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$\mathscr{H}_2$ optimal control for sampled-data systems

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Abstract: This paper considers an $\mathscr{H}_2$ optimal control problem for sampled-data systems. After defining a new $\mathscr{H}_2$ norm for sampled-data systems, we give a state space solution to the optimal controller synthesis problem. We show that the $\mathscr{H}_2$ optimal control problem for a sampled-data system is equivalent to a standard $\mathscr{H}_2$ optimal control problem for a related discrete-time system.

Keywords: Sampled-data systems; digital control; $\mathscr{H}_2$ optimal control; periodic systems; discrete-time systems.

1. Introduction

For analytical design of control systems, it is often convenient to measure system performance in terms of norm of the closed loop system from the exogenous signals to the regulated variables. After choosing a suitable norm and a synthesis model for the plant, the control system synthesis proceeds by finding a controller that optimizes the norm of the closed loop system. The synthesis model includes the plant as well as weighting functions that reflect the design objectives. This approach has turned out to be quite successful as shown by the development of $\mathscr{H}_2$ or LQG (Linear Quadratic Gaussian), $\mathscr{H}_\infty$, ℓ_1 control theories.

While the $\mathscr{H}_2$, $\mathscr{H}_\infty$, and ℓ_1 control theories are well understood for finite dimensional linear time-invariant (FDLTI) systems, there has been relatively little work on the corresponding theories for sampled-data systems. By a sampled-data system, we do not mean a discrete-time system. Rather, a sampled-data control system means a continuous-time plant connected to a discrete-time controller using D/A and A/D devices. (In this paper, we will assume that the D/A and A/D devices are ideal zero order hold and ideal sampler respectively. We will not take into account the fact that these devices also involve quantization in magnitude.) Thus, in the study of sampled-data systems, it is important to analyze the behavior of the closed loop system with continuous-time inputs and outputs. This in particular implies that the inter-sample behavior must be taken into account.

There have been some studies of the linear-quadratic regulator problem for sampled-data systems taking inter-sample behavior into account, see for example [15,6,13]. Recently, Chen and Francis [5] have formulated and solved an $\mathscr{H}_2$ optimal control problem for sampled-data systems, while the $\mathscr{H}_\infty$ optimal control problem has been investigated by Kabamba and Hara [9,10], Chen and Francis [4], Toivonen [17], and Bamieh and Pearson [2]. Recently, we have given an explicit formula for the $\mathscr{L}_\infty$-induced norm for sampled-data systems [16].

One common measure of performance for a linear time-invariant system is the $\mathscr{H}_2$-norm of its transfer function. This is the norm that is optimized in the LQG controller design. For linear time-invariant systems, there are many (equivalent) ways of defining the $\mathscr{H}_2$ norm. One deterministic definition is to

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take the input to be a Dirac delta function at $t = 0$ and define the $\mathcal{H}_2$ norm to be the square root of the integral square of the output. (This is for a single input system. For a multi-input system, one applies delta functions at each input channel and then take the square root of the sum of integral squares of the resulting outputs.) Chen and Francis [5] generalize precisely this concept of the $\mathcal{H}_2$ norm to sampled-data systems and solve the corresponding optimal control problem.

However, the interconnection of an FDLTI continuous-time plant and a finite dimensional linear shift-invariant (FDLSI) discrete-time controller via sample and hold devices leads to a closed loop system which is *periodically time-varying*. In view of this fact, it seems unnatural to apply the impulsive input only at $t = 0$. Rather, it is more natural to examine the effect of an impulsive input at any arbitrary time.

Motivated by the recent work of Chen and Francis [5], in this paper, we define a new $\mathcal{H}_2$ norm for sampled-data systems. We begin by considering a general exponentially stable linear periodic input-output system, and define an $\mathcal{H}_2$ norm for it. This notion seems to be a natural generalization of the $\mathcal{H}_2$ norm for linear time-invariant systems. Although in this paper we will take a purely deterministic approach, it is interesting to note that this definition is also equivalent to the stochastic definition of the $\mathcal{H}_2$ norm. This notion of the $\mathcal{H}_2$ norm for periodic systems leads to a suitable definition of the $\mathcal{H}_2$ norm for sampled-data systems. Our notion of the $\mathcal{H}_2$ norm is also closely related to the work of Juan and Kabamba [7] who have investigated the use of Generalized Sampled-Data Hold Function (GSHF) control to optimize quadratic performance measures for sampled-data systems.

We then consider the synthesis problem for $\mathcal{H}_2$ optimal control of sampled-data systems. More specifically, we give a complete state space solution to the problem of finding a stabilizing FDLSI discrete-time controller for an FDLTI continuous-time plant such that the $\mathcal{H}_2$-norm of the closed loop system is minimized. In the problem formulation, we also allow for (discrete-time) measurement noise. We show that this problem is equivalent to a standard FDLSI discrete-time $\mathcal{H}_2$ synthesis problem. Under standard assumptions on the given continuous-time plant, the resulting FDLSI discrete-time $\mathcal{H}_2$ synthesis problem turns out to be an LQG problem for which an optimal controller exists. Thus, if we find the $\mathcal{H}_2$ optimal FDLSI discrete-time controller for the equivalent FDLSI discrete-time plant, then we have a solution to the main synthesis problem. The problem of finding $\mathcal{H}_2$ (or LQG) optimal controllers for FDLSI discrete-time systems is by now a classical problem [14,1].

In the next section, we define the $\mathcal{H}_2$-norm for sampled-data systems and also pose the main synthesis problem. In Section 3, we solve the optimal control problem in state-space form. We conclude the paper with some remarks regarding current and future research directions.

We end this section with some remarks on notation. Let $\mathcal{C}^n$ denote the space of continuous functions from the time set $[0, \infty)$ to $\mathbb{R}^n$, and let $\mathcal{PC}^n$ denote the space of piecewise-continuous functions from the time set $[0, \infty)$ to $\mathbb{R}^n$ that are bounded on compact sets of $[0, \infty)$ and are continuous from the left at every point except the origin. As usual, $\mathcal{L}_2^n[0, \infty)$ denotes the Lebesgue space of measurable functions $f(t)$ from $[0, \infty)$ to $\mathbb{R}^n$ which satisfy

$$\| f \|_{\mathcal{L}_2} := \left(\int_0^\infty \| f(t) \|^2 \, \mathrm{d}t \right)^{1/2} < \infty$$

where $\| \cdot \|$ is an appropriate spatial norm on $\mathbb{R}^n$. Similarly, in discrete-time $\mathcal{S}^n$ denotes the space of $\mathbb{R}^n$-valued sequences defined on the time set $\{0, 1, 2, \dots\}$, l_2^n denotes the set of all sequences ξ in $\mathcal{S}^n$ which satisfy

$$\| \xi \|_{l_2} := \left(\sum_{k=0}^{\infty} \| \xi(k) \|^2 \right)^{1/2} < \infty .$$

Let $\mathcal{B}$ denote a banach space. Now define

$$l_2^{\mathcal{B}} := \{ x_i : x_i \in \mathcal{B}; \| x_i \|_{l_2} < \infty \} .$$

where the spatial norm is the appropriate norm on $\mathcal{B}$. If $\mathcal{B} = \mathbb{R}^n$ then we write l_2^n instead of $l_2^{\mathbb{R}^n}$. We will be mainly interested in $l_2^{\mathscr{L}_2^n[0,T]}$ in this paper. We will drop the superscript n in the subsequent sections as the dimension of the signal space will be clear from the context. As usual the transpose of a matrix A is denoted as A' and the adjoint of an operator E is denoted as E^*.

2. Problem formulation

Since sampled-data systems are time-varying periodic systems, we begin by defining the $\mathscr{H}_2$ norm for periodic systems. Later, we will derive an explicit expression for the $\mathscr{H}_2$ norm of sampled-data systems.

2.1. $\mathscr{H}_2$ norm for periodic systems

Consider an exponentially stable linear continuous-time periodic system $\mathscr{T}$ whose input–output equation is given by

$$z(t) = \int_0^t h(t, \tau) w(\tau) \, \mathrm{d}\tau.$$

Here, $h(t, \tau)$ is the weighting function or the impulse response of the system. Let T be the period of the system. The $\mathscr{H}_2$ norm of this linear time-varying periodic system is defined as

$$\| \mathscr{T} \|_2 := \left[\frac{1}{T} \int_0^T \left\{ \int_\tau^\infty \operatorname{trace}(h(t, \tau) h'(t, \tau)) \, \mathrm{d}t \right\} \mathrm{d}\tau \right]^{1/2}. \tag{1}$$

The $\mathscr{H}_2$ norm defined above has the following interpretation as an average value of the $\mathscr{L}_2$ norm of the output. For simplicity, consider a single input system. Suppose we apply a Dirac delta function at the input at time $t \in [0, T]$. Then the $\mathscr{H}_2$ norm defined above is the square root of the average of the integral square of $\| z(t) \|$. Also, if the input is a zero mean Gaussian stochastic process of unit covariance, then the above norm is the average variance of z in steady state.

Notice that if $\mathscr{T}$ is an LTI (linear time-invariant) system, then the above definition gives us the standard formula for the $\mathscr{H}_2$ norm of an LTI system, namely

$$\| \mathscr{T} \|_2 = \left\{ \int_0^\infty \operatorname{trace}(h(t) h'(t)) \, \mathrm{d}t \right\}^{1/2}.$$

For sampled-data systems, we need a somewhat more general framework. In particular, we will consider a periodic system with a continuous-time input w and a discrete-time input v. For simplicity, and this will be sufficient for our purposes, we will assume that the discrete-time input is also of period T. (The discrete-time input v will represent an exogenous signal entering the controller.) Now consider the input-output equation for such a linear periodic system:

$$z(t) = \int_0^t h_w(t, \tau) w(\tau) \, \mathrm{d}\tau + \sum_{i=0}^{k} h_v(t - iT) v(i).$$

Here, $h_w(t, \tau)$ is the weighting function for the continuous-time input and $h_v(t)$ is the pulse response function for the discrete-time input v. Assuming that the system is exponentially stable, the $\mathscr{H}_2$ norm of this sampled-data periodic system is defined as

$$\| \mathscr{T} \|_2 := \left[\frac{1}{T} \int_0^T \left\{ \int_\tau^\infty \operatorname{trace}(h_w(t, \tau) h_w'(t, \tau)) \, \mathrm{d}t \right\} \mathrm{d}\tau + \int_0^\infty \operatorname{trace}(h_v(t) h_v'(t)) \, \mathrm{d}t \right]^{1/2}. \tag{2}$$

2.2. Sampled-data systems

Consider the sampled-data feedback system in Figure 1. Here G is a FDLTI causal continuous-time plant, K is a FDLSI causal discrete-time controller, $w(t) \in \mathbb{R}^{m_1}$ is the exogenous input, $u(t) \in \mathbb{R}^{m_2}$ is the control input, $z(t) \in \mathbb{R}^{p_1}$ is the controlled output, $y(t) \in \mathbb{R}^{p_2}$ is the measured output, and $v(k) \in \mathbb{R}^{m_3}$ is the discrete-time measurement noise. The block labeled as S_T represents the sampling operator with time period T defined as follows:

$$S_T : \mathscr{C}^{p_2} \to \mathscr{S}^{p_2} : y \mapsto S_T y : (S_T y)(k) = y(kT).$$

The system block denoted by H_T represents the (zero-order) hold operator with time period T:

$$H_T : \mathscr{S}^{m_2} \to \mathscr{P}\mathscr{C}^{m_2} : \psi \mapsto H_T \psi : (H_T \psi)(t) = \psi(k), \; kT < t \le (k+1)T.$$

Consider the transfer function representation of G:

$$z = G_{11} w + G_{12} u, \qquad y = G_{21} w + G_{22} u.$$

In Figure 1, since the measured output y is sampled, it must be continuous. To ensure this, it is sufficient to assume that G_{21} is strictly proper. We will also assume throughout this paper that G_{22} is strictly proper to ensure the well-posedness of the feedback system. Also, for the $\mathscr{H}_2$ norm of the closed loop system to be bounded, it is necessary that G_{11} be strictly proper.

Now consider a state space representation of the systems given in Figure 1:

$$G: \quad \dot{x} = Ax + B_1 w + B_2 u, \qquad z = C_1 x + D_{12} u, \qquad y = C_2 x, \tag{3}$$

$$K: \quad \xi(k+1) = \Phi \xi(k) + \Gamma \eta(k) \qquad \psi(k) = \Theta \xi(k) + \Upsilon \eta(k). \tag{4}$$

Let the state-dimension of G in (3) be n and that of K in (4) be $\hat{n}$. The input to the controller, $\eta(k)$, is the sampled output measurement corrupted by discrete by *discrete-time noise*, i.e.,

$$\eta(k) = C_2 x(kT) + D_2 v(k).$$

And the control input is generated by a zero-order hold:

$$u(t) = \psi(k), \quad kT < t \le (k+1)T. \tag{5}$$

The feedback interconnection $(G, H_T K S_T)$ is called internally asymptotically stable if the associated autonomous discrete-time system with the state

$$\begin{pmatrix} x(k) \\ \xi(k) \end{pmatrix} := \begin{pmatrix} x(kT) \\ \xi(k) \end{pmatrix}$$

is asymptotically stable. If the sampled-data system is internally asymptotically stable then it is also exponentially stable.

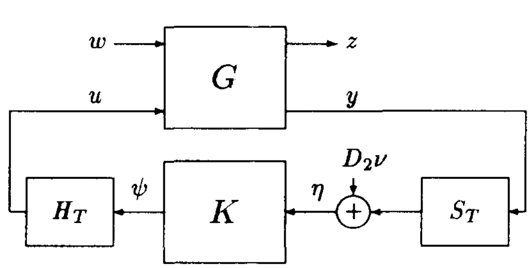

Fig. 1.

In Figure 1, since u is the output of a (zero-order) hold operator as given in (5) it follows that

$$x((k+1)T) = e^{AT}x(kT) + \int_0^T e^{A(T-s)}B_1 w(kT+s)\, ds + \phi(T)B_2 u(kT),$$

$$z(kT+t) = C_1 e^{At}x(kT) + \int_0^t C_1\, e^{A(t-s)}B_1 w(kT+s)\, ds + [C_1\phi(t)B_2 + D_{12}]u(kT),$$

where $\phi(t) := \int_0^t e^{A\tau}\, d\tau$ and $t \in [0, T]$.

A compact way of writing the above system is as follows:

$$x((k+1)T) = e^{AT}x(kT) + \overline{B}_1 w_k + \phi(T)B_2 u(kT),$$
$$z_k = \overline{C}_1 x(kT) + \overline{D}_{11} w_k + \overline{D}_{12} u(kT),$$

where w_k, z_k belong to $\mathscr{L}_2^{m_1}[0, T]$, $\mathscr{L}_2^{p_1}[0, T]$ respectively, and are given by $w_k(t) := w(kT+t)$, $z_k(t) := z(kT+t)$, $t \in [0, T]$; and $\overline{B}_1$, $\overline{C}_1$, $\overline{D}_{11}$ and $\overline{D}_{12}$ are linear operators defined as follows:

$$\overline{B}_1 : \mathscr{L}_2^{m_1}[0, T] \to \mathbb{R}^n \quad \text{and} \quad \overline{B}_1 w = \int_0^T e^{A(T-s)}B_1 w(s)\, ds,$$
$$\overline{C}_1 : \mathbb{R}^n \to \mathscr{L}_2^{p_1}[0, T] \quad \text{and} \quad (\overline{C}_1 x)(t) = C_1\, e^{At}x,$$
$$\overline{D}_{11} : \mathscr{L}_2^{m_1}[0, T] \to \mathscr{L}_2^{p_1}[0, T] \quad \text{and} \quad (\overline{D}_{11} w)(t) = \int_0^t C_1\, e^{A(t-s)}B_1 w(s)\, ds,$$
$$\overline{D}_{12} : \mathbb{R}^{m_2} \to \mathscr{L}_2^{p_1}[0, T] \quad \text{and} \quad (\overline{D}_{12} u)(t) = [C_1\phi(t)B_2 + D_{12}]u.$$

This is quite similar to the lifting technique described in [2,17,18].

If the controller is given by (4) then it is easy to verify that the closed loop system with inputs w_k, $v(k)$ and output z_k and a combined state vector $(x'(kT)\xi'(k))'$ has the form (in the packed matrix notation):

$$\mathscr{T} := \left[\begin{array}{c|cc} F & E_w & E_v \\ \hline H & J_w & J_v \end{array}\right], \tag{6}$$

where

$$F := \begin{bmatrix} e^{AT} + \phi(T)B_2\Upsilon C_2 & \phi(T)B_2\Theta \\ \Gamma C_2 & \Phi \end{bmatrix}, \quad E_w := \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}, \quad E_v := \begin{bmatrix} \phi(T)B_2\Upsilon \\ \Gamma \end{bmatrix} D_2,$$
$$H := [\overline{C}_1 + \overline{D}_{12}\Upsilon C_2 \quad \overline{D}_{12}\Theta], \qquad J_w := \overline{D}_{11}, \qquad J_v := \overline{D}_{12}\Upsilon D_2.$$

2.3. $\mathscr{H}_2$ norm for sampled-data systems

Now for all t, $\tau \in [0, T]$ define (with a slight but suggestive abuse of notation):

$$\overline{D}_{12}(t) := [C_1\phi(t)B_2 + D_{12}], \tag{7}$$
$$H(t) := [C_1\, e^{At} + \overline{D}_{12}(t)\Upsilon C_2 \quad \overline{D}_{12}(t)\Theta], \tag{8}$$
$$J_v(t) := \overline{D}_{12}(t)\Upsilon D_2, \tag{9}$$
$$J_w J_w^*(t, \tau) := \int_0^{\min(t, \tau)} C_1\, e^{A(t-s)}B_1 B_1'\, e^{A'(\tau-s)}C_1'\, ds, \tag{10}$$

where $J_w J_w^*(t, \tau)$ is the kernel of the operator $J_w J_w^*$:

$$J_w J_w^* : \mathscr{L}_2^{p_1}[0, T] \to \mathscr{L}_2^{p_1}[0, T].$$

Now consider the asymptotically stable discrete-time system (6). The inputs to this system are w_k, v. The *Controllability Gramian* of the system in (6) with respect to the input w_k is defined to be the unique solution L_{cw} to the Lyapunov equation

$$FL_{cw}F' - L_{cw} + E_w E_w^* = 0. \tag{11}$$

Note that

$$E_w^* : \mathbb{R}^{n+\hat{n}} \to \mathscr{L}_2^{m_1}[0, T] : \begin{pmatrix} x \\ \xi \end{pmatrix} \mapsto \begin{pmatrix} B_1' \, e^{A's} & 0 \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} = B_1' \, e^{A's} x, \text{ and } 0 \le s \le T.$$

Thus, $E_w E_w^*$ is an $(n+\hat{n}) \times (n+\hat{n})$ square matrix of the form

$$E_w E_w^* = \int_0^T \begin{pmatrix} e^{As} B_1 \\ 0 \end{pmatrix} \begin{pmatrix} B_1' \, e^{A's} & 0 \end{pmatrix} ds = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}. \tag{12}$$

where

$$U := \int_0^T e^{As} B_1 B_1' \, e^{A's} \, ds.$$

Similarly, define the controllability Gramian of the system in (6) with respect to the discrete-time input v as the unique solution L_{cv} to the Lyapunov equation

$$FL_{cv}F' - L_{cv} + E_v E_v' = 0. \tag{13}$$

The Gramians L_{cw} and L_{cv} can be easily computed using standard matrix algebra software.

The next result gives a formula for the $\mathscr{H}_2$ norm of the sampled-data system in terms of the plant and controller data.

Lemma 2.1. *Consider the system given in Figure* 1, *where G and K are as defined by* (3)–(4), *and suppose it is internally asymptotically stable. Let $\mathscr{T}$ denote the closed loop input–output operator with inputs w, v and output z. Then the $\mathscr{H}_2$ norm of the closed loop system is*

$$\|\mathscr{T}\|_2 := \Bigg[\text{trace}\left\{ \int_0^T J_v'(t) J_v(t) \, dt \right\} + \text{trace}\left\{ \int_0^T H(t) L_{cv} H'(t) \, dt \right\} + \text{trace}\left\{ \frac{1}{T} \int_0^T (J_w J_w^*(t, t) + H(t) L_{cw} H'(t)) \, dt \right\} \Bigg]^{1/2}, \tag{14}$$

where L_{cw} and L_{cv} are the controllability Gramians of the closed loop feedback system as defined above in (11) *and* (13) *respectively and $H(t)$, $J_v(t)$ and $J_w J_w^*(t, t)$ are as defined in equations* (8)–(10).

Proof. This lemma follows by a direct application of (2). It is straightforward to verify that

$$h_v(t) = \begin{cases} J_v(t) & \text{for } 0 \le t \le T, \\ H(t - kT) F^{k-1} E_v & \text{for } kT \le t \le (k+1)T \text{ and } k \ge 1. \end{cases}$$

Then by the property of trace$(\cdot)$ and the definition of controllability Gramian L_{cw}, it follows that

$$\begin{aligned} &\int_0^\infty \text{trace}(h_v(t) h_v'(t)) \, dt \\ &= \text{trace} \int_0^T J_v(t) J_v'(t) \, dt + \text{trace} \sum_{k=1}^\infty \int_{kT}^{(k+1)T} H(t - kT) F^{k-1} E_v E_v' (F^{k-1})' H'(t - kT) \, dt \\ &= \int_0^T \text{trace}(J_v'(t) J_v(t)) \, dt + \text{trace} \int_0^T H(\tau) \sum_{k=1}^\infty F^{k-1} E_v E_v' (F^{k-1})' H'(\tau) \, d\tau \\ &= \text{trace} \int_0^T (J_v'(t) J_v(t)) \, dt + \text{trace} \int_0^T H(\tau) L_{cv} H'(\tau) \, d\tau. \end{aligned}$$

Similarly, notice that for $0 \le \tau \le T$,

$$h_w(t, \tau) = \begin{cases} C_1 \, e^{A(t-\tau)} B_1 & \text{for } 0 \le \tau \le t \le T, \\ H(t-kT) F^{k-1} \begin{pmatrix} e^{A(T-\tau)} B_1 \\ 0 \end{pmatrix} & \text{for } kT \le t \le (k+1)T \text{ and } k \ge 1. \end{cases}$$

Then, using the definition of controllability Gramian L_{cw}, a calculation as above gives

$$\frac{1}{T} \int_0^T \left\{ \int_\tau^\infty \operatorname{trace}(h_w(t, \tau) h_w'(t, \tau)) \, dt \right\} d\tau = \operatorname{trace}\left\{ \frac{1}{T} \int_0^T (J_w J_w^*(t, t) + H(t) L_{cw} H'(t)) \, dt \right\}.$$

Proof is completed by using the definition of the $\mathscr{H}_2$ norm for periodic systems in (2). □

In case there is no measurement noise, which can be obtained by taking $D_2 = 0$, the above formula for the $\mathscr{H}_2$ norm simplifies to

$$\| \mathscr{T} \|_2 := \left[\operatorname{trace} \frac{1}{T} \int_0^T (J_w J_w^*(t, t) + H(t) L_{cw} H'(t)) \, dt \right]^{1/2}. \tag{15}$$

Now we can pose the main synthesis problem addressed in this paper.

Given a continuous-time plant G as in Figure 1, *find a discrete-time controller K such that the closed loop feedback system is internally asymptotically stable and the $\mathscr{H}_2$ norm of the closed loop system is minimized.*

A solution to this problem is given in the next section.

3. $\mathscr{H}_2$ optimal controller synthesis

Consider the sampled-data system in Figure 1. Given a continuous-time plant G with an internally stabilizing discrete-time controller K as in Figure 1, we will analyze the $\mathscr{H}_2$ performance of this system. We will see that a solution to the synthesis problem posed in the previous section falls out naturally from this analysis.

Let $G_{22d} := S_T G_{22} H_T$ represent the discretized version of G_{22}. Clearly, G_{22d} is a linear finite-dimensional discrete-time system. It is standard to verify that the state space representation of G_{22d} is:

$$G_{22d} = \left[\begin{array}{cc} e^{AT} & \phi(T) B_2 \\ C_2 & 0 \end{array} \right].$$

Suppose (A, B_2, C_2) in (3) is stabilizable and detectable. Let **T** be the set of sampling periods T_i such that either $(e^{AT_i}, \phi(T_i)B_2)$ is not stabilizable or (C_2, e^{AT_i}) is not detectable (in discrete-time). The set **T** is a discrete set (see [11]). Thus, if the sampling time T is chosen outside **T** then G_{22d} is also stabilizable and detectable. In this case, G_{22d} admits an internally stabilizing controller. The following result follows directly from [5].

Theorem 3.1. *Consider the system in* (3). *Assume that* (A, B_2) *is stabilizable and* (C_2, A) *is detectable and the sampling time T is outside* **T**. *Then there exists a discrete-time controller K which internally stabilizes G. Moreover, a discrete-time controller K internally stabilizes G iff K internally stabilizes G_{22d} in discrete time.*

We will next introduce a few matrix factorizations. Let S_1 be a $p_1 \times r_1$ real matrix such that

$$S_1 S_1' = \frac{1}{T} \int_0^T J_w J_w^*(t, t) \, dt = \frac{1}{T} \int_0^T \int_0^t C_1 \, e^{A\tau} B_1 B_1' \, e^{A'\tau} C_1' \, d\tau \, dt. \tag{16}$$

Define

$$H_i(t) := \begin{pmatrix}\overline{C}_1(t) & \overline{D}_{12}(t)\end{pmatrix} = \begin{pmatrix}C_1\,\mathrm{e}^{At} & C_1\phi(t)B_2 + D_{12}\end{pmatrix} \tag{17}$$

and let W be an $r_2 \times (n+m_2)$ real matrix such that

$$W'W = \int_0^T H_i'(t)H_i(t)\,\mathrm{d}t. \tag{18}$$

Now partition W as

$$W =: \begin{pmatrix}R & S_2\end{pmatrix} \tag{19}$$

where R is $r_2 \times n$ and S_2 is $r_2 \times m_2$. Finally, let Q be an $n \times r_3$ real matrix such that

$$QQ' = \frac{1}{T}U = \frac{1}{T}\int_0^T \mathrm{e}^{As}B_1B_1'\,\mathrm{e}^{A's}\,\mathrm{d}s. \tag{20}$$

Note that the matrices R, Q, S_1 and S_2 depend only on the plant data given in (3) and the sampling time T.

Introduce the FDLSI discrete-time plant $\tilde{G}$ given by:

$$\tilde{G}:\quad \begin{cases} \tilde{x}(k+1) = \mathrm{e}^{AT}\tilde{x}(k) + (Q\ 0)\tilde{w}(k) + \phi(T)B_2\tilde{u}(k), \\ \tilde{z}(k) = \begin{pmatrix}R\\0\end{pmatrix}\tilde{x}(k) + \begin{pmatrix}0 & 0\\0 & S_1\end{pmatrix}\tilde{w}(k) + \begin{pmatrix}S_2\\0\end{pmatrix}\tilde{u}(k), \\ \tilde{y}(k) = C_2\tilde{x}(k) + D_2v(k). \end{cases} \tag{21}$$

Here $\tilde{w}(k)$ is an (r_3+r_1)-vector, $\tilde{z}$ is an (r_2+p_1)-vector, and the zero matrices are of compatible dimensions. Now we have the following proposition.

Proposition 3.2. *Consider the continuous-time plant G given by* (3) *and the discrete-time plant $\tilde{G}$ given by* (21) *where the matrices R, S_1, S_2, and Q are as defined in* (19), (16), (20). *Suppose the discrete-time controller K given by* (4) *internally asymptotically stabilizes G. Then*

$$\|\mathscr{T}\|_2 = \|\mathscr{T}_\mathrm{d}\|_2. \tag{22}$$

where $\mathscr{T}_\mathrm{d}$ is the closed loop feedback system of $\tilde{G}$ and K as represented by Figure 2.

Remark. Note that $\mathscr{T}_\mathrm{d}$ is a standard FDLSI discrete-time system and $\|\mathscr{T}_\mathrm{d}\|_2$ is the usual $\mathscr{H}_2$-norm for FDLSI discrete-time systems.

Proof. The proof is in two steps. The first step is to compute $\|\mathscr{T}\|_2$ and the second is to compute $\|\mathscr{T}_\mathrm{d}\|_2$.

Step 1. Set

$$H_\mathrm{f} = \begin{pmatrix} I & 0 \\ \Upsilon C_2 & \Theta \end{pmatrix}.$$

Let $L_{\mathrm{c}w}$ and $L_{\mathrm{c}v}$ be the controllability Gramians defined in (11) and (13) respectively.

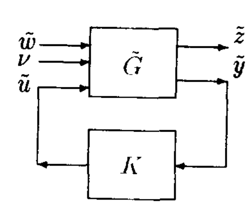

Fig. 2.

From Lemma 2.1, we can express the square of the $\mathscr{H}_2$ norm of the closed loop system as

$$\|\mathscr{T}\|_2^2 = \left[\text{trace}\left\{\int_0^T J_v'(t) J_v(t)\ \mathrm{d}t\right\} + \text{trace}\left\{\int_0^T H(t) L_{cv} H'(t)\ \mathrm{d}t\right\} + \text{trace}\left\{\frac{1}{T}\int_0^T (J_w J_w^*(t, t) + H(t) L_{cw} H'(t))\ \mathrm{d}t\right\}\right].$$

First note that

$$\int_0^T J_v'(t) J_v(t)\ \mathrm{d}t = D_2' \Upsilon' \left[\int_0^T (C_1\phi(t) B_2 + D_{12})' (C_1\phi(t) B_2 + D_{12})\ \mathrm{d}t\right] \Upsilon D_2. \tag{23}$$

Then using the factorization defined in (18)–(19),

$$\text{trace}\int_0^T J_v'(t) J_v(t)\ \mathrm{d}t = \text{trace}(D_2' \Upsilon' S_2' S_2 \Upsilon D_2).$$

Similarly, using (16),

$$\text{trace}\frac{1}{T}\int_0^T J_w J_w^*(t, t)\ \mathrm{d}t = \text{trace}(S_1 S_1').$$

Next, note that

$$\begin{aligned}\text{trace}\frac{1}{T}\int_0^T H(t) L_{cw} H'(t)\ \mathrm{d}t &= \text{trace}\frac{1}{T}\int_0^T H_i(t) H_f L_{cw} H_f' H_i'(t)\ \mathrm{d}t \\ &= \frac{1}{T}\int_0^T \text{trace}(H_i'(t) H_i(t) H_f L_{cw} H_f')\ \mathrm{d}t = \text{trace}\left(W' W H_f \frac{L_{cw}}{T} H_f'\right) \\ &= \text{trace}\left(W H_f \left(\frac{L_{cw}}{T}\right) H_f' W'\right)\end{aligned}$$

where the third equality follows from (18). Similarly, it is easy to verify that

$$\text{trace}\int_0^T H(t) L_{cv} H'(t)\ \mathrm{d}t = \text{trace}(W H_f L_{cv} H_f' W').$$

Thus,

$$\|\mathscr{T}\|_2^2 = \text{trace}(D_2' \Upsilon' S_2' S_2 \Upsilon D_2) + \text{trace}(W H_f L_{cv} H_f' W') + \text{trace}(S_1 S_1') + \text{trace}\left(W H_f \left(\frac{L_{cw}}{T}\right) H_f' W'\right). \tag{24}$$

Step 2. The closed loop system consisting of $(\tilde{G}, K)$ in Figure 2 has the following state space representation:

$$\begin{pmatrix}\tilde{x}(k+1)\\ \xi(k+1)\end{pmatrix} = \begin{bmatrix} e^{AT} + \phi(T) B_2 \Upsilon C_2 & \phi(T) B_2 \Theta \\ \Gamma C_2 & \Phi \end{bmatrix}\begin{pmatrix}\tilde{x}(k)\\ \xi(k)\end{pmatrix} + \begin{bmatrix} (Q\ \ 0) & \phi(T) B_2 \Upsilon D_2 \\ 0 & \Gamma D_2 \end{bmatrix}\begin{pmatrix}\tilde{w}(k)\\ v(k)\end{pmatrix},$$

$$\tilde{z}(k) = \begin{pmatrix} R + S_2 \Upsilon C_2 & S_2 \Theta \\ 0 & 0 \end{pmatrix}\begin{pmatrix}\tilde{x}(k)\\ \xi(k)\end{pmatrix} + \begin{bmatrix} 0 & S_2 \Upsilon D_2 \\ (0\ \ S_1) & 0 \end{bmatrix}\begin{pmatrix}\tilde{w}(k)\\ v(k)\end{pmatrix}. \tag{25}$$

Set

$$L := \frac{L_{cw}}{T} + L_{cv}.$$

We claim that L is the controllability Gramian of the system (25). To see this note that using (11) and (13) we get

$$FLF' - L = F\frac{L_{cw}}{T}F' - \frac{L_{cw}}{T} + FL_{cv}F' - L_{cv}$$
$$= -\begin{pmatrix} U/T & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \phi(T)B_2\Upsilon D_2 \\ \Gamma D_2 \end{pmatrix}\begin{pmatrix} D_2'\Upsilon'B_2'\phi'(T) & D_2'\Gamma' \end{pmatrix}$$
$$= -\begin{bmatrix} (Q\ 0) & \phi(T)B_2\Upsilon D_2 \\ 0 & \Gamma D_2 \end{bmatrix}\begin{bmatrix} (Q\ 0) & \phi(T)B_2\Upsilon D_2 \\ 0 & \Gamma D_2 \end{bmatrix}'.$$

Now using the standard definition of the $\mathcal{H}_2$ norm for discrete-time systems, the square of the $\mathcal{H}_2$ norm of the system in (25) is

$$\| \mathcal{T}_d \|_2^2 = \operatorname{trace}(C_d L C_d' + D_d D_d') \tag{26}$$

where

$$C_d := \begin{pmatrix} R + S_2\Upsilon C_2 & S_2\Theta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad D_d := \begin{bmatrix} 0 & S_2\Upsilon D_2 \\ (0\ S_1) & 0 \end{bmatrix}.$$

Some simple algebra shows that the right hand sides of (26) and (24) are equal. This completes the proof. □

The following theorem gives a solution to the optimal $\mathcal{H}_2$ sampled-data controller synthesis problem.

Theorem 3.3. *Consider the* FDLTI *continuous-time plant G given by* (3). *Let* (A, B_2) *be stabilizable and* (C_2, A) *be detectable. Construct the* FDLSI *discrete-time plant* $\tilde{G}$ *given by* (21). *Assume that the sampling period T is outside the set* **T**. *Then the following statements are equivalent*:

(i) *The discrete-time controller K described by* (4) *internally asymptotically stabilizes G as in Figure* 1 *and minimizes* $\mathcal{H}_2$ *norm of the closed loop system.*

(ii) *The discrete-time controller K described by* (4) *internally stabilizes* $\tilde{G}$ *as in Figure* 2 *and minimizes the (standard discrete-time)* $\mathcal{H}_2$ *norm of the closed loop system.*

This result follows from Theorem 3.1 and Proposition 3.2 by noticing that K internally stabilizes $\tilde{G}$ iff it internally stabilizes $\tilde{G}_{22} = G_{22d}$. From the above theorem the solution to the synthesis problem is obvious.

Given a continuous-time plant G, construct the discrete-time plant $\tilde{G}$ *as in* (21). *This construction only uses the plant parameters and the sampling period T. Now solve the* $\mathcal{H}_2$ *optimal control problem for this discrete-time plant* $\tilde{G}$. *Then the controller that we obtain is also the* $\mathcal{H}_2$ *optimal discrete-time controller for the plant G.*

Although we have stated the above result for optimal controllers, it is easy to see using Proposition 3.2 that a similar result also holds for suboptimal controllers.

Existence of the optimal controller

Consider the problem of finding an internally stabilizing controller K for the discrete-time plant $\tilde{G}$ that minimizes the $\mathcal{H}_2$ norm of the closed loop system with exogenous inputs $\tilde{w}$, v and the controlled output $\tilde{z}$. If the system for $\tilde{u}$ to $\tilde{z}$ has no invariant zeros on the unit circle and if the system from $(\tilde{w}, v)$ to $\tilde{y}$ has no invariant zeros on the unit circle, then it is a well known fact that the $\mathcal{H}_2$ optimal controller exists. If in addition, the transfer function from $(\tilde{w}, v)$ to $\tilde{y}$ has full row rank on the unit circle and the transfer function from $\tilde{u}$ to $\tilde{z}$ has full column rank on the unit circle, then the optimal controller is unique, and can be obtained by solving two discrete-time algebraic Riccati equations.

Thus, a relevant question here is under what conditions on the continuous-time plant G are the above invariant zeros and rank conditions on $\tilde{G}$ satisfied.

Suppose the continuous-time plant (3) satisfies the following (standard) assumptions:

(a) (A, B_2, C_2) is stabilizable and detectable and the sampling time T is chosen outside the set $\mathbf{T}$.

(b) D_{12} is full column rank, $C_1'D_{12} = 0$, and (C_1, A) has no unobservable modes on the imaginary axis.

(c) D_2 is full row rank and (A, B_1) has no uncontrollable modes on the imaginary axis.

Then it is not difficult to verify that for the resulting FDLSI discrete-time system $\tilde{G}$, the aforementioned invariant zeros and rank conditions are satisfied and the $\mathscr{H}_2$ optimal controller exists and is unique. Of course, these are only sufficient conditions for the existence of optimal controllers.

4. Conclusions

We have formulated and solved an $\mathscr{H}_2$ optimal control problem for sampled-data systems. We have shown that this problem is equivalent to a standard discrete-time $\mathscr{H}_2$ or LQG optimal control problem. Explicit state equations for this equivalent discrete-time system are given.

Chen and Francis [3] have shown that, in general, a time varying controller gives a better $\mathscr{H}_2$ performance than the optimal LSI controller for their definition of $\mathscr{H}_2$-norm. Now if the controller is time-varying then the resulting closed loop sampled-data system is not necessarily periodically time-varying. Thus, the definition of the $\mathscr{H}_2$ norm should be suitably changed. One possibility would be to apply an impulsive input at an arbitrary instant of time (not necessarily during the first sampling interval), and then take the square root of the average of the $\mathscr{L}_2$ norm of the resulting output. It is believed that with such a definition of the $\mathscr{H}_2$ norm, linear time-invariant controllers yield the best possible performance. Also, one could pose the multi-rate sampled-data synthesis problem where the measured outputs and control inputs of the plant operate at different sampling frequencies. A complete investigation of this is left for future research.

After this paper was completed, we received [19] where results similar to those in this paper are obtained.

References

[1] B.D.O. Anderson and J.B. Moore, *Optimal Control – Linear Quadratic Methods*, (Prentice-Hall, Englewood Cliffs, NJ, 1989).

[2] B. Bamieh and J.B. Pearson, A general framework for linear periodic systems with applications to $\mathscr{H}_\infty$ sampled-data control, Report No. 9021, Department of Electrical and Computer Engineering, Rice University, TX (Nov. 1990).

[3] T. Chen and B.A. Francis, Linear time varying $\mathscr{H}_2$-optimal control of sampled-data systems, Systems Control Group Report No. 9005, Department of Electrical Engineering, University of Toronto, Canada (1990).

[4] T. Chen and B.A. Francis, On the $\mathscr{L}_2$-induced norm of a sampled-data system, *Systems Control Lett.* **15** (1990) 211–219.

[5] T. Chen and B.A. Francis, $\mathscr{H}_2$ optimal sampled-data control, *IEEE Trans. Automat. Control* **36** (1991) 387–397.

[6] P. Dorato and A.H. Levis, Optimal linear regulators: the discrete-time case, *IEEE Trans. Automat. Control* **16** (1971) 613–620.

[7] Y.C. Juan and P.T. Kabamba, Optimal hold functions for sampled data regulation, *Automatica* **27** (1991) 177–181.

[8] E.I. Jury and F.J. Mullin, The analysis of sampled-data control systems with a periodically time-varying sampling rate, *IRE Trans. Automat. Control* **24** (1959) 735–741.

[9] P.T. Kabamba and S. Hara, On computing the induced norm of a sampled-data system, *Proceedings of American Control Conference*, San Diego, CA (1990) 319–320.

[10] P.T. Kabamba and S. Hara, Worst case analysis and design of sampled data control systems, Preprint (1990).

[11] R.E. Kalman, B.L. Ho and K. Narendra, Controllability of linear dynamical systems, *Contributions to Differential Equations* Vol. 1 (Interscience, New York, 1963).

[12] P.P. Khargonekar, K. Poolla and A. Tannenbaum, Robust control of linear time-invariant plants using periodic compensation, *IEEE Trans. Automatic Control* **30** (1985) 1088–1096.

[13] R. Kondo and K. Furuta, Sampled-data optimal control of continuous systems for quadratic criterion function taking account of delayed control action, *Internat. J. Control* **41** (1985) 1051–1060.

[14] H. Kwakernaak and R. Sivan, *Linear Optimal Control Systems* (Wiley Interscience, New York, 1972).

[15] A.H. Levis, R.A. Schlueter and M. Athans, On the behaviour of optimal linear sampled-data regulators, *Internat. J. Control* **13** (1971) 343–361.

[16] N. Sivashankar, P.P. Khargonekar, $\mathcal{L}_\infty$-induced norm of sampled-data systems, *Proceedings of American Control Conference*, Boston, MA (1991) 167–172.

[17] H.T. Toivonen, Sampled-data control of continuous-time system with an $\mathcal{H}_\infty$ optimality criterion, Process Control Laboratory Report No. 90-1, Department of Chemical Engineering, Åbo Akademi, Finland (1990).

[18] Y. Yamamoto, New approach to sampled-data control systems – a function space method, *Proceedings of Conference on Decision and Control*, Honolulu (1990) 1882–1887.

[19] B. Bamieh and J.B. Pearson, The $\mathcal{H}_2$ problem for sampled-data systems, Report No. 9104, Department of Electrical and Computer Engineering, Rice University, TX (March 1991).