

A Lie Algebra Variation on a Theorem of Wedderburn

I. N. HERSTEIN

*Department of Mathematics, University of Chicago,
Chicago, Illinois 60637*

L. SMALL

*Department of Mathematics, University of California at San Diego,
La Jolla, California 92093*

AND

D. J. WINTER

*Department of Mathematics, University of Michigan,
Ann Arbor, Michigan 48109*

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1. INTRODUCTION

In one of his last published papers, Wedderburn proved the following beautiful little theorem, which relates nilpotency of a finite dimensional associative algebra to that of the elements in a basis for it. A proof of this theorem based on Wedderburn's structure theorem for semisimple Artinian rings, as well as two nice applications of it, can be found in Herstein [4, pp. 56–60].

THEOREM. *Let A be a finite dimensional associative algebra over a field F . Suppose that A has a basis over F consisting of nilpotent elements. Then A itself must be nilpotent.*

Given a finite dimensional Lie algebra L which is a Lie subalgebra of an associative algebra A (possibly infinite dimensional), we can ask what results from supposing that L has a basis over F consisting of elements that are nilpotent in the associative algebra. We cannot conclude that L is nilpotent and finite dimensional. In fact, this is rarely the case. For example,

if V is the complex Lie algebra with basis f, h, e such that $[h, f] = -f$, $[h, e] = e$, $[f, e] = h$ and A is the algebra $\text{Hom } V$ of linear transformations of V , then the Lie subalgebra L of A having as basis the three nilpotent linear transformations

$$a_1 = \text{ad } f, \quad a_2 = \text{ad } e, \quad a_3 = \text{ad } \alpha(e) = \text{ad}(e + h + f/2),$$

α being the automorphism $\text{Exp}(\text{ad } f) = I + \text{ad } f + (\text{ad } f)^2/2$ of V , is not nilpotent. Rather, L is the three dimensional simple Lie algebra. Since such three dimensional Lie algebras abound in complex semisimple Lie algebras, it is hopeless to try to find a true Lie algebra counterpart of this theorem of Wedderburn.

Nevertheless, we do get the following result, with a much weaker hypothesis (we replace "basis of nilpotent elements" in Wedderburn's theorem by "generating set of nilpotent elements" here) and the conclusion that the ring generated by the radical (maximal solvable ideal) of L is nilpotent. This is a very useful result when one needs only this conclusion, since Lie algebras satisfying the hypothesis are very easy to come by. (See, for instance, Theorem 6.) The following result is the main theorem of the paper.

THEOREM 8. *Let L be a finite dimensional Lie subalgebra of an associative algebra A (possibly infinite dimensional) with identity element 1_A over a field F of characteristic 0. Suppose that L has a Lie algebra generating set a_1, \dots, a_k consisting of associatively nilpotent elements. Then*

1. *the algebra $\langle L \rangle_{\text{Algebra}}$ with identity 1_A generated by L in A is finite dimensional;*
2. *the ring $\langle R \rangle_{\text{Ring}}$ generated by the radical R of L must be nilpotent.*

We start with theorems on graded algebras and on the universal enveloping algebra of L in Sections 2 and 3. This will enable us to prove results on finite and infinite dimensional L -modules in Section 4. The first of these is the linear version of part of the main theorem. In Section 5, we prove the main theorem for finite dimensional Lie algebras. In Section 6, we extend the main theorem to infinite dimensional Lie algebras L in the case in which L is solvable.

2. A THEOREM ON FILTERED ALGEBRAS

Let R be a filtered algebra over a field F , so R has a filtration

$$R_0 = F \subseteq R_1 \subseteq \dots \subseteq R_n \subseteq \dots$$

of subspaces such that $R_r R_s \subseteq R_{r+s}$ for all $r, s \geq 0$. Let

$$\text{Gr}(R) = \sum_{r=1}^{\infty} \oplus R_r/R_{r-1}$$

be the associated graded algebra, and let δ be the mapping from R to $\text{Gr}(R)$ which sends $a \in R$ to $\delta(a) = a + R_{r-1} \in R_r/R_{r-1}$, where $a \in R_r - R_{r-1}$.

Suppose that $\text{Gr}(R)$ is commutative. Then for $x, y \in R$, we take r and s so that $\delta(x) \in R_r/R_{r-1}$, $\delta(y) \in R_s/R_{s-1}$ and get

$$\delta(xy) = \delta(x) \delta(y) = \delta(y) \delta(x) = \delta(yx).$$

But then the element $xy = yx$ of R_{r+s} is actually in R_{r+s-1} , so we can define $\{x, y\} = xy - yx + R_{r+s-2} \in R_{r+s-1}/R_{r+s-2}$. We can extend $\{x, y\}$ additively so that it is defined for all $x, y \in \text{Gr}(R)$. Then $\text{Gr}(R)$ with the product $\{x, y\}$ (the Poisson bracket) is a Lie algebra over F .

When $\text{Gr}(R)$ is of characteristic 0, we have the following theorem. In the case where $\text{Gr}(R)$ is Noetherian, this theorem is an important special case of a theorem of Gabber [1] on left ideals of R . It is based on the following lemma, which is well known. So that our paper stays reasonably self-contained, we include a short proof.

LEMMA. *The nil radical of a commutative algebra A of characteristic 0 is mapped into itself by every derivation of A .*

Proof. For $x^n = 0$, we have the following for any i :

$$\begin{aligned} nx^{n-1}D(x) &= 0, \\ n(n-1)x^{n-2}D(x)^2 &= 0 \\ &\vdots \\ n \cdots (n-i+1)x^{n-i}D(x)^{2i-1} &= 0. \end{aligned}$$

This is clear if $i = 1$. And, if it is true for i , then it is true for $i + 1$:

$$\begin{aligned} 0 &= n \cdots (n-i+1)[x^{n-i}(2i-1)D(x)^{2i-2}D^2(x) \\ &\quad + (n-i)x^{n-i-1}D(x)^{2i}] \\ 0 &= 0D(x) = n \cdots (n-i+1)[x^{n-i}(2i-1)D(x)^{2i-1}D^2(x) \\ &\quad + (n-i)x^{n-i-1}D(x)^{2i+1}] \\ 0 &= n \cdots (n-i)x^{n-(i+1)}D(x)^{2(i+1)-1}. \end{aligned}$$

So, it is true for all i . Taking $i = n$, $0 = n! D(x)^{2n-1}$, which shows that $D(x)$ is nilpotent.

THEOREM 1. *Let T be a two sided ideal in R , T^{Gr} the corresponding ideal of $Gr(R)$, $\sqrt{T^{Gr}}$ its radical. Then $\sqrt{T^{Gr}}$ is a Lie ideal of $Gr(R)$.*

Proof. In the ring $Gr(R)/T^{Gr}$, the radical is $\sqrt{T^{Gr}/T^{Gr}}$. Since the derivation $\text{ad } \delta(r) = \{\delta(r), \}$ maps T^{Gr} into T^{Gr} for any $r \in R$, it induces a derivation D of $Gr(R)/T^{Gr}$. By a theorem of A. Seidenberg [6], this derivation preserves the radical $\sqrt{T^{Gr}/T^{Gr}}$. But then $\text{ad } \delta(r)$ preserves $\sqrt{T^{Gr}}$, that is, $\{\delta(r), \sqrt{T^{Gr}}\} \subseteq \sqrt{T^{Gr}}$. Since this is true for all $r \in R$, we conclude that $\sqrt{T^{Gr}}$ is a Lie ideal of $Gr(R)$. ■

3. A THEOREM ON THE ENVELOPING ASSOCIATIVE ALGEBRA OF L

Let L be a finite dimensional Lie algebra over a field F of characteristic 0 and let U_L be the universal enveloping associative algebra of L . Let ρ be the canonical mapping from L into U_L . If $a \in L$, and T is a left ideal of U_L , we say that $\rho(a)$ is algebraic over T if $f(\rho(a)) \in T$ for some monic polynomial $f(x)$ over F .

We now have the following special case of a theorem of Kostant which is contained in the paper of Guillemin, Quillen and Sternberg [3, Theorem 8.1, p. 71] on left ideals of U_L . Again, to keep our paper self-contained, we now give a proof for this special case. It is possible that this proof also may be known to others.

THEOREM 2. *Let T be a two sided ideal of U_L . Suppose that L has a generating set $\{a_1, \dots, a_k\}$ whose corresponding elements $\rho(a_s)$ in U_L are algebraic over T . Then T is of finite codimension in U_L .*

Proof. From the natural filtration

$$\begin{aligned} U_L^0 &= F \\ U_L^1 &= U_L^0 + \rho(L) \\ &\vdots \\ U_L^r &= U_L^1 \cdots U_L^1 \quad (r \text{ times}) \end{aligned}$$

we get the associated graded algebra

$$Gr(U_L) = \sum_{r=1}^{\infty} \oplus U_L^r / U_L^{r-1}.$$

To show that U_L/T is finite dimensional, it suffices to show that $\text{Gr}(U_L)/T^{\text{Gr}}$ is finite dimensional. Since U_L is the subring with 1 generated by the finite dimensional Lie algebra $\rho(L)$, and since the elements $\rho(x)\rho(y) - \rho(y)\rho(x)$ are of degree 1 for all $x, y \in L$, $\text{Gr}(U_L)/T^{\text{Gr}}$ is a commutative Noetherian algebra. Since each $\rho(a_s)$ is algebraic over T , the corresponding $\delta(\rho(a_s))$ in $\text{Gr}(U_L)$ are in fact in $\sqrt{T^{\text{Gr}}}$. By Theorem 1, $\{\sqrt{T^{\text{Gr}}}, \sqrt{T^{\text{Gr}}}\} \subseteq \sqrt{T^{\text{Gr}}}$. Since the a_s generate the Lie algebra L , this implies that $\delta(\rho(L)) \subseteq \sqrt{T^{\text{Gr}}}$. But then $\sqrt{T^{\text{Gr}}}$ is the augmentation ideal of $\text{Gr}(U_L)$. So, since $\text{Gr}(U_L)$ is Noetherian, $\sqrt{T^{\text{Gr}}}/T^{\text{Gr}}$ is finite dimensional. Since $\sqrt{T^{\text{Gr}}}$ is the augmentation ideal of $\text{Gr}(U_L)$, this implies that $\text{Gr}(U_L)/T^{\text{Gr}}$ is finite dimensional. So, T is of finite codimension in U_L . ■

COROLLARY 3. *Let V be an L -module and regard V as a U_L -module in the usual way. Suppose that L has a generating set $\{a_1, \dots, a_k\}$ whose corresponding elements $\rho(a_s)$ in U_L are nilpotent over the annihilator $T = \{t \in U_L \mid tV = 0\}$. Then T is of finite codimension in U_L .*

Proof. Since T is an ideal, this follows directly from the above theorem. ■

4. FINITE AND INFINITE DIMENSIONAL L -MODULES

Let L be a finite dimensional Lie algebra of characteristic 0, let W be a module for L (possibly infinite dimensional), and let ρ be the associated representation. If every vector in W is contained in a finite dimensional submodule, then we say that W is *locally finite dimensional*. If every vector of W is annihilated by some power of a linear transformation S , we say that S is *locally nilpotent*. Given this, we now have

THEOREM 4. *Suppose that L has elements e_1, \dots, e_n which generate L as Lie algebra and such that $\rho(e_1), \dots, \rho(e_n)$ are locally nilpotent. Then W is locally finite dimensional and the transformations $\rho(x)$ with x in the radical of L are locally nilpotent. Moreover, if $\rho(e_1), \dots, \rho(e_n)$ are nilpotent, then $\rho(x)$ is nilpotent for all x in the radical of L .*

Proof. Since the hypothesis is preserved under ascent from the ground field F to its algebraic closure F' , and since the conclusion is preserved under descent from F' to F , we may assume with no loss of generality that the ground field F is algebraically closed.

We may extend ρ to a representation of the enveloping associative algebra U_L of L . If $W = \{0\}$, there is nothing to prove, so we assume that W has nonzero elements. Take a nonzero element $v \in W$ and let V be the

submodule $V = U_L v$. Since v was chosen arbitrarily, it suffices to prove that V is finite dimensional and that the transformations $\rho(x)$ with x in the radical of L satisfy $\rho(x)^d = 0$ for some d independent of v . The first assertion follows from Corollary 3, since it says that the ideal $T_v = \{t \in U_L \mid tV = 0\}$ is of finite codimension in U_L , which implies that V is of finite dimension. In fact, Corollary 3 assures that the ideal $T = \{t \in U_L \mid tW = 0\}$ is of some finite codimension d , so that the dimensions of the submodules $U_L v$ are all at most d .

For the second assertion, we proceed as follows. Since any L -module $V = U_L v$ is of dimension at most d , it has a composition series

$$V = V_1 \supset V_2 \supset \cdots \supset V_n = \{0\}$$

We consider the irreducible quotient modules

$$W_i = V_i / V_{i+1}$$

and associated representations ρ_i for $1 \leq i \leq n-1$. By Lie's theorem, there is a linear functional λ_i on the radical R of L such that the λ_i weight space of W_i as R -module is nonzero. By the *proof* of Lie's theorem (Humphreys [5]), this weight space is L -stable, so it is all of W_i by L -irreducibility. Since $\rho_i(x)$ is in the Lie algebra generated by the nilpotent linear transformations $\rho(e_1), \dots, \rho(e_n)$, and since the latter are contained in the Lie algebra of trace 0 linear transformations, so is $\rho_i(x)$. But then $\rho_i(x)$ has trace 0. Since this trace is a multiple of $\lambda_i(x)$, $\lambda_i(x)$ is 0. Since this is true for all i , x is nilpotent and $\rho(x)^d = 0$. This is true for all $x \in R$, as we set out to prove. ■

When stated for linear Lie algebras over algebraically closed fields of characteristic 0, the converse of Theorem 4 is also true, as we show in

COROLLARY 5. *Let L be a finite dimensional linear Lie algebra over an algebraically closed field F of characteristic 0. Then L has locally nilpotent elements e_1, \dots, e_n which generate L as Lie algebra if and only if the underlying vector space is locally finite dimensional and the radical of L consists of locally nilpotent linear transformations.*

Proof. One direction is precisely Theorem 4 for the given context. For the other, suppose that the radical N of L consists of nilpotent elements. The quotient L/N is a semisimple Lie algebra, and we let T be a Cartan subalgebra of a Levi factor S of L , $S = \sum S_a$ be the root space decomposition of S with respect to T . Since V is locally finite dimensional, we can decompose it as $V = \sum V_b$, the weight space decomposition of V with respect to T . We then see that $\text{ad } S_a$ maps V_b to V_{b+a} for all a, b . It follows that for $a \neq 0$, $\text{ad } L_a$ consists of locally nilpotent linear transformations of

V . Since S is semisimple with split Cartan decomposition $\sum S_a$, L is generated by

$$N \cup \{ \text{ad } S_a \mid a \neq 0 \}$$

and therefore by a finite subset thereof. Since this set consists of nilpotent elements, we have proved the converse. ■

Theorem 4 and Corollary 5 cannot be generalized to fields of characteristic $p > 0$. The proof breaks down over such fields because

1. Jacobson's theorem is stated only in characteristic 0 (and, in characteristic p , holds only for sufficiently large values of p);
2. the trace argument in the proof of Theorem 1 works only if p is greater than the dimensions of the composition factors of V .

We now give an example to show that these difficulties cannot be circumvented.

EXAMPLE. Let F be a field of characteristic $p > 0$ and let L be a Lie algebra of the form $L = Ff + Fh + Fe$, where $[f, e] = h$ and h is central. Define a representation $\rho: L \rightarrow \text{Hom}_F(V)$, where V has basis v_0, \dots, v_{p-1} and

$$\begin{aligned} \rho(f) v_i &= v_{i-1} & \text{for } 1 \leq i \leq p-1 \\ \rho(f) v_0 &= v_{p-1} \\ \rho(e) v_{i-1} &= iv_1 & \text{for } 1 \leq i \leq p-1 \\ \rho(e) v_{p-1} &= 0. \end{aligned}$$

Then

$$\begin{aligned} \rho(h) v_i &= \rho(f) \rho(e) v_i - \rho(e) \rho(f) v_i \\ &= (i+1) v_i - iv_i \\ &= v_i \end{aligned}$$

for $1 \leq i \leq p-1$, and $\rho(h)$ is the identity. It follows that $\rho(e)$ and $\rho(f-h)$ are nilpotent, since $\rho(f)^p$ is also the identity and f and h commute:

$$\begin{aligned} \rho(e)^p &= 0 \\ \rho(f-h)^p &= (\rho(f) - \rho(h))^p = \rho(f)^p - \rho(h)^p = I - I = 0. \end{aligned}$$

Since $[f-h, e] = h$, e and $f-h$ generate L . If Theorem 1 were true in this context, then it would say that $\rho(x)$ is nilpotent for all x in the radical of L , which in this case is L itself. But this is not true if $x = h$.

Following the language of Jacobson [6], a *split Cartan subalgebra* of a Lie algebra L is a Cartan subalgebra L_0 of L such that L is the direct sum of the spaces $L_a = \{x \in L \mid (\text{ad } h - a(h)I)^i x = 0 \text{ for all } h \text{ in } L_0 \text{ for some integer } i \geq 0\}$. Here, a is in the dual space of L_0 and a is called a *root* and L_a a *root space* if L_a is nonzero. Theorem 4 can be applied easily by way of

THEOREM 6. *Let L_0 be a split Cartan subalgebra of a finite dimensional Lie algebra L of characteristic 0, $L = \sum L_a$ the corresponding Cartan decomposition of L . Let ρ be any finite dimensional representation of L . Then for any subalgebra M of L generated by any number of elements taken from the various L_a with a nonzero, $\rho(x)$ is nilpotent for x in the radical of M .*

Proof. This follows directly from Theorem 1 as soon as we know that $\rho(x)$ is nilpotent for any $x \in L_a$. To see this, we may first extend the ground field to the algebraic closure. This enables us to take the weight space decomposition

$$V = \sum V_b, V_b = \{v \in V \mid (\rho(h) - b(h))^i v = 0 \text{ for all } h \in L_0 \\ \text{for some integer } i \geq 0\}$$

of the module V afforded by ρ with respect to L_0 . When V_b is nonzero, b is a *weight*. Since $\rho(x)^m V_b \subseteq V_{b+ma}$ and since $b+ma$ is not a weight for any b for m sufficiently large, $\rho(x)^m = 0$ for some m . Thus, $\rho(x)$ is nilpotent. ■

As an instance of this, we now prove the following theorem, for which we know no other proof.

THEOREM 7. *Let L be a finite dimensional Lie algebra of characteristic 0, ρ a finite dimensional representation of L . Then $\rho(x)$ is nilpotent for all x in the radical of $L^\infty = \bigcap_{i=1}^\infty L^i$.*

Proof. We may assume with no loss of generality that L is a Lie algebra of $n \times n$ matrices over F and ρ is the identity. Letting K be the algebraic closure of F , we adopt the notation that KS denote the span of S over K . We then have

$$K(L^\infty) = (KL)^\infty$$

since $L^\infty = L^n$ for any integer for which $L^n = L^{n+1}$. Since any x in the radical of L^∞ is in the radical of $(KL)^\infty = K(L^\infty)$, we may assume with no loss of generality that the original ground field F is algebraically closed.

Since the ground field F is algebraically closed, we can take a Cartan

decomposition $L = \sum L_a$ for L over F . Let M be the subalgebra of L generated by the L_a for which a is nonzero. Then M is an ideal, since it is normalized by L_0 ; and the quotient algebra L/M is nilpotent, since L_0 is nilpotent. It follows that $L^\infty \subseteq M$. On the other hand, we have $M \subseteq L^\infty$ since $[L, M] = M$. It follows that $L^\infty = M$. But then the elements of the radical of $L^\infty = M$ are nilpotent by Theorem 6. ■

5. A VARIATION OF WEDDERBURN'S THEOREM FOR L

Theorem 4 on representations of Lie algebras has the following consequence for Lie subalgebras L of an associative algebra A . We get it by applying Theorem 4 to the left regular representation of A restricted to L .

THEOREM 8. *Let L be a finite dimensional Lie subalgebra of an associative algebra A (possibly infinite dimensional) with identity element 1_A over a field F of characteristic 0. Suppose that L has a Lie algebra generating set a_1, \dots, a_k consisting of associatively nilpotent elements. Then*

1. *the algebra $\langle L \rangle_{\text{Algebra}}$ with identity 1_A generated by L in A is finite dimensional;*
2. *the ring $\langle R \rangle_{\text{Ring}}$ generated by the radical R of L must be nilpotent.*

Proof. Let V be A regarded as the left regular module for A and let ρ be the associated representation. By way of ρ , V is then also a Lie algebra module for L . Since the a 's are associatively nilpotent elements of A , the $\rho(a)$'s are nilpotent linear transformations. So, by Theorem 4, the L -module V is locally finite dimensional and the elements of the radical R act on V as nilpotent linear transformations.

Since $\langle L \rangle_{\text{Algebra}}$ is the L -submodule of V generated by the finite dimensional subspace $L + Fl_A$ of V , it is finite dimensional by the local finite dimensionality of the L -module V . Letting λ be the left regular representation of $\langle L \rangle_{\text{Algebra}}$, we may regard λ as a Lie algebra representation of L . Of course, these representations of $\langle L \rangle_{\text{Algebra}}$ and L are both faithful. Since the elements of $\lambda(R)$ are nilpotent, they all have upper triangular matrices with respect to a suitable basis. But then all elements of the ring $\langle \lambda(R) \rangle_{\text{Ring}} = \lambda(\langle R \rangle_{\text{Ring}})$ generated by them are also upper triangular. Since λ is faithful, it follows that the ring $\langle R \rangle_{\text{Ring}}$ is nilpotent. ■

A straightforward application of Theorem 8 is

COROLLARY 9. *Let A be a finite dimensional associative algebra over a field F of characteristic 0. Suppose that A is generated by nilpotent elements a_1, \dots, a_k . Let L be the Lie subalgebra of A generated by the a 's. Then*

1. the radical R of L is the intersection of L and the radical of A ;
2. if S is a Levi factor of L , so $L = S \oplus R$, then $A = \langle S \rangle_{\text{Ring}} \oplus A \langle R \rangle_{\text{Ring}}$, where $\langle S \rangle_{\text{Ring}}$ is a semisimple subalgebra of A and $A \langle R \rangle_{\text{Ring}}$ is the radical of A .

Proof. Let J be the radical of A . Then A/J has a faithful finite dimensional completely reducible representation, so that $(L+J)/J = L/J \cap L$ does as well. Since the radical of $(L+J)/J$ consists of nilpotent elements of A/J by Theorem 8, it follows that $L/J \cap L$ is semisimple and $J \cap L$ is the radical R of L . So, J contains $A \langle R \rangle_{\text{Ring}}$. Since $[L, A \langle R \rangle_{\text{Ring}}] \subseteq A \langle R \rangle_{\text{Ring}}$ and $A = \langle L \rangle_{\text{Ring}}$, $A \langle R \rangle_{\text{Ring}}$ is an ideal of A , which must be contained in J . Since $L = S + R$, it follows that $\langle S \rangle_{\text{Ring}} + A \langle R \rangle_{\text{Ring}}$ is a subring of A containing L . Since L generates A , this implies that $A = \langle S \rangle_{\text{Ring}} + A \langle R \rangle_{\text{Ring}}$. Since S is semisimple, $\langle S \rangle_{\text{Ring}}$ is semisimple since any faithful finite dimensional $\langle S \rangle_{\text{Ring}}$ -module is S -completely reducible, hence $\langle S \rangle_{\text{Ring}}$ -completely reducible. Since $\langle S \rangle_{\text{Ring}}$ is semisimple and $A \langle R \rangle_{\text{Ring}}$ is a nilpotent ideal, we now have a direct sum $A = \langle S \rangle_{\text{Ring}} \oplus A \langle R \rangle_{\text{Ring}}$. Since $A \langle R \rangle_{\text{Ring}} \subseteq J$ and $A = \langle S \rangle_{\text{Ring}} + J$, it follows that $A \langle R \rangle_{\text{Ring}}$ is the radical J of A . ■

6. INFINITE DIMENSIONAL LIE ALGEBRAS

For solvable Lie algebras, we do not need to assume that the Lie algebra is finite dimensional, as we now see in Theorem 12 below. First, however, we need to prepare the way for its proof.

DEFINITION. A Lie algebra L is *locally finite dimensional* if any finite subset of L is contained in a finite dimensional subalgebra of L . And a representation ρ of a Lie algebra L on a vector space V is *locally locally finite dimensional* if for any finite subset S of L and any finite subset T of V , V has a finite dimensional subspace W containing T which is mapped into itself by $\rho(x)$ for all $x \in S$.

By this definition, ρ is locally locally finite dimensional if and only if for every finite subset S of L the associated module for the Lie algebra generated by S is locally finite dimensional as defined in Section 4. Of course, locally finite dimensional representations are locally locally finite dimensional.

THEOREM 10. Let L be a solvable Lie algebra over a field of characteristic 0, ρ a representation of L on a vector space V such that $\rho(x)$ is nilpotent for all $x \in L$. Then the Lie algebra L is locally finite dimensional and the representation ρ is locally locally finite dimensional.

Proof. Since L is solvable, $L^{(n+1)} = \{0\}$ while $A = L^{(n)}$ is nonzero for some integer n . Then A is an Abelian ideal. We prove the theorem by induction on n .

If $n=0$, then $A=L$ and L is Abelian. In this case, for any finite subset S of L and any finite subset T of V , we have:

1. The span M of S is a finite dimensional subalgebra of L ;
2. If one lets W_v be the span of

$$\{x_{i_1} \cdots x_{i_r} v \mid 0 \leq r \leq s, x_{i_1}, \dots, x_{i_r} \in S\},$$

where s is sufficiently large, and lets

$$W = \sum_{v \in T} W_v,$$

W is a finite dimensional subspace of V which is mapped into itself by $\rho(x)$ for all $x \in S$. In this argument, it is the nilpotency of the finitely many commuting elements of S that assures the existence of a sufficiently large s . This proves the theorem for $n=0$.

Next, suppose that $n \geq 1$ and that the theorem has been proved for $n-1$. Then A is Abelian and nonzero. We may, with no loss of generality, assume that the representation ρ is the identity, by replacing L by $\rho(L)$. Then the condition of nilpotence of $\rho(x)=x$ implies the condition of nilpotence of $\text{ad } L$. We may regard L/A as a Lie algebra of nilpotent linear transformations on A . If we invoke the induction hypothesis, L/A is locally finite dimensional and the L/A -module A is locally finite dimensional. It follows that L is locally finite dimensional as a Lie algebra:

Any finite subset S of L leads to a subalgebra of L/A with basis $b_1 + A, \dots, b_m + A$ such that S is contained in the span of the b 's. If one then takes a finite dimensional subspace C of A stabilized by the b 's which has the property that $[b_r, b_s]$ is in the span of C and the b 's for all r, s , then S is contained in the finite dimensional algebra spanned by the b 's and C .

By the local finite dimensionality of L , we may assume, without loss of generality, that L is finite dimensional. Each element of A is nilpotent on V and the elements of A commute. It follows from this that there is a finite series

$$V = V_0 \supset AV = V_1 \supset \cdots \supset AV_n = V_{n+1} = \{0\}$$

with last nonzero term $W = V_n$. Since $AW=0$, L/A acts on W . Similarly,

L/A acts on all of the other quotients V_m/V_{m+1} . By induction, it follows that the modules V_m/V_{m+1} are locally finite dimensional for L/A , hence for the action of L that is factored through L/A . That V is a locally finite dimensional L -module now follows in the same way that we showed that L is locally finite dimensional since L/A is a locally finite dimensional algebra acting on the locally finite dimensional module A . ■

Taking the representation ρ in Theorem 10 to be the adjoint representation, we get

COROLLARY 11. *If L is a solvable Lie algebra such that $\text{ad } x$ is nilpotent for all $x \in L$, then L is locally finite dimensional.*

Now we can prove the infinite dimensional version of Theorem 4 for solvable Lie algebras.

THEOREM 12. *Let L be a solvable Lie algebra, V an L -module, ρ the associated representation. Suppose that for every $x \in L$, $\rho(L)$ has nilpotent elements $\rho(e_1), \dots, \rho(e_n)$ such that the Lie subalgebra generated by $\rho(e_1), \dots, \rho(e_n)$ contains $\rho(x)$. Then L and ρ are both locally finite dimensional and $\rho(x)$ is nilpotent for all $x \in L$.*

Proof. Suppose that $\rho(x)$ is not nilpotent for all $x \in L$. Then we can take a counterexample L and representation ρ of L on V with $L^{(n+1)} = \{0\}$, $L^{(n)} = A$ nonzero and n a minimal nonnegative integer for which $\rho(w)$ is not nilpotent for some $w \in L$. Having done this, we may replace L by $\rho(L)$, w by $\rho(w)$, and ρ by the identity.

Note that if $x \in L$ is nilpotent, then $\text{ad } x$ is also, since it is just the difference $x_L - x_R$ of the two commuting nilpotent transformations x_L (left translation) and x_R (right translation). It follows that the Lie algebra L/A with the representation ad satisfies the hypothesis of Theorem 12. By our minimality hypothesis, then, it follows that the Lie algebra L/A has the property that $\text{ad}(x + A)$ is nilpotent as a linear transformation of L/A for all $x + A \in L/A$. By the same reasoning, applied a second time to a second representation, it follows that L/A has the property that $\text{ad}(x + A)$ is nilpotent as linear transformation of A defined by the equations.

$$\text{ad}(x + A)(b) = [x, b] \quad (b \in A)$$

for all $x + A \in L/A$. From this, it follows that $\text{ad } x$ is nilpotent on L for all $x \in L$. So, by Corollary 11, we may conclude that L is locally finite dimensional.

Since the problematic element w which is not nilpotent is in L , our hypothesis assures us that $\rho(L) = L$ has nilpotent elements e_1, \dots, e_n such

that the Lie subalgebra generated by e_1, \dots, e_n contains w . By the local finite dimensionality of L which we have just established, we can assume without loss of generality that L is finite dimensional and is generated by e_1, \dots, e_n . But then we can apply Theorem 4 to get that all elements of L are nilpotent. Since $w \in L$, this contradicts our hypothesis to the contrary that w is not nilpotent. Therefore, we conclude that $\rho(x)$ is nilpotent for all x in the Lie algebra L taken by us at the beginning of this proof.

Finally, having shown that $\rho(x)$ is nilpotent for all $x \in L$, we can now invoke Theorem 10 to conclude that ρ is locally finite dimensional. ■

The above theorem on representations of solvable Lie algebras L has the following consequence for Lie subalgebras (possibly infinite dimensional) of an associative algebra A . Following in the footsteps of what we did in part of Section 5, we get it by applying the corresponding theorem on representations to the left regular representation of A restricted to L . Note that the conclusion (1) implies conclusion (2), using Wedderburn's theorem, although we prove (2) instead by invoking Theorem 8.

THEOREM 13. *Let L be a solvable Lie subalgebra of an associative algebra A (possibly infinite dimensional) over a field F of characteristic 0. Suppose that L has a Lie algebra generating set a_1, \dots, a_k consisting of associatively nilpotent elements. Then*

1. *the algebra $\langle L \rangle_{\text{Algebra}}$ is finite dimensional;*
2. *the ring $\langle L \rangle_{\text{Ring}}$ consists of nilpotent elements.*

Proof. With no loss in generality, we may assume that A has an identity, for otherwise we could adjoin one.

Let V be A regarded as the left regular module for A and let ρ be the associated representation. By way of ρ , V is then also a Lie algebra module for L . Since the a 's are associatively nilpotent elements of A , the $\rho(a)$'s are nilpotent linear transformations. So, by Theorem 10, L and the Lie algebra representation ρ of L are locally finite dimensional and the elements $\rho(L)$ are nilpotent transformations.

By the local finite dimensionality of ρ , the elements $a_1, \dots, a_k 1_A$ are contained in a finite dimensional L -submodule W of V . There is a unique smallest such W , namely the algebra $\langle L \rangle_{\text{Algebra}}$ generated by L . So, the finite dimensionality of $W = \langle L \rangle_{\text{Algebra}}$ proves (1).

Since $\langle L \rangle_{\text{Algebra}}$ is finite dimensional, so is L and we may now invoke Theorem 8 to prove (2), that the elements of L are nilpotent. ■

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