On the Vibrations of Overdamped Systems

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ABSTRACT: *Vibration systems are characterized by the two parameters: ω_n, the undamped natural frequency, and ζ, the damping ratio. In the case of lightly damped systems, these parameters are determined by means of the logarithmic decrement. A simple method is presented here which applies to overdamped systems, $\zeta > 1$. The method involves simple measurements, and a criterion for the accuracy of the method is given.*

I. Introduction

The fundamental parameters of a single degree of freedom vibratory system are ω_n, the undamped natural frequency, and ζ, the damping ratio. If the system is lightly damped, these parameters can be determined from the ratio of successive maxima of the free motion:

$$\delta = \ln (A_0/A_1) = \frac{1}{n} \ln (A_0/A_n).$$

Then

$$\zeta = \frac{\delta}{\sqrt{4\pi^2 + \delta^2}} \quad \text{and } \omega_n = \frac{2\pi}{\tau_d \sqrt{1-\zeta^2}},$$

where τ_d is the damped period of the free motion.

If $\zeta < 1$, ω_n is shown by the oscillatory nature of the free motion while ζ is exhibited by the exponential decay of successive maxima. In an overdamped system, the motion is exponential in nature. While the motion can be viewed as a harmonic motion in which the damping is so heavy that the motion never completes a half cycle, clearly the logarithmic decrement method is not applicable. Most authors of vibrations and control texts discuss the overdamped case, (**1**, **2**), but the discussion of determining the vibration parameters ω_n and ζ from an experimental plot is limited to lightly damped systems. Consider the plot of an overdamped system in Fig. 1.

In this paper, we give a method for finding ω_n and ζ from the free motion of an overdamped system. The method is as accurate as the measurements used in its application. In any event, the method gives a very quick and accurate idea of the range of the vibration parameters in an experimental plot to the vibration engineer.

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FIG. 1. An overdamped system in free motion.

At the end of the paper, we establish several properties of overdamped systems one of which will demonstrate the range of applicability of the method.

II. Overdamped Systems

The differential equation which defines the free motion of an overdamped single degree of freedom system is

$$\ddot{x}+2\zeta\omega_n\dot{x}+\omega_n^2 x = 0, \quad \zeta > 1. \tag{1}$$

If the initial conditions are $x(0) = x_0$ and $\dot{x}(0) = v_0$, the solution to (1) is:

$$x = \mathrm{e}^{-\omega_n\zeta t}\left[\left\{\frac{v_0+\omega_n\zeta x_0}{\omega_d}\right\}\sinh\omega_d t+x_0\cosh\omega_d t\right] \tag{2}$$

where

$$\omega_d = \omega_n\sqrt{\zeta^2-1}.$$

Figure 1 is a typical plot of (2).

For the calculations, we take $t = 0$ to be the first point at which $x(t)$ crosses the time axis. Thus we take the specific initial conditions:

$$x(0) = x_0 = 0, \quad \dot{x}(0) = v_0,$$

and by (2), $x(t)$ becomes

$$x = \frac{v_0}{\omega_d}\mathrm{e}^{-\omega_n\zeta t}\sinh\omega_d t. \tag{2'}$$

Consider the plot of (2′) shown in Fig. 2.

There are several points on the plot in Fig. 2 which we use in the method proposed:

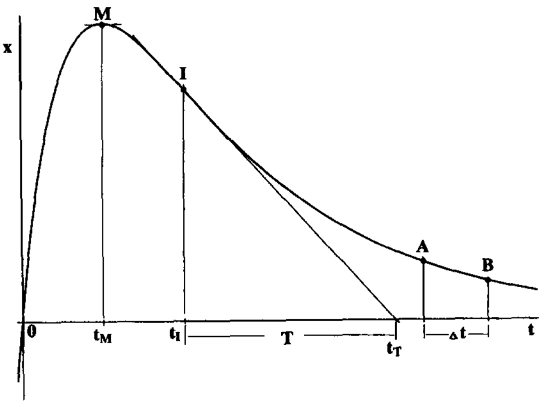

FIG. 2. The points M, I, A and B and the times T and Δt.

M—the maximum point $(\mathrm{d}x/\mathrm{d}t) = 0$ at M

I—the inflection point $(\mathrm{d}^2x/\mathrm{d}t^2) = 0$ at I

A, B—two points at large values of t: t_A, t_B.

III. The Procedure

We begin by locating the point M and the time at which it occurs, t_M. The inflection point is shown to occur at

$$t_I = 2t_M.$$

We now proceed as follows:

(1) Draw a tangent line to $x(t)$ through I. Denote the time at which this line intersects the time axis by t_T. Then set $T = t_T - t_I$.
(2) Select two points A and B as far as to the right of t_I as the data are meaningful. Then set

$$\Delta t = t_B - t_A.$$

(3) Let

$$\omega_1 = \frac{1}{\Delta t} \ln (x_A/x_B). \tag{3}$$

(4) Then

$$\zeta = \frac{T\omega_1}{2\sqrt{T\omega_1 - 1}}, \tag{4}$$

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and

$$\omega_n = 2\zeta/T. \tag{5}$$

IV. Proof of the Method

(a) *Location of the Maximum and Inflection Points*

Differentiating (2′), we get

$$\dot{x} = \frac{v_0}{\omega_d} e^{-\omega_n \zeta t} (-\omega_n \zeta \sinh \omega_d t + \omega_d \cosh \omega_d t) \tag{a}$$

$$\ddot{x} = \frac{v_0}{\omega_d} e^{-\omega_n \zeta t} [(\omega_d^2 + \omega_n^2 \zeta^2) \sinh \omega_d t - 2\omega_n \omega_d \zeta \cosh \omega_d t]. \tag{a′}$$

Setting $\dot{x} = 0$ and solving for t gives t_M. Similarly setting $\ddot{x} = 0$ and solving for t gives t_I:

$$t_M = \frac{1}{\omega_d} \tanh^{-1} \left\{ \frac{\sqrt{\zeta^2 - 1}}{\zeta} \right\} \tag{b}$$

$$t_I = \frac{1}{\omega_d} \tanh^{-1} \left\{ \frac{2\zeta\sqrt{\zeta^2 - 1}}{2\zeta^2 - 1} \right\}. \tag{c}$$

We can show that $t_I = 2t_M$ by noting the hyperbolic identity:

$$\tanh 2\theta = \frac{2 \tanh \theta}{1 + \tanh^2 \theta}.$$

Thus

$$\tanh (2\omega_d t_M) = \frac{\dfrac{2\sqrt{\zeta^2 - 1}}{\zeta}}{1 + \left\{ \dfrac{\sqrt{\zeta^2 - 1}}{\zeta} \right\}^2} = \frac{2\zeta\sqrt{\zeta^2 - 1}}{2\zeta^2 - 1}$$

which by (c) shows that

$$t_I = 2t_M. \tag{d}$$

In practice, the point M will be obtained from measurement rather than analytical computation, so (d) allows us to determine the time t_I and thus the point I itself.

(b) *Significance of the Inflection Point*

At the inflection point, I, the acceleration $\ddot{x}$ is zero. Thus by (1):

$$0 + 2\zeta\omega_n \dot{x}_I + \omega_n^2 x_I = 0.$$

Thus

$$\frac{2\zeta}{\omega_n} = -\frac{x_I}{\dot{x}_I}.$$

From Fig. 2, we see that

$$\dot{x}_I = x_I/T$$

and thus

$$\frac{2\zeta}{\omega_n} = T,$$

which establishes (5).

(c) *The Motion* $x_{(t)}$ *for Large Values of Time*

Suppose we rewrite (2′) in terms of exponential functions:

$$x(t) = \frac{v_0}{2\omega_d}(e^{-\omega_1 t} - e^{-\omega_2 t}), \tag{2''}$$

where

$$\omega_1 = \omega_n(\zeta - \sqrt{\zeta^2 - 1}) \tag{6a}$$

$$\omega_2 = \omega_n(\zeta + \sqrt{\zeta^2 - 1}). \tag{6b}$$

Both ω_1 and ω_2 are positive but $\omega_2 > \omega_1$. Thus for large values of time, $e^{-\omega_2 t}$ is negligibly small compared to $e^{-\omega_1 t}$, and we have

$$x(t) \approx \frac{v_0}{2\omega_d} e^{-\omega_1 t} \quad (\text{large} \ldots t). \tag{7}$$

Now suppose that we select two points A and B at times t_A and t_B. Then

$$x(t_A) = x_A \approx \frac{v_0}{2\omega_d} e^{-\omega_1 t_A}$$

$$x(t_B) = x_B \approx \frac{v_0}{2\omega_d} e^{-\omega_1 t_B},$$

and thus

$$x_A/x_B \approx e^{\omega_1(t_B - t_A)} = e^{\omega_1 \Delta t}.$$

Inverting this relation, we get

$$\omega_1 \approx \frac{1}{\Delta t} \ln (x_A/x_B),$$

which demonstrates (3).

(d) *Proof of Equation* (4)

Multiplying (6a) by T and using (5), we get

$$T\omega_1 = 2\zeta(\zeta - \sqrt{\zeta^2 - 1}),$$

or

$$T\omega_1 - 2\zeta^2 = -2\zeta\sqrt{\zeta^2 - 1}.$$

Squaring both sides, we obtain

$$T^2\omega_1^2 - 4T\omega_1\zeta^2 + 4\zeta^4 = 4\zeta^4 - 4\zeta^2,$$

or

$$4\zeta^2(T\omega_1 - 1) = T^2\omega_1^2.$$

And finally

$$\zeta = \frac{T\omega_1}{2\sqrt{T\omega_1 - 1}},$$

which establishes (4).

V. Example

Consider the plot of $x(t)$ shown in Fig. 3.

In this case, we measure $T = 0.38$ (s) and we take $\Delta t = 0.1$ (s). Then from (3):

$$\omega_1 = \frac{1}{0.1} \ln \left\{\frac{1.78}{1.32}\right\} = 2.99 \text{ (rad/s)}.$$

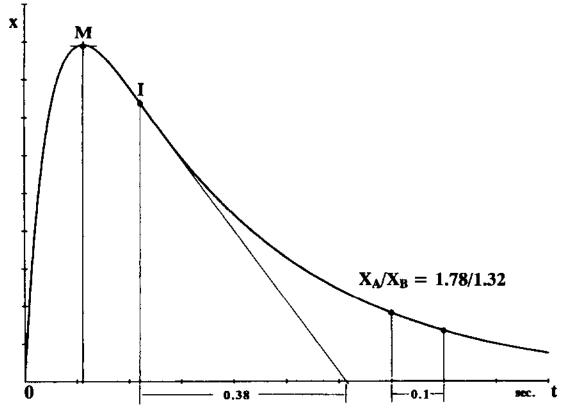

FIG. 3. Example.

Then (4) and (5) give:

$$\zeta = \frac{1.14}{2\sqrt{1.14-1.00}} = 1.52$$

$$\omega_n = \frac{2(1.52)}{0.38} = 8.00\,(\text{rad/s}).$$

The exact values in this particular case are

$$\omega_1 = 3.00\,(\text{rad/s}),$$
$$\zeta = 1.49,$$
$$\omega_n = 7.75\,(\text{rad/s}).$$

Clearly the accuracy of our results depends on the accuracy of the measurements taken. However, from the point of view of the vibration engineer, the accuracy in these results is more than acceptable.

VI. Some Properties of the Overdamped Motion

There are three properties of the overdamped free motion of a system which we will establish here:

(A) *ω_n is the geometric mean of ω_1 and ω_2;*
(B) *ζ is the ratio of the arithmetic mean to the geometric mean of ω_1 and ω_2;*
(C) *Let $\tau_1 = 1/\omega_1$ and $\tau_2 = 1/\omega_2$ be the* Exponential Time Constants. *Then the total time constant* T *is the sum of the exponential time constants.*

(a) *Proof of (A)*
Multiplying (6a) and (6b), we get

$$\omega_1\omega_2 = \omega_n^2[\zeta^2-(\zeta^2-1)] = \omega_n^2,$$

or

$$\omega_n = \sqrt{\omega_1\omega_2}. \tag{8}$$

(b) *Proof of (B)*
Adding (6a) to (6b):

$$\omega_1+\omega_2 = 2\omega_n\zeta.$$

And thus by (8):

$$\zeta = \frac{\omega_1+\omega_2}{2\sqrt{\omega_1\omega_2}}. \tag{9}$$

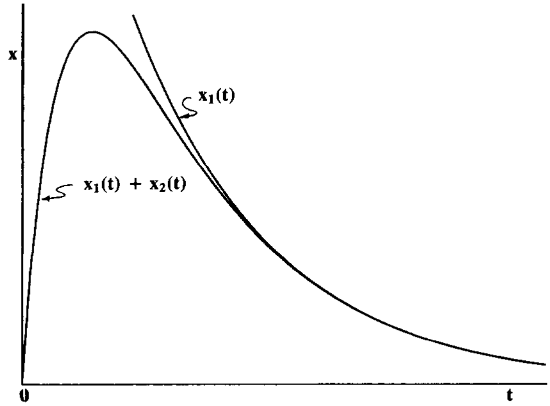

FIG. 4. Plots of $x_1(t)$ and $[x_1(t)+x_2(t)]$.

(c) *Proof of (C)*
From (5) with (8) and (9), we get

$$T = \frac{2\zeta}{\omega_n} = \frac{2(\omega_1+\omega_2)}{2\sqrt{\omega_1\omega_2}}\,\frac{1}{\sqrt{\omega_1\omega_2}} = \frac{\omega_1+\omega_2}{\omega_1\omega_2},$$

or

$$T = \frac{1}{\omega_1} + \frac{1}{\omega_2} = \tau_1+\tau_2. \tag{10}$$

VII. Accuracy of the Method

The accuracy of the method given here is highly dependent upon the spread between the frequencies ω_1 and ω_2. Consider the plot in Fig. 4. The first plot is (2′); the second is (7). The points A and B should be chosen for those points at which the two curves converge. That is, we wish to use points A and B where $e^{-\omega_2 t} \approx 0$.

We can use the property (9) to determine the spread in the frequencies for a given value of ζ. Let $\omega_2 = \alpha\omega_1$. Then (9) becomes

$$\zeta = \frac{1+\alpha}{2\sqrt{\alpha}}, \tag{11}$$

or

$$\alpha = (2\zeta^2-1)+2\zeta\sqrt{\zeta^2-1}. \tag{11′}$$

From (11) and (11′), we see that we have the following values:

α	ζ
2.0	1.06
4.0	1.25
10.0	1.74
100.0	5.05

VIII. Conclusions

The method presented here is both simple and accurate. The point M is easily located. This leads to the point I and the time T. Since we require only the ratio x_A/x_B, we need only scale the time axis. While the method can be extremely accurate, a design engineer generally needs to know approximate values of ω_n and ζ. The procedure given here presents a very quick method for providing that information.

References

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