cos(m+n)\,\phi$, and

$$f(z)=\prod_{k=1}^{m}\left(z-\cos\frac{(2k-1)\,\pi}{2(m+n)}\right).$$

To finish this section, we observe the estimate

$$C_2(\infty,m,n)=\prod_{k=1}^{\inf(m,n)}\tan^2\frac{(2k-1)\,\pi}{4(m+n)}\geqslant\frac{1}{4^{m+n}}.$$

We now turn to polynomials which are not necessarily homogeneous.

2. Polynomials with Concentration at Low Degrees

Let, as before,

$$P(x_1, ..., x_N) = \sum_\alpha a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$$

be a polynomial in many variables. We set $\alpha = (\alpha_1, ..., \alpha_N)$, $|\alpha| = \alpha_1 + \cdots + \alpha_N$, and, for $k \in \mathbb{N}$, we put

$$P|^k = \sum_{|\alpha| \leqslant k} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

Let $0 < d \leqslant 1$. We say that P has concentration d at (total) degree k, in a norm $\|\cdot\|$, if

$$\|P|^k\| \geqslant d \, \|P\|.$$

This applies for instance to any of the norms $|\cdot|_p$, $\|\cdot\|_p$ defined in the previous paragraph.

We extend to this new frame the results previously obtained for homogeneous polynomials. First, we extend them to polynomials of fixed degrees, with bounds dependent on the degree.

Theorem 2.1. *For any p, $1 \leqslant p \leqslant \infty$, any polynomials P, Q of degrees m, n, respectively,*

$$C_1(m, n) \, |P|_p \, |Q|_p \leqslant |PQ|_p \leqslant 2^{(m+n)(1-1/p)} \, |P|_p \, |Q|_p,$$

where $C_1(m, n)$ is the constant defined in Theorem 1.1.

Theorem 2.2. *For any p, $1 \leqslant p < \infty$, any polynomials P, Q of degrees m, n, respectively,*

$$C_2(p, m, n) \, \|P\|_p \, \|Q\|_p \leqslant \|PQ\|_p \leqslant C_2'(p, m, n) \, \|P\|_p \, \|Q\|_p,$$

and in the case $p = \infty$,

$$C_2(\infty, m, n) \, \|P\|_\infty \, \|Q\|_\infty \leqslant \|PQ\|_\infty \leqslant \|P\|_\infty \, \|Q\|_\infty,$$

where $C_2(p, m, n)$, $C_2'(p, m, n)$ are the constants defined in Theorem 1.2.

These theorems are immediately deduced from Theorems 1.1 and 1.2. Indeed, if $P(x_1, ..., x_N)$ is any polynomial of degree m, we put

$$P^*(x_0, x_1, ..., x_N) = x_0^m P\left(\frac{x_1}{x_0}, ..., \frac{x_N}{x_0}\right),$$

and P^* is homogeneous of degree m. Moreover, $(PQ)^* = P^*Q^*$, $|P|_p = |P^*|_p$, $\|P\|_p = \|P^*\|_p$. The theorems follow.

We now turn to polynomials with concentration at low degrees. Let

$$P(x_1, ..., x_N) = \sum_\alpha a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N},$$

$$Q(x_1, ..., x_N) = \sum_\beta b_\beta x_1^{\beta_1} \cdots x_N^{\beta_N}$$

satisfy

$$|P|^m|_p \geqslant d\, |P|_p \tag{1}$$

$$|Q|^n|_p \geqslant d'\, |Q|_p. \tag{2}$$

THEOREM 2.3. *Let* $1 \leqslant p \leqslant \infty$. *For* $m, n \in \mathbb{N}$, $0 < d$, $d' \leqslant 1$, *there is a constant* $\lambda(p; d, d'; m, n) > 0$ *such that, for any polynomials* P, Q *satisfying* (1), (2),

$$|(PQ)|^{m+n}|_p \geqslant \lambda\, |P|_p\, |Q|_p.$$

Proof of Theorem 2.3. We may assume $|P|_p = |Q|_p = 1$. For $i \in \mathbb{N}$, we write P_i for the homogeneous part of P of degree i, that is,

$$P_i = \sum_{|\alpha| = i} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

We give the proof for $p < \infty$ and leave it to the reader for $p = \infty$.

Condition (1) implies

$$\sum_{i \leqslant m} |P_i|_p^p \geqslant d^p.$$

Therefore, for some $i_1 \leqslant m$,

$$|P_{i_1}|_p \geqslant \frac{d}{(m+1)^{1/p}}. \tag{3}$$

The same way, for some $j_1 \leqslant n$,

$$|Q_{j_1}|_p \geqslant \frac{d'}{(n+1)^{1/p}}. \tag{4}$$

By Theorem 1.1, with $\lambda_1 = \lambda_1(p; m, n; d, d')$,

$$|P_{i_1} Q_{j_1}|_p \geqslant \lambda_1. \tag{5}$$

Set $R = PQ$. Then

$$R_{i_1+j_1} = \sum_{i+j=i_1+j_1} P_i Q_j. \tag{6}$$

In this sum, either $i \leqslant i_1$ or $j \leqslant j_1$, and $i_1 + j_1 \leqslant m+n$. So the sum has at most $(m+n)^2$ terms. We have two cases:

— Case 0:

$$|R_{i_1+j_1}|_p \geqslant \tfrac{1}{2}\lambda_1,$$

and the conclusion is reached, or

— Case 1:

$$|R_{i_1+j_1}|_p < \tfrac{1}{2}\lambda_1.$$

In this case, by (6),

$$\sum_{(i,j) \neq (i_1, j_1),\, i+j=i_1+j_1} |P_i Q_j|_p \geqslant \frac{1}{2}\lambda_1.$$

So we can find i_2, j_2, with either $i_2 < i_1$ or $j_2 < j_1$, such that

$$|P_{i_2} Q_{j_2}|_p \geqslant \lambda_2, \tag{7}$$

with

$$\lambda_2 = \frac{1}{2}\frac{\lambda_1}{(m+n)^2}.$$

By Lemma 1.A.3, since $i_2 \leqslant m+n$, $j_2 \leqslant m+n$,

$$|P_{i_2} Q_{j_2}|_p \leqslant C(p, m, n)\, |P_{i_2}|_p\, |Q_{j_2}|_p, \tag{8}$$

with $C(p, m, n) = 2^{2(m+n)(1-1/p)}$, and, with $\lambda_2' = \lambda_2 / C(p, m, n)$,

$$|P_{i_2}|_p \geqslant \lambda_2', \qquad |Q_{j_2}|_p \geqslant \lambda_2'. \tag{9}$$

Assume first $i_2 < i_1$. We look at the product $P_{i_2} Q_{j_1}$, and we have, by Theorem 1.1,

$$|P_{i_2} Q_{j_1}|_p \geqslant \lambda_3,$$

and we have two cases:

— Case 1.0:

$$|R_{i_2+j_1}|_p \geqslant \tfrac{1}{2}\lambda_3,$$

and

— Case 1.1:

$$|R_{i_2+j_1}|_p < \tfrac{1}{2}\lambda_3.$$

If $i_2 > i_1$, then $j_2 < j_1$, and we do the same with $P_{i_1}Q_{j_2}$ instead.

We repeat this process. If we fall into a Case 0, the conclusion is reached. Otherwise, we find indexes i_k, j_k, such that for every k either $i_k < i_{k-1}$ or $j_k < j_{k-1}$. Therefore, the number of steps is at most $m+n$: then we reach polynomials of degree 0, for which the conclusion is clear.

We now look at the L_p norm. There is a substantial difference from the l_p norm, which has to be mentioned first. Obviously, one has

$$|P|^m|_p \leqslant |P|_p.$$

A similar estimate,

$$\|P|^m\|_p \leqslant C\,\|P\|_p,$$

with an absolute constant C, cannot hold in the L_p norm. Indeed, the exponential system is not a 1-unconditional basis in L_p, so there is a polynomial f in one variable such that

$$\|f|^m\|_p > \|f\|_p,$$

and one considers $P(x_1, ..., x_N) = f(x_1) \cdots f(x_N)$.

On the other hand, we have:

LEMMA 2.4. *For every p, $1 \leqslant p \leqslant \infty$, every polynomial P with N variables,*

$$\|P|^m\|_p \leqslant (2^{m+1}+1)\,\|P\|_p.$$

Proof of Lemma 2.4. We assume $\|P\|_p = 1$. We first consider the case $p \geqslant 2$. By the Marcinkiewicz interpolation theorem, to prove our formula for $p < \infty$ it is enough to prove it for $p = 2, 4, ...$ Then, letting $p \to \infty$, we also get it for $p = \infty$.

So we take $p = 2k$, $k \geqslant 1$. We have the obvious formula:

$$\|P\|_{2k} = \|P^k\|_2^{1/2}. \tag{10}$$

Let, as before, P_m be the homogeneous part of P of degree m. Let

$$R_m = P_m + P_{m+1} + \cdots$$

In R_m^k, the only terms of degree km are P_m^k. Since the L_2 norm is unconditional, we get

$$\|P_m^k\|_2^{1/k} \leqslant \|R_m^k\|_2^{1/k},$$

and by (10)

$$\|P_m\|_p \leqslant \|R_m\|_p.$$

Since $R_{m+1} = R_m - P_m$, we deduce

$$\|R_{m+1}\|_p \leqslant 2\ \|R_m\|_p,$$

and inductively

$$\|R_m\|_p \leqslant 2^m\ \|R_0\|_p = 2^m.$$

Since $P|^m = P - R_{m+1}$, we obtain

$$\|P|^m\|_p \leqslant 1 + 2^{m+1},$$

which is our claim. So the lemma is proved for $p \geqslant 2$. The case $1 \leqslant p \leqslant 2$ follows if we consider the operator $P \to P|^m$, from L_p into itself: its transpose is the same operator, from L_q into itself $(1/p + 1/q = 1)$.

We may now state our result for the L_p norms:

THEOREM 2.5. *Let* $1 \leqslant p \leqslant \infty$. *For* $m, n \in \mathbb{N}$, $0 < d$, $d' \leqslant 1$, *there is a constant* $\lambda'(p; d, d'; m, n) > 0$ *such that, for any polynomials* P, Q *satisfying*

$$\|P|^m\|_p \geqslant d\ \|P\|_p,$$

$$\|Q|^n\|_p \geqslant d'\ \|Q\|_p,$$

one has

$$\|(PQ)|^{m+n}\|_p \geqslant \lambda'\ \|P\|_p\ \|Q\|_p.$$

The proof is identical to the one of Theorem 2.3, except that, in order to get (9) in this proof, one uses Lemma 2.4.

We now turn to the following question: when does the product PQ have a large coefficient?

3. Large Coefficients

Let again P and Q be polynomials in many variables. Assume that P has a large coefficient and that Q has a large coefficient at low degrees. Then we prove that the product PQ has a large coefficient, with estimates independent of the number of the variables. We obtain precisely:

THEOREM 3.1. *Let $0 < d, d' \leqslant 1$, and $k \in \mathbb{N}$. There is a number $\lambda(d, d'; k)$ such that for any polynomials P and Q with*

$$|P|_\infty \geqslant d\,|P|_1, \tag{1}$$

$$|Q|^k|_\infty \geqslant d'\,|Q|_2, \tag{2}$$

the product satisfies

$$|PQ|_\infty \geqslant \lambda\,|P|_2 \cdot |Q|_2.$$

In the case of one-variable polynomials, a statement of the same nature was obtained by Beauzamy and Enflo [1]. There is, however, a difference between the present assumptions and the ones in [1], where we required only $|Q|^k|_2 \geqslant |Q|_2$. In the present case, dealing with many variables, we have to require also that Q have a large coefficient, and not only some concentration at low degrees, otherwise $Q = (x_1 + \cdots + x_N)/N$, together with $P = 1$, would provide a counter-example to our statement.

The proof uses most of the ideas of [1], but some refinements are needed to go from the one-variable case to the many-variable case.

We denote by m the normalized Haar measure on Π^N.

LEMMA 3.2. *Let P be a polynomial satisfying*

$$\|P\|_2 \geqslant d\,\|P\|_\infty.$$

Let

$$E = \{(\theta_1, ..., \theta_N);\ |P(e^{i\theta_1}, ..., e^{i\theta_N})| \geqslant d\,\|P\|_2\}.$$

Then $m(E) \geqslant d^2/2$.

Proof. We may of course assume that $\|P\|_2 = 1$, and therefore $\|P\|_\infty \leqslant 1/d$. So we have

$$1 = \int |P|^2 \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_N}{2\pi} = \int_E + \int_{E^c} \leqslant \frac{1}{d^2} m(E) + d^2(1 - m(E)),$$

and the result follows.

The next lemma provides an extension of Jensen's inequality, for polynomials with concentration at low degrees, in the case of a fixed number of variables. The proof is quite similar to that of Lemma 3 in [1].

LEMMA 3.3. *Let $k \in \mathbb{N}$, d', with $0 < d' \leqslant 1$. There exists a constant $C(d', k)$ such that any polynomial Q in k variables with*

$$|Q|^k|_\infty \geqslant d', \qquad \|Q\|_2 \leqslant 1 \tag{3}$$

satisfies

$$\int \log |Q(e^{i\theta_1}, ..., e^{i\theta_k})| \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \geqslant C(d', k). \tag{4}$$

Proof. For any $\beta = (\beta_1, ..., \beta_k)$ and $0 < r_1, ..., r_k \leqslant 1$, we have

$$b_\beta = \int \frac{e^{-i(\beta_1\theta_1 + \cdots + \beta_k\theta_k)}}{r_1^{\beta_1} \cdots r_k^{\beta_k}} Q(r_1 e^{i\theta_1}, ..., r_k e^{i\theta_k}) \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi}, \tag{5}$$

and therefore, for some points $z_1, ..., z_k$, with $|z_1| = r_1, ..., |z_k| = r_k$,

$$d' \leqslant \left(\sum_{|\beta| \leqslant k} |b_\beta|^2 \right)^{1/2} \leqslant \left(\sum_{|\beta| \leqslant k} r_1^{-2\beta_1} \cdots r_k^{-2\beta_k} \right)^{1/2} |Q(z_1, ..., z_k)|. \tag{6}$$

But repeated applications of the classical Jensen's inequality give

$$\begin{aligned}
&\int \log \left| Q\left(\frac{e^{i\theta_1} + z_1}{1 + \bar{z}_1 e^{i\theta_1}}, ..., \frac{e^{i\theta_k} + z_k}{1 + \bar{z}_k e^{i\theta_k}} \right) \right| \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \\
&\quad \geqslant \int \log \left| Q\left(\frac{e^{i\theta_1} + z_1}{1 + \bar{z}_1 e^{i\theta_1}}, ..., \frac{e^{i\theta_{k-1}} + z_{k-1}}{1 + \bar{z}_{k-1} e^{i\theta_{k-1}}}, z_k \right) \right| \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_{k-1}}{2\pi} \\
&\quad \geqslant \log |Q(z_1, ..., z_k)|.
\end{aligned}$$

Making k changes of variables, we can write

$$\begin{aligned}
&\int \log \left| Q\left(\frac{e^{i\theta_1} + z_1}{1 + \bar{z}_1 e^{i\theta_1}}, ..., \frac{e^{i\theta_k} + z_k}{1 + \bar{z}_k e^{i\theta_k}} \right) \right| \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \\
&\quad = \int \log |Q(e^{i\theta_1}, ..., e^{i\theta_k})| \frac{1 - r_1^2}{|1 - \bar{z}_1 e^{i\theta_1}|^2} \cdots \frac{1 - r_k^2}{|1 - \bar{z}_k e^{i\theta_k}|^2} \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \\
&\quad = \int_{\log |Q| < 0} + \int_{\log |Q| \geqslant 0} \\
&\quad \leqslant \frac{1 - r_1}{1 + r_1} \cdots \frac{1 - r_k}{1 + r_k} \int_{\log |Q| < 0} \log |Q(e^{i\theta_1}, ..., e^{i\theta_k})| \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi} \\
&\qquad + \frac{1 + r_1}{1 - r_1} \cdots \frac{1 + r_k}{1 - r_k} \int_{\log |Q| \geqslant 0} \log |Q(e^{i\theta_1}, ..., e^{i\theta_k})| \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_k}{2\pi}.
\end{aligned}$$

But $\int_{\log|Q|\geqslant 0}\log|Q|\leqslant\frac{1}{2}$. Taking $r_1=\cdots=r_k=\frac{1}{3}$, we obtain

$$\begin{aligned}\log|Q(z_1,...,z_k)| &\leqslant \frac{1}{2^k}\int_{\log|Q|<0}\log|Q|+2^{k-1}\\ &\leqslant \frac{1}{2^k}\int\log|Q|+2^{k-1},\end{aligned}$$

and the result follows. We observe that here the constant $C(d',k)$ might be computed explicitly.

LEMMA 3.4. *Let Q be a polynomial in any number of variables, satisfying* (3). *Then*

$$\int\log|Q(e^{i\theta_1},...,e^{i\theta_N})|\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_N}{2\pi}\geqslant C(d',k),$$

where $C(d',k)$ is the constant given by the previous lemma.

Proof. By assumption, we know that in Q there is a coefficient b_β, with $|\beta|\leqslant k$ and $|b_\beta|\geqslant d'$. Therefore, the term $b_\beta x_1^{\beta_1}\cdots x_N^{\beta_N}$ contains at most k variables, and we can of course assume that only $x_1,...,x_k$ appear.

Let $\tilde{Q}$ be the part of Q containing only these variables. Then $\tilde{Q}$ satisfies

$$||\tilde{Q}|^k|_\infty\geqslant d',\qquad \|\tilde{Q}\|_2\leqslant 1.$$

So by the previous lemma

$$\int\log|\tilde{Q}(e^{i\theta_1},...,e^{i\theta_k})|\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_k}{2\pi}\geqslant C(d',k).$$

Let us denote by $\tilde{Q}_{k+1}$ the part of Q which contains only the variables $x_1,...,x_{k+1}$. Then $\tilde{Q}_{k+1}$ can be written as

$$\tilde{Q}_{k+1}=\tilde{Q}+x_{k+1}R_1+x_{k+1}^2R_2+\cdots,$$

where $R_1, R_2, ...$ are polynomials in $x_1,...,x_k$ only. Therefore, by the classical Jensen's inequality,

$$\int\log|\tilde{Q}_{k+1}(e^{i\theta_1},...,e^{i\theta_{k+1}})|\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_{k+1}}{2\pi}\geqslant\int\log|\tilde{Q}(e^{i\theta_1},...,e^{i\theta_k})|\frac{d\theta_1}{2\pi}\cdots\frac{d\theta_k}{2\pi}. \tag{7}$$

We now introduce $\tilde{Q}_{k+2}$, part of Q containing only the $k+2$ first variables, and repeat (7), and so on: the estimates remain the same at each stage.

LEMMA 3.5. *Let* $0<a\leqslant 1$, $k\in\mathbb{N}$, $0<d'\leqslant 1$. *There is a constant* $\lambda_1(a,d',k)>0$ *such that, for any polynomial* Q *satisfying* (3), *any subset* E *of* Π^N *with* $m(E)\geqslant a$,

$$m(E\cap\{|Q|\geqslant\lambda_1\})\geqslant a/2.$$

Proof. Let A be any measurable subset of Π^N. We have

$$2\int_A \log|Q| = \int_A \log|Q|^2 \leqslant \int_A |Q|^2 \leqslant 1, \tag{8}$$

and from Lemma 3.4,

$$\int_{A^c} \log|Q| \geqslant C(d',k)-1/2.$$

Let now $\varepsilon>0$, and take $A=\{|Q|\geqslant\varepsilon\}$. Then

$$(\log\varepsilon)\, m(A^c)\geqslant C(d',k)-\tfrac{1}{2}.$$

So

$$m(A)\geqslant 1-\frac{C(d',k)-\frac{1}{2}}{\log\varepsilon}.$$

If now ε is taken sufficiently small (depending only on a, d', k) to ensure that

$$\frac{C(d',k)-\frac{1}{2}}{\log\varepsilon}\leqslant a/2,$$

the result follows.

We now make the convenient definitions and normalizations for the proof of the theorem. We write P under the form

$$P(x_1,\ldots,x_N)=\sum_{\alpha\in\mathbb{Z}^N} a_\alpha x_1^{\alpha_1}\cdots x_N^{\alpha_N}, \tag{9}$$

with $a_{0,\ldots,0}=1$. So by (1)

$$\sum_\alpha |a_\alpha|\leqslant 1/d.$$

We put $P_0=a_{0,\ldots,0}$.

We know also that in Q written as before there is a coefficient b_β of a term containing only $x_1,\ldots,x_k$ (at most), with $|\beta|\leqslant k$. We can assume that this $b_\beta=1$, and we get

$$\|Q\|_2\leqslant 1/d'. \tag{10}$$

We also denote by Q_0 the above term $b_\beta x_1^{\beta_1}\cdots x_k^{\beta_k}$.

We say that P' is a *part* of P if it can be written as $\sum_{\alpha \in A} a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}$, for some subset A of $\mathbb{Z}^N$. Similarly, we define a part of Q. We say that a part P' of P is disjoint from P_0 if A does not contain $(0, ..., 0)$, and that Q' is disjoint from Q_0 if A does not contain the above index β.

With these definitions, we can now state:

LEMMA 3.6. *There is a constant $\lambda(d, d', k) > 0$ such that for any polynomials P, Q as above, any parts P' and Q' disjoint from P_0 and Q_0, respectively, one has*

$$\|(P_0 + P')(Q_0 + Q')\|_2 \geqslant \lambda.$$

Proof. We observe that

$$\|P_0 + P'\|_2 = |P_0 + P'|_2 \geqslant |P_0 + P'|_\infty \geqslant d\,|P_0 + P'|_1 \geqslant d\,\|P_0 + P'\|_\infty,$$

and therefore $P_0 + P'$ satisfies the assumptions of Lemma 3.2. Let E be the subset of Π^N where $|P_0 + P'| \geqslant d$. It follows from Lemma 3.2 that $m(E) \geqslant d^2/2$.

We can apply Lemma 3.5 to $Q_0 + Q'$, since it also satisfies (3), and to the subset E previously found. We deduce from Lemma 3.5 that there is a subset E_1 of E, with $m(E_1) \geqslant d^2/4$, on which we have simultaneously

$$|P_0 + P'| \geqslant d, \qquad |Q_0 + Q'| \geqslant \lambda_1.$$

The lemma follows obviously.

We now turn to the proof of the theorem, which follows the lines of [1]. If A is a subset of $\mathbb{Z}^N$ or $\mathbb{N}^N$, we denote by π_A the restriction projection

$$\pi_A(P) = \sum_A a_\alpha x_1^{\alpha_1} \cdots x_N^{\alpha_N}.$$

The *support* A of a polynomial P is the set $A = \{\alpha; a_\alpha \neq 0\}$, and $\mathrm{card}(A)$ is the number of its elements.

Let E_0 be the support of $P_0 Q_0$: it has only 1 element. Let π_0, instead of π_{E_0}, be the projection on it. By our normalizations,

$$\|\pi_0(P_0 Q_0)\|_2 = 1 \geqslant \lambda.$$

Several cases can occur:

— If $\|\pi_0(PQ)\|_2 \geqslant \lambda/2$, then $|PQ|_\infty \geqslant \lambda/2$, and the theorem is obtained. We call this Case 0.

— Or $\|\pi_0(PQ)\|_2 < \lambda/2$. Then we decompose P and Q into disjoint pieces:

$$P = P_0 + P'; \qquad Q = Q_0 + Q'.$$

Then we have:

— either $\|\pi_0(P'Q_0)\|_2 \geqslant \lambda/4$ (Case 1),

— or $\|\pi_0(PQ')\|_2 \geqslant \lambda/4$ (Case 2).

We first look at Case 1. We set $\varepsilon_1 = \lambda/8$. We decompose P' into two disjoint pieces, $P_1 + P''$, where in P_1 all coefficients satisfy $|a_\alpha| \geqslant \varepsilon_1$, and in P'' all $|a_\alpha| < \varepsilon_1$. Then

$$\|\pi_0(P''Q_0)\|_2 \leqslant |\pi_0(P''Q_0)|_\infty \leqslant \varepsilon_1 \, |Q_0|_1 \leqslant \lambda/8,$$

and therefore

$$\|\pi_0(P_1 Q_0)\|_2 \geqslant \lambda/8.$$

The part P_1, since $|P|_1 \leqslant 1/d$, has at most $K_1 = 1/\varepsilon_1 d$ terms. Moreover

$$\lambda/8 \leqslant \|P_1 Q_0\|_2 \leqslant \|P_1\|_\infty \cdot \|Q_0\|_2 \leqslant |P_1|_1.$$

We denote by E_1 the support of $(P_0 + P_1)\, Q_0$. It has at most $1 + K_1$ elements. By Lemma 3.6, we have $\|(P_0 + P_1)\, Q_0\|_2 \geqslant \lambda$, and we look at $\|\pi_1(PQ)\|_2$, with π_1 written instead of π_{E_1}.

We now turn to Case 2. We put $\varepsilon_1' = \lambda d/8$, we decompose Q' into $Q_1 + Q''$ as in Case 1, and we get in the same way

$$\|\pi_0(PQ_1)\|_2 \geqslant \lambda/8.$$

There are in Q_1 at most $K_1' = 1/(\varepsilon_1' d')^2$ terms, and

$$\|Q_1\|_2 \geqslant \lambda d/8.$$

The support E_1 of $P_0(Q_0 + Q_1)$ has at most $1 + K_1'$ terms, and we look at $\|\pi_1(PQ)\|_2$.

Assume now that we have repeated this process n times, and that we have obtained a sequence, denoted by u_n, of Case 1's or 2's, in some order, with for instance a Case 1's and b Case 2's $(a + b = n)$.

Each Case 1 produces in P disjoint parts $P_1, ..., P_a$, with $|P_i|_1 \geqslant \lambda d'/8$. Each term in P_i is greater than ε_i, and there are at most K_i such terms.

Each Case 2 produces in Q disjoint parts $Q_1, ..., Q_b$, with $\|Q_j\|_2 \geqslant \lambda d/8$. Each term in Q_j is greater than ε_j' and there are at most K_j' such terms.

Let E_n be the support of $(P_0 + \cdots + P_a)(Q_0 + \cdots + Q_b)$. This set has a. cardinal $\sigma_n \leqslant (1 + K_1 + \cdots + K_a)(1 + K_1' + \cdots + K_b')$. We look at $\|\pi_n(PQ)\|_2$.

— If $\|\pi_n(PQ)\|_2 \geqslant \lambda/2$, we have $|PQ|_\infty \geqslant \lambda/2\sqrt{\sigma_n}$: this is Case u_n, 0.

— If not, we apply Lemma 3.6, and we write

$$P = P_0 + \cdots + P_a + P', \qquad Q = Q_0 + \cdots + Q_b + Q',$$

we obtain two cases:

— either $\|\pi_n(P'(Q_0 + \cdots + Q_b))\|_2 \geqslant \lambda/4$, Case u_n, 1,

— or $\|\pi_n(PQ')\|_2 \geqslant \lambda/4$, Case u_n, 2.

In Case u_n, 1, we set $\varepsilon_{a+1} = \lambda d'/8\sqrt{\sigma_n}$. We decompose P' into $P_{a+1} + P''$, where all terms in P_{a+1} are $\geqslant \varepsilon_{a+1}$, and all terms in P'' are $< \varepsilon_{a+1}$. We obtain

$$\|\pi_n(P_{a+1}(Q_0 + \cdots + Q_b))\|_2 \geqslant \lambda/8.$$

The part P_{a+1} has at most $K_{a+1} \leqslant 1/d\varepsilon_{a+1}$ terms, and

$$|P_{a+1}|_1 \geqslant \lambda d'/8.$$

In Case u_n, 2, we set $\varepsilon'_{b+1} = \lambda d/8\sqrt{\sigma_n}$. We decompose Q' into $Q_{b+1} + Q''$, as above. We obtain

$$\|\pi_n(PQ_{b+1})\|_2 \geqslant \lambda/8.$$

The part Q_{b+1} has at most $K'_{b+1} \leqslant 1/d'^2\varepsilon'^2_{b+1}$ terms, and $\|Q_{b+1}\|_2 \geqslant \lambda d/8$.

Since $|P|_1 \leqslant 1/d$, the total number of Case 1's is at most $8/\lambda dd'$. Since $\|Q\|_2 \leqslant 1/d'$, the total number of Case 2's is at most $(8/\lambda dd')^2$. Therefore, for $n \leqslant 8/\lambda dd' + (8/\lambda dd')^2$, a Case 0 occurs and the result is proved.

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