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H^2-optimal Control with an H^∞-constraint: The State Feedback Case*

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A state-space approach solves the problem of finding among all state feedback controllers that minimize an H^2-performance measure one that also satisfies an H^∞-norm bound.

Key Words—Linear optimal control; multiobjective optimization; robust control; state-feedback.

Abstract—In this paper we consider a mixed H^2/H^∞-optimal control problem. It is assumed that the plant as well as the feedback controller are finite-dimensional and linear time-invariant, and that the plant state is available for feedback. More specifically, among all the state-feedback controllers that minimize the H^2-norm of a closed loop transfer matrix, we give necessary and sufficient conditions for the existence of a controller that also satisfies a prescribed H^∞-norm bound on some other closed loop transfer matrix. When these conditions are met, the solution to the above problem is also a global solution to the contrained optimization problem of minimizing an H^2-norm performance measure subject to an H^∞-norm constraint. We also give state-space formulae for computing the solutions. Some easily checkable sufficient conditions for the existence of solutions are given. Finally we give an example in which all solutions to the constrained optimization problem are necessarily dynamic, i.e. there is no static gain solution even though plant state is available for feedback.

1. INTRODUCTION AND PROBLEM FORMULATION

THE CONTROL problem addressed in this paper concerns the finite-dimensional linear time-invariant feedback system depicted in Fig. 1. The signals w_1 and w_2 denote exogenous inputs while z_1 and z_2 denote controlled (i.e. regulated) signals. The signals u and y denote the control input and the measured output, respectively. The transfer matrices of the plant and the controller are denoted by G and K, respectively. It is also assumed that both G and K are real-rational and proper transfer matrices. Finally, for given a real-rational and proper controller K, we let $T_1(K)$ and $T_2(K)$ denote the closed loop transfer matrices from w_1 to z_1 and w_2 to z_2, respectively. When there is no possibility of confusion, the dependence of T_1 and T_2 on K will be omitted.

In this paper we assume that the state of the generalized plant G is available for feedback. To be more precise let a state-space description of G be given by:

$$\frac{dx}{dt} = Ax + B_1w_1 + B_2w_2 + B_3u \qquad (1.1a)$$

$$z_1 = C_1x + D_1u \qquad (1.1b)$$

$$z_2 = C_2x + D_2u \qquad (1.1c)$$

$$y = x \qquad (1.1d)$$

where all the matrices in (1.1) are real matrices of compatible dimensions. Although no explicit frequency dependent weights were introduced, it is assumed that all weighting functions have been absorbed in the generalized plant G. Note that there are no feedthrough terms from the exogenous signals w to the controlled signals z. Although it is possible to include these terms, we have chosen not to include them in order to keep the presentation as simple as possible.

A given controller K is called *admissible* (*for the plant* G) if K is real-rational proper, and the minimal realization of K internally stabilizes the state-space realization (1.1) of G. Let $\|.\|_2$ and $\|.\|_\infty$ denote the usual H^2 and H^∞ norms, respectively. The two problems considered in this paper are defined as follows:

Problem A: Minimal H^2-norm subject to an H^∞-norm constraint. For the plant G defined in (1.1), find an admissible controller K that

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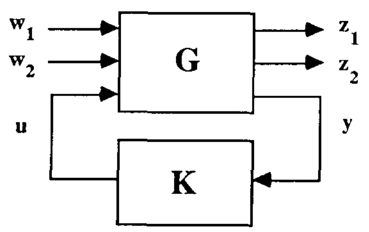


Fig. 1. The synthesis framework.

achieves

$$\inf\{\|T_1(K)\|_2 : K \text{ admissible and } \|T_2(K)\|_\infty < 1\}.$$

Problem B: Simultaneous H^2/H^∞ optimal control. For the plant G defined in (1.1) find an admissible controller K that achieves

$$\inf\{\|T_1(K)\|_2 : K \text{ admissible}\}$$

and such that $\|T_2(K)\|_\infty < 1$.

Note that while Problem A represents a constrained optimization problem, Problem B is to find (if it exists) a solution to the unconstrained problem of minimizing an H^2-performance measure that also satisfies an H^∞-norm bound. The key point is that a solution to Problem B is also a solution to Problem A but the converse need not be true.

Recently Problem A has received a great deal of attention, mainly because it represents a problem of optimal nominal performance with robust stability [see, for example, Bernstein and Haddad (1989), Mustafa and Glover (1988), Doyle *et al.* (1989b)]. Indeed, if we consider that a stable (possibly nonlinear) perturbation Δ is connected from z_2 to w_2 (see Fig. 1), then the small gain theorem ensures stability of the perturbed system if the nominal system ($\Delta = 0$) is internally stable and $\|T_2\|_\infty < 1$, provided that the induced operator norm of Δ is less than or equal to one. Among all the admissible controllers K that provide robust stability, Problem A is to find a controller that minimizes the variance of the output z_1 (with $\Delta = 0$) when w_1 is zero mean unit variance white noise.

Currently, no analytic solution to Problem A is known. Some attempts have been made to solve "modified" versions of this optimization problem. Mustafa and Glover (1988) and Glover and Mustafa (1989) have considered the special case in which $B_1 = B_2$, $C_1 = C_2$, $D_1 = D_2$, and hence $T_1 = T_2$ (see Fig. 1). For this case, they have solved the problem of maximizing an entropy functional subject to an H^∞-norm constraint. This problem formulation is related to Problem A in that the (negative of the) entropy of a transfer matrix is an upper bound for its H^2-norm. Bernstein and Haddad (1989) have considered the case of one exogeneous signal. In our setting, this means $B_1 = B_2$. They have also considered the minimization of an upper bound ("auxiliary cost", as defined by them) for $\|T_1\|_2$ subject to an H^∞-norm constraint on T_2. Using a Lagrange multiplier technique, and under the assumption that the order of the controller is specified, they have derived necessary conditions for optimality. See also Bernstein *et al.* (1989) for more recent work on this approach and Mustafa (1989) for an explicit connection between the entropy and the auxiliary cost for the special case $T_1 = T_2$. Finally, Doyle *et al.* (1989b) have considered a similar problem with one controlled output, i.e. $C_1 = C_2$ and $D_1 = D_2$. They have derived necessary conditions and sufficient conditions for this modified problem to have a solution. As shown in Doyle *et al.* (1989b), there may be a gap between these conditions. It is important to note that these papers address the more general and interesting situation of output feedback.

Problem B has not been considered before. Our objective in this paper is twofold. Firstly, we want to parametrize the set of all solutions for the (unconstrained) H^2-optimal control problem $\inf\{\|T_1(K)\|_2 : K \text{ admissible}\}$. Secondly, we want to find necessary and sufficient conditions for the existence of a solution to Problem B. Since a solution to Problem B is also a solution to Problem A, these conditions are sufficient for Problem A to have a solution. While it may seem that the solvability of Problem B is a very strong sufficient condition for the solvability of Problem A, it will be seen that if $\mathrm{im}B_1$ and $\mathrm{im}B_2$ are linearly independent, then Problem B and Problem A become equivalent.

It is important to note that the problems considered in this paper can also be approached with the aid of convex nonlinear programming; see, for example, Boyd *et al.* (1988). Since our results are analytical, they complement the numerical optimization approach taken by Boyd *et al.* (1988) and related papers.

A brief summary of our results and the organization of the paper is as follows. In Section 2 we give a parametrization of all H^2-optimal state feedback controllers. This parametrization is obtained in terms of a free transfer matrix in RH^2. Furthermore, any closed loop transfer matrix is *affine* in this free parameter. Using this parametrization along with the recent solution to the standard H^∞ problem given in Glover and Doyle (1988) and Doyle *et al.* (1989a), Section 3 gives necessary and sufficient conditions for the existence of solutions to Problem B (cf. Theorem 2). These conditions involve two algebraic Riccati equations (AREs) and a coupling condition. The first ARE reflects the fact that there must exist an admissible (state-feedback) controller such that $\|T_2\|_\infty < 1$. The second ARE and the coupling condition arise due to the requirement of the H^2 optimization. Since a solution to Problem B is also a solution to Problem A, these conditions are sufficient for Problem A to have a solution (cf. Corollary 1). When these conditions are satisfied, a "dynamic" state-feedback controller that solves Problem B (and Problem A) is given. Finally, in Section 4, we consider the special case in which $\mathrm{im}B_1$ and $\mathrm{im}B_2$ are linearly independent. In this case we show that Problem A has a solution if and only if there exists an admissible controller such that $\|T_2\|_\infty < 1$. Thus, under the mild condition of linear independence of $\mathrm{im}B_1$ and $\mathrm{im}B_2$, one gets a complete solution to Problem A. In this section we also show (by example) that there are situations in which any solution to either Problem A or Problem B must necessarily be dynamic. This is in significant contrast to the fact that in either H^2 or H^∞ optimal control problems, when states are available for feedback, the controller can be chosen to be a memoryless gain [see, for example Kalman (1960) and Khargonekar *et al.* (1988)]. This example appears to indicate that the mixed H^2/H^∞ problems are likely to be much more complicated than standard H^2 and H^∞ problems. Finally in Section 5 the conclusions of this work are given.

The notation is fairly standard. The identity matrix is denoted by I. For a constant matrix M, let $\mathrm{im}M$ and $\ker M$ denote its range and null space, respectively. The spectral radius of M is denoted by $\rho(M)$. The transpose of M is denoted by M'. Let M^+ denote the Moore–Penrose inverse of M. If $M = 0$ we shall define $M^+ := 0$. The orthogonal complement of a subspace $\mathbf{S} \subset \mathbf{R^p}$ is denoted by $\mathbf{S}^\perp$. Packed matrix notation is used to represent state-space realizations, i.e.

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := C(sI - A)^{-1}B + D.$$

For a transfer matrix G, we define $G^\sim$ as $G^\sim(s) := G'(-s)$ for all complex s. The spaces H^2 and $H^{2\perp}$ denote the Hardy spaces of matrix-valued functions that are square integrable on the imaginary axis with analytic extensions into the right and left half plane, respectively. The Hardy space H^∞ consists of matrix-valued functions that are bounded on the imaginary axis with analytic extension into the right half plane. The norms in these spaces are defined in the usual way:

$$\|G\|_2 := \sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{trace}\,(G^\sim G)(\mathrm{j}\omega)\,\mathrm{d}\omega}$$

$$\|G\|_\infty := \sup_{\omega\in R} \sigma\{G(\mathrm{j}\omega)\},\ (\sigma := \text{max singular value}).$$

When R is used as a prefix, it denotes real rational. Given real matrices A, $R = R'$, and $Q = Q'$, will say that the algebraic Riccati equation (ARE) $A'X + XA + XRX + Q = 0$ admits the (unique) stabilizing solution X_s if X_s is real and symmetric, X_s satisfies the ARE, and $A + RX_s$ has eigenvalues in the open left half plane. A similar definition is used for the antistabilizing solution with the obvious modifications.

In the rest of the paper, the following standard assumptions on the plant G are made:

(i) $\{A, B_3\}$ is stabilizable. (1.2a)
(ii) D_1 and D_2 both have full column rank. (1.2b)
(iii) For each $\omega \in \mathbf{R}$ and for $k = 1, 2$

$$\begin{bmatrix} A - \mathrm{j}\omega I & B_3 \\ C_k & D_k \end{bmatrix} \tag{1.2c}$$

has full column rank.

Note that assumption (1.2a) ensures that the set of admissible controllers is non-empty while assumptions (1.2b, c) are standard and guarantee that the LQR problem corresponding to the quadratic cost $\|z_k\|_2$ has an admissible solution. Without loss of generality we further assume that

(iv) $D_2'[C_2\ \ D_2] = [0\ \ I]$. (1.2d)

In fact, as is well known, a preliminary feedback transformation will enforce this last equation.

2. PARAMETRIZATION OF ALL H^2-OPTIMAL CONTROLLERS

In this section we parametrize the set of all admissible unconstrained H^2-optimal state-feedback controllers for the interconnection of Fig. 1. The development is carried out using a frequency domain approach. More specifically, the YJBK parametrization of all stabilizing compensators is used to solve this problem (see for example Vidyasagar, 1984). The final formula for the solution to this problem is given in state-space (cf. Theorem 1).

Consider the block diagram of Fig. 1, where the plant G is as in (1.1). It is a classical fact that [under assumptions (1.2a–c)] there exists an admissible controller that minimizes $\|T_1\|_2$. One such admissible controller is $K_0 = F$, where the constant real matrix F is computed according to

$$F := -(D_1'D_1)^{-1}(D_1'C_1 + B_3'X_1), \quad (2.1a)$$

and the constant matrix X_1 is the unique stabilizing solution of the (LQR) ARE

$$A'X + XA - (D_1'C_1 + B_3'X)'(D_1'D_1)^{-1} \times (D_1'C_1 + B_3'X) + C_1'C_1 = 0. \quad (2.1b)$$

Perhaps, it is less well known that (2.1) is not the only admissible controller that minimizes $\|T_1\|_2$. Our first result (Theorem 1) gives a complete parametrization of all solutions to this optimization problem. With reference to the realization of the plant G given in (1.1) and with F given by (2.1), define

$$\Pi_1 := I - B_1B_1^+, \quad (2.2)$$

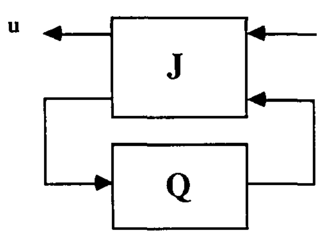
$$A_F := A + B_3F, \quad (2.3)$$

$$C_{kF} := C_k + D_kF; \quad k = 1, 2. \quad (2.4)$$

The matrix Π_1 is the orthogonal projection onto $(\mathrm{im}B_1)^\perp$. Note that A_F is a stability matrix. Define the set of transfer matrices:

$$S := \{Q \in RH^\infty : Q = W\Pi_1(sI - A_F),\ W \in RH^2\}. \quad (2.5)$$

Theorem 1. Consider the feedback system of Fig. 1, with the plant G given by (1.1). Let **S** be defined by (2.5). Let K denote an admissible controller and T_1 the corresponding closed loop transfer matrix from w_1 to z_1. Then, K minimizes $\|T_1\|_2$ if and only if K equals the transfer matrix from y to u in



$$J := \left[\begin{array}{c|cc} A_F & 0 & B_3 \\ \hline 0 & F & I \\ -I & I & 0 \end{array}\right] \quad (2.6)$$

for some $Q \in S$.

Note that if $\mathrm{im}B_1 = \mathbf{R}^n$ (n := state dimension) then $\Pi_1 = 0$ and (2.7) reduces to the single state-feedback controller $K_0 = F$. On the other hand, if $\mathrm{im}B_1$ is a proper subspace of $\mathbf{R}^n$ then (2.6) generates a family of controllers parametrized by W. This extra freedom can be used to satisfy some additional constraints.

The next lemma will be useful for establishing Theorem 1. It provides state-space formulae for the YJBK parametrization of all admissible controllers. The formulae given below are more appropriate for our setting (state-feedback) than the well known formulae in terms of an "observer-based" stabilizing compensator.

Lemma 1. Consider the feedback system of Fig. 1, where G is given by (1.1). Then, a given controller K is admissible if and only if there exists $Q \in RH^\infty$ such that K equals the transfer matrix from y to u in (2.6).

Proof. First, note that a given controller K is admissible for G if and only if K is admissible for $G_{yu} := (sI - A)^{-1}B_3$. With F given by (2.1), define the RH^∞ matrices

$$N := \left[\begin{array}{c|c} A_F & B_3 \\ \hline I & 0 \end{array}\right],$$

$$M := I + FN, \quad \tilde{N} := N, \quad \tilde{M} := I + NF,$$

$$X := F, \quad Y := I, \quad \tilde{X} := F, \quad \tilde{Y} := I.$$

Then, it is straightforward to verify that

$$G_{yu} = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & X \\ N & Y \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}.$$

These equations provide a (doubly) coprime factorization over RH^∞ for G_{yu}. It now follows (see, for example, Vidyasagar, 1984) that K is admissible for G_{yu} if and only if there exists $Q \in RH^\infty$ such that

$$K = (X + MQ)(Y + NQ)^{-1} = F + Q(I + NQ)^{-1},$$

which is of the form (2.6). Q.E.D.

Proof of Theorem 1. From Lemma 1 and after some standard algebraic manipulations, it follows that the set of all admissible closed loop transfer matrices from w_1 to z_1 (i.e. those that are generated by admissible controllers) can be parametrized by the formula

$$T_1 := S_1 + U_1QV_1, \quad Q \in RH^\infty, \quad (2.7)$$

$$S_1 := \left[\begin{array}{c|c} A_F & B_1 \\ \hline C_{1F} & 0 \end{array}\right], \quad U_1 := \left[\begin{array}{c|c} A_F & B_3 \\ \hline C_{1F} & D_1 \end{array}\right],$$

$$V_1 := \left[\begin{array}{c|c} A_F & B_1 \\ \hline I & 0 \end{array}\right],$$

where A_F and C_{1F} are defined in (2.3) and (2.4), respectively.

It is now standard to show (using some simple algebra and (2.1)) that $R_1 := D_1'D_1 = U_1^\sim U_1$ and that $U_1^\sim S_1 \in H^{2\perp}$. Using these properties, we conclude from (2.9) that if $Q \in RH^\infty$ then

$$\|T_1\|_2^2 = \|S_1\|_2^2 + \|U_1QV\|_2^2 = \|S_1\|_2^2 + \|\sqrt{R_1}\,QV_1\|_2^2.$$

It is now clear that $Q \in RH^\infty$ minimizes $\|T_1\|_2$ if and only if Q satisfies

$$\sqrt{R_1}\, QV_1 = 0 \Leftrightarrow Q(sI - A_F)^{-1}B_1 = 0. \quad (2.8)$$

To complete the proof we must show that $Q \in RH^\infty$ satisfies (2.8) if and only if $Q \in S$. Clearly, if $Q \in S$ then $Q \in RH^\infty$ and satisfies (2.8). The converse is as follows. Let $Q \in RH^\infty$ be given and suppose that it satisfies (2.8). Define $W := Q(sI - A_F)^{-1}$. Note that $W \in RH^2$, for A_F is a stability matrix. Moreover, from (2.2) and (2.8), it follows that

$$W\Pi_1 = Q(sI - A_F)^{-1} \Rightarrow W\Pi_1(sI - A_F) = Q.$$

Therefore, $Q \in S$. Q.E.D.

We conclude this section with a state-space representation for the controller of Theorem 1 that will be useful for establishing the main result of this paper (Theorem 2). Let $W \in RH^2$ be given by

$$W = \left[\begin{array}{c|c} A_w & B_w \\ \hline C_w & 0 \end{array}\right].$$

Then it is easy to show that $Q = W\Pi_1(sI - A_F)$ is given by

$$Q = \left[\begin{array}{c|c} A_w & A_wB_w\Pi_1 - B_w\Pi_1A_F \\ \hline C_w & C_wB_w\Pi_1 \end{array}\right].$$

Substituting this realization of Q in (2.6), and after deleting unobservable modes, the controller K of Theorem 1 is given by

$$A_K = A_w - B_w\Pi_1B_3C_w, \quad (2.9a)$$

$$K = \left[\begin{array}{c|c} A_K & A_KB_w\Pi_1 - B_w\Pi_1A_F \\ \hline C_w & F + C_wB_w\Pi_1 \end{array}\right]. \quad (2.9b)$$

3. THE SIMULTANEOUS H^2/H^∞ PROBLEM

In this section we solve the simultaneous H^2/H^∞ optimization problem (Problem B) defined in Section 1. The development is carried out using Theorem 1 along with the recent solution to the standard H^∞-optimization problem given in Glover and Doyle (1988) and Doyle *et al.* (1989a). For the sake of completeness, a slightly modified statement (suitable for our purposes) of the main result of Glover and Doyle (1988) has been included in a separate appendix.

Our first result in this section gives necessary and sufficient conditions for the existence of solutions to Problem B. Consider the plant G given by (1.1) and let Π_1 denote the projection matrix defined in (2.2). Define also

$$V_2 := \Pi_1B_2. \quad (3.1)$$

Note that $V_2 = 0$ if and only if $\mathrm{im}B_2 \subset \mathrm{im}B_1$. The following algebraic Riccati equations (for X and Y) will play an important role in stating our conditions for the existence of solutions to Problem B:

$$A'X + XA + X(B_2B_2' - B_3B_3')X + C_2'C_2 = 0 \quad (3.2)$$

$$YA_F' + A_FY + Y(C_{2F}'C_{2F})Y + B_2(I - V_2^+V_2)B_2' = 0 \quad (3.3)$$

where F, A_F, and C_{2F} are defined in (2.1), (2.3) and (2.4), respectively.

Theorem 2. Consider the feedback system of Fig. 1, with the plant G given by (1.1). Then there exists an admissible controller K that solves Problem B if and only if the following conditions hold:

(i) The ARE (3.2) admits the stabilizing solution X_2, and $X_2 \geq 0$. (3.5a)
(ii) The ARE (3.3) admits the stabilizing solution Y_2. (3.5b)
(iii) $\rho(Y_2X_2) < 1$. (3.5c)

Moreover, when these conditions are met, one solution to Problem B is given by

$$K := \left[\begin{array}{c|c} A_0 & A_0\Sigma - \Sigma A_F \\ \hline H - F & F(I - \Sigma) + H\Sigma \end{array}\right], \quad (3.6a)$$

where

$$A_0 := A + (I - \Sigma)B_3H + \Sigma B_3F + (I - \Sigma)B_2B_2'X_2, \quad (3.6b)$$

$$\Sigma := Z_2B_2V_2^+\Pi_1, \quad H := -B_3'X_2, \quad Z_2 := (I - Y_2X_2)^{-1}, \quad (3.6c)$$

and F, A_F are defined in (2.1) and (2.3), respectively.

It should be noted that, under assumption (1.2d), condition (3.5a) is equivalent to the existence of an admissible controller K such that $\|T_2\|_\infty < 1$ (Doyle *et al.*, 1989a). The other two conditions, namely (3.5b, c), reflect the fact that one of these admissible controllers must also minimize $\|T_1\|_2$. As it will become clear from the proof of Theorem 2, when $\mathrm{im}B_2 \subset \mathrm{im}B_1$, Problem B has a solution if and only if condition (3.5b) holds. In this case, the constant gain F defined in (2.1) is a solution to Problem B.

Since a solution to Problem B is also a solution to Problem A, Theorem 2 may be used to produce a sufficient condition under which Problem A can be solved. It is obvious that Problem A makes sense only if condition (3.5a) holds. In other words, if (3.5a) is not satisfied then, there is no state-feedback controller that internally stabilizes the plant G and yields

$\|T_2\|_\infty < 1$. An immediate consequence of Theorem 2 is the following.

Corollary 1. Consider the feedback system of Fig. 1, with the plant G given by (1.1). Suppose that conditions (3.5) hold. Then the controller given by (3.6) is an admissible controller that solves Problem A. Furthermore,

$$\inf\{\|T_1\|_2 : K \text{ admissible and } \|T_2\|_\infty < 1\} = \inf\{\|T_1\|_2 : K \text{ admissible}\}.$$

We conclude this section with a number of intermediate results that will lead to a proof of Theorem 2. In the light of Theorem 1, solving Problem B reduces to that of finding a controller of the form (2.7) such that $\|T_2\|_\infty < 1$. Let F, A_F, and C_{2F} be defined by (2.1), (2.3) and (2.4), respectively. An easy calculation shows that when the controller given in Theorem 1 is connected to the plant G, the closed loop transfer matrix from w_2 to z_2 equals

$$T_2 := S_2 + U_2 W V_2, \quad W \in RH^2 \text{ (free)}, \quad (3.8a)$$

where

$$S_2 := \left[\begin{array}{c|c} A_F & B_2 \\ \hline C_{2F} & 0 \end{array}\right], \quad U_2 := \left[\begin{array}{c|c} A_F & B_3 \\ \hline C_{2F} & D_2 \end{array}\right],$$

$$V_2 := \Pi_1 B_2. \quad (3.8b)$$

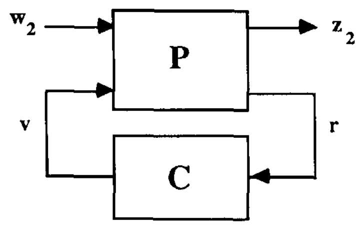
From (3.8b) we observe that if $\operatorname{im} B_2 \subset \operatorname{im} B_1$ then $V_2 = 0$, and $T_2 = S_2$ for all $W \in RH^2$. Therefore, in this particular case, we can not exploit the transfer matrix W to reduce $\|T_2\|_\infty$. In other words, for all $W \in RH^2$ it follows that $T_2 = T_2(F)$, where F is the constant gain given in (2.1). The following lemma gives necessary and sufficient conditions for the existence of $W \in RH^2$ such that $\|T_2\|_\infty < 1$ and it will become useful for establishing Theorem 2.

Lemma 2. Consider the transfer matrix T_2 given in (3.8). Assume that $V_2 \neq 0$. Then, there exists $W \in RH^2$ such that $\|T_2\|_\infty < 1$ if and only if conditions (3.5) hold. In this case, a transfer matrix $W \in RH^2$ such that $\|T_2\|_\infty < 1$ is given by

$$W := \left[\begin{array}{c|c} A_H + (I - \Sigma)B_2 B_2' X_2 & Z_2 B_2 V_2^+ \\ \hline H - F & 0 \end{array}\right], \quad (3.10)$$

where $A_H := A + B_3 H$, and H, Σ and Z_2 are defined in (3.6c).

Proof. First we factor the non-zero constant matrix V_2 as $V_2 = M_0 M_i$, where M_0 is a full column rank matrix and M_i satisfies $M_i M_i' = I$. (Note that this factorization always exists.) Now it is easy to see that the closed loop transfer matrix T_2 given in (3.8) equals the transfer matrix from w_2 to z_2 in the following diagram:



$$P := \left[\begin{array}{c|cc} A_F & B_2 & B_3 \\ \hline C_{2F} & 0 & D_2 \\ 0 & M_i & 0 \end{array}\right] \quad C := W M_0. \quad (3.11)$$

Note that the full column rank property of M_0 guarantees that the existence of $W \in RH^2$ such that $\|T_2\|_\infty < 1$ is equivalent to the existence of $C \in RH^2$ such that $\|T_2\|_\infty < 1$.

Next, we show that conditions (3.5) are necessary and sufficient for the existence of such a transfer matrix C. First note that since the open loop transfer matrix P_{rv} (from v to r) in (3.11) is identically zero, and since A_F is a stability matrix, it is obvious that a given controller C is admissible for P if and only if $C \in RH^\infty$. We claim that the auxiliary plant P defined in (3.11) satisfies all the assumptions of Theorem A.7 (see the Appendix). This claim will be verified later.

Now, applying the result of Theorem A.7 to the auxiliary plant P and after some algebra, one concludes that there exists $C \in RH^\infty$ such that $\|T_2\|_\infty < 1$ if and only if the following conditions are met:

(a) The ARE (3.2) admits the stabilizing solution X_2, and $X_2 \geq 0$. [Here we have used assumption (1.2d).]
(b) The ARE (3.3) admits the stabilizing solution Y_2, and $Y_2 \geq 0$. [Here we have used the identity $M_i' M_i = V_2^+ V_2$.]
(c) $\rho(Y_2 X_2) < 1$.

The equivalence between the condition (b) above and (3.5b) is obtained by observing that the stability of A_F implies that any symmetric solution to the ARE (3.3) is positive semidefinite. We must also show that when the above conditions are met there is a choice of C not only in RH^∞ but also in RH^2. This immediately follows from the construction of the controller given in Theorem A.7. In fact, from (A.8a), we observe that C can be chosen to be strictly proper.

Finally, assuming that conditions (3.5) hold, the formula for W given in (3.10) follows from (A.8) (to obtain a formula for C), and solving the linear equation indicated in (3.11) for the transfer matrix W. In this step we have used the fact that there always exist a choice of M_i and M_0 such that $V_2^+ = M_i' M_0^-$ (where M_0^- denotes a left inverse of M_0).

To complete the proof, we must verify that the auxiliary plant P in (3.11) satisfies all the assumptions of Theorem A.7. Clearly, (A.1) follows from the stability property of A_F. Assumption (A.2) follows from (1.2d) and the identity $M_iM_i' = I$. Finally, using (1.2c) and the identity

$$\begin{bmatrix} A_F - j\omega I & B_3 \\ C_{2F} & D_2 \end{bmatrix} = \begin{bmatrix} A - j\omega I & B_3 \\ C_2 & D_2 \end{bmatrix} \begin{bmatrix} I & 0 \\ F & I \end{bmatrix},$$

we conclude that (A.3) is satisfied. The fact that assumption (A.4) holds is a consequence of the nonsingularity of A_F and the full row rank property of M_i. Q.E.D.

Proof of Theorem 2. We consider two cases.

Case 1. Suppose that $\mathrm{im} B_2 \not\subset \mathrm{im} B_1$. In this case $V_2 \neq 0$ (cf. (3.1)). Combining the results of Theorem 1 and Lemma 2 it is clear that among the admissible controllers that minimize $\|T_1\|_2$ there exists one such that $\|T_2\|_\infty < 1$ if and only if conditions (3.5) hold. Suppose that these conditions hold and let W be given by (3.10). Define

$$A_w := A_H + (I - \Sigma)B_2B_2'X_2,$$

$$B_w := Z_2B_2V_2^+, \quad C_w := H - F.$$

Using these equations and the fact that $\Sigma := B_w\Pi_1$ [cf. (3.6c)], it follows from (2.11) that a solution to Problem B is given by (3.6).

Case 2. Suppose that $\mathrm{im} B_2 \subset \mathrm{im} B_1$. In this case V_2 and V_2^+ are both zero [cf. (3.1)]. Therefore, from (3.8a), we conclude that Problem B has a solution if and only if

$$\|S_2\|_\infty < 1, \tag{3.12}$$

where the transfer matrix S_2 is defined in (3.8b). We claim that (3.12) and (3.5b) are equivalent. In fact, since A_F is a stability matrix, from Lemma 4 in Doyle *et al.* (1989a) it follows that (3.12) is satisfied if and only if the ARE

$$YA_F' + A_FY + Y(C_{2F}'C_{2F})Y + B_2B_2' = 0, \tag{3.13}$$

admits the stabilizing solution. The claim is finally established by observing that $V_2^+V_2 = 0$ implies that the AREs (3.3) and (3.13) are the same. Next, we show that conditions (3.5a) and (3.5c) are necessary for Problem B to have a solution.

Suppose that Problem B has a solution; then (3.12) holds. From (3.8a) it follows that the admissible controller F defined in (2.1) yields the closed loop transfer matrix $T_2(F) = S_2$. Hence, from (3.12) and item FI.4 in Doyle *et al.* (1989a), we conclude that condition (3.5a) must hold.

The necessity of condition (3.5c) will be proved under the technical assumption that the pair (C_2, A) is observable. (Even if this is not the case, the result is still true, and can be obtained by factoring the unobservable subspace out.) Suppose that Problem B has a solution, then conditions (3.5a, b) are satisfied. It is easy to show [using the observability of the pair (C_2, A)] that the stabilizing solution to the ARE (3.2) satisfies $X_2 > 0$. Define $Y_+ := X_2^{-1}$. From (3.2) it follows that Y_+ satisfies

$$AY_+ + Y_+A' + Y_+C_2'C_2Y_+ + B_2B_2' - B_3B_3' = 0,$$

$$(A + Y_+C_2'C_2)' = -Y_+^{-1}(A + (B_2B_2' - B_3B_3')X_2)Y_+. \tag{3.14}$$

Hence, Y_+ is the (unique) *anti-stabilizing* solution to the ARE (3.14). We now show that if Y denotes any real symmetric solution to the ARE (3.3) then

$$Y_+ \geq Y. \tag{3.15}$$

Indeed, using a "completion of squares" argument, it follows that any solution Y of (3.3) satisfies the quadratic matrix inequality

$$AY + YA' + YC_2'C_2Y + B_2B_2' - B_3B_3' = -(B_3' + FY)'(B_3' + FY) \leq 0.$$

Hence, from Ran and Vreugdenhil (1988), it follows that inequality (3.15) must hold.

Note that the pair (C_{2F}, A_F) is observable. This follows from the observability of (C_2, A) and assumption (1.2d). Since (3.5b) holds we conclude from Willems (1971) that the anti-stabilizing solution to the ARE (3.3), say Y_-, exists and

$$Y_- > Y_2, \tag{3.16}$$

where Y_2 denotes the stabilizing solution to the ARE (3.3). Combining (3.15) and (3.16) we obtain that $Y_+ \geq Y_- > Y_2$, which implies that $X_2^{-1} > Y_2$. Therefore, condition (3.5c) is satisfied.

To complete the proof we must show that when conditions (3.5) hold, the controller K in (3.6) solves Problem B. This is immediate since $V_2^+ = 0$ implies that $\Sigma = 0$ [cf. (3.6c)]. Thus (3.6a) reduces to

$$K = \left[\begin{array}{c|c} A + (B_2B_2' - B_3B_3')X_2 & 0 \\ \hline H - F & F \end{array}\right] = F.$$

Since F is an admissible controller, the result follows from (3.12) and the fact that $T_2(F) = S_2$. Q.E.D.

4. SPECIAL CASES

In this section we will focus on Problem A. Corollary 1 tells us that if conditions (3.5) hold, then there is a solution to the unconstrained problem $\inf\{\|T_1(K)\|_2 : K \text{ admissible}\}$ that also solves Problem A. Therefore, one might be tempted to conclude that these conditions are too restrictive. We claim that this is not the case. In this section we show that if $\mathrm{im}B_1 \cap \mathrm{im}B_2 = 0$, then conditions (3.5b, c) hold. In fact, under this geometric condition, a much stronger result is true. As before, let $\Pi_1 := I - B_1B_1^+$ and $V_2 := \Pi_1 B_2$. (4.1)

Lemma 3. Consider the feedback system of Fig. 1, with the plant G given by (1.1). Assume that $\mathrm{im}B_1 \cap \mathrm{im}B_2 = 0$. Let F_1 and F_2 denote two (arbitrary) constant state-feedback matrices for the plant G. Then the dynamic state-feedback controller

$$K := \left[\begin{array}{c|c} A_1 & A_1\Delta - \Delta A_{F_1} \\ \hline F_2 - F_1 & F_1(I-\Delta) + F_2\Delta \end{array}\right], \quad (4.2a)$$

where

$$A_1 := A + (I-\Delta)B_3F_2 + \Delta B_3 F_1, \quad (4.2b)$$

$$A_{F_1} := A + B_3F_1, \quad \Delta := B_2V_2^+\Pi_1, \quad (4.2c)$$

achieves the following closed loop transfer matrices: $T_1(K) = T_1(F_1)$ and $T_2(K) = T_2(F_2)$. Moreover, K is admissible if and only if both F_1 and F_2 are admissible.

Proof. Note that $\Delta B_1 = 0$, since Π_1 is the orthogonal projection onto $(\mathrm{im}B_1)^\perp$. Next we show that $\Delta B_2 = B_2$. It is easy to see that $\mathrm{im}B_1 \cap \mathrm{im}B_2 = 0$ implies that $\ker V_2 = \ker B_2$. Since $V_2^+V_2$ and $B_2^+B_2$ are the orthogonal projections onto $(\ker V_2)^\perp$ and $(\ker B_2)^\perp$ respectively, and since orthogonal projections are unique, we conclude that

$$V_2^+V_2 = B_2^+B_2.$$

Therefore,

$$\Delta B_2 = B_2V_2^+V_2 = B_2B_2^+B_2 = B_2.$$

(Actually, Δ is a projection onto $\mathrm{im}B_2$.)

Let the controller K be given by (4.2). Let x and ξ denote the states of G and K respectively. Consider the interconnection of Fig. 1 and define new coordinates x_n and ξ_n according to $x_n := (I-\Delta)x - \xi$ and $\xi_n := \Delta x + \xi$. It is easy to verify that the transformation from (x, ξ) to (x_n, ξ_n) is invertible. Using the fact that $\Delta B_1 = 0$ and $\Delta B_2 = B_2$, a trivial computation shows that the closed loop equation for the block diagram of Fig. 1 is given by

$$\frac{dx_n}{dt} = (A + B_3F_1)x_n + B_1w_1$$

$$\frac{d\xi_n}{dt} = (A + B_3F_2)\xi_n + B_2w_2$$

$$z_1 = (C_1 + D_1F_1)x_n + (C_1 + D_1F_2)\xi_n$$

$$z_2 = (C_2 + D_2F_1)x_n + (C_2 + D_2F_2)\xi_n$$

which completes the proof. Q.E.D.

Lemma 3 tells us that whenever the two subspaces $\mathrm{im}B_1$ and $\mathrm{im}B_2$ are independent, there exists a dynamic state-feedback controller that simultaneously achieves closed loop transfer matrices T_1 and T_2, provided there exist constant state feedback matrices F_1 and F_2 such that $T_1 = T_1(F_1)$ and $T_2 = T_2(F_2)$. The connection between Lemma 3 and Problem A is given by the following lemma.

Lemma 4. Consider the feedback system of Fig. 1, with the plant G given by (1.1). Suppose that $\mathrm{im}B_1 \cap \mathrm{im}B_2 = 0$. Then Problem A is solvable if and only if there exists an admissible controller K such that $\|T_2(K)\|_\infty < 1$. In this case, a solution to Problem A is given by (4.2) with $F_1 := F$ and $F_2 := H$, where F and H are defined in (2.1) and (3.6c), respectively.

Proof. The necessity immediately follows from the definition of Problem A. The sufficiency part is as follows. First, note that the existence of an admissible controller K such that $\|T_2(K)\|_\infty < 1$ implies that condition (3.5a) holds. In this case, the constant matrix H in (3.6c) is an admissible controller such that $\|T_2(H)\|_\infty < 1$. (See Doyle *et al.*, 1989a). Note also that the constant matrix F in (2.1) is an admissible controller that minimizes $\|T_1\|_2$. Choosing $F_1 := F$ and $F_2 := H$, the result follows from Lemma 3. Q.E.D.

It is also interesting to establish a connection between Lemma 4 and Corollary 1. Suppose that $\mathrm{im}B_1 \cap \mathrm{im}B_2 = 0$. We will show that conditions (3.5b, c) are automatically satisfied. Recall from the proof of Lemma 3 that under this geometric condition, $V_2^+V_2 = B_2^+B_2$. Hence, the ARE (3.3) reduces to

$$YA_F' + A_FY + Y(C_{2F}'C_{2F})Y = 0, \quad (4.3)$$

which has the unique stabilizing solution $Y_2 = 0$. (Recall that A_F is a stability matrix.) Thus, conditions (3.5b, c) hold and from Corollary 1 we conclude that Problem A has a solution if and only if condition (3.5a) holds. In this case, a trivial computation shows that the controllers of Corollary 1 and Lemma 4 are the same.

We conclude this section with an example. Our objective in this example is to show that although Problem A need not have a unique solution, there are situations in which any solution must necessarily be "dynamic". With reference to Fig. 1, let G be given by [note that

G satisfies Assumptions (1.2)]

$$G := \begin{array}{c} \\ \\ z_1\{ \\ z_2\{ \\ \\ y\{ \end{array} \left[\begin{array}{c|ccc} & w_1 & w_2 & u \\ \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \hline [0 \;\; 0] & 0 & 0 & 1 \\ \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{array}\right]. \tag{4.4}$$

First, note that the ARE (3.2) admits the stabilizing solution

$$X_2 = \begin{bmatrix} 1 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

which is positive definite. Thus, the set of admissible controllers K such that $\|T_2\|_\infty < 1$ is non-empty. Since $\mathrm{im}B_1 \cap \mathrm{im}B_2 = 0$, Lemma 4 may be used to solve Problem A. From (2.1) it follows that $F = [0 \;\; 0]$. This is obvious since $z_1 = u$ and the plant G is stable. Next, we compute the gain $H := -B_3' X_2$. In this case we obtain $H = -[0.5 \;\; 0.5]$. Finally, from (4.2) it follows that a solution to Problem A is given by

$$K = \left[\begin{array}{cc|cc} -1 & 1 & -1 & 1 \\ 0 & -1 & 0 & 1 \\ \hline -0.5 & -0.5 & 0.5 & -0.5 \end{array}\right]$$

which obviously has McMillan degree equal to two. We now show that in this example any other solution to Problem A must be dynamic. First, note that

$$\inf\{\|T_1(K)\|_2 : K \text{ admissible and } \|T_2(K)\|_\infty < 1\} = 0.$$

Using some simple algebra it is easily seen that the unique state feedback "gain" that achieves this optimal performance is $F = [0 \;\; 0]$. From (4.4) it follows that $\|T_2(F)\|_\infty = \sqrt{2}$. Thus we conclude that for this particular example any solution to Problem A must be necessarily dynamic.

5. CONCLUSIONS

For the state-feedback case we have completely solved a mixed H^2/H^∞ control problem (Problem B). Necessary and sufficient conditions for the existence of solutions to Problem B were given in terms of solutions to certain AREs. A closed form expression for a solution was also provided. A solution to Problem B (when it exists) also solves the constrained optimal control problem of minimizing an H^2 performance measure subject to an H^∞ constraint (Problem A). This problem is well motivated since it models a problem of optimal nominal performance with robust stability. Previous authors have only considered the minimization of an upper bound for the H^2 design objective. In this sense, the results of this work constitute the first results on this problem.

From Lemma 3 it follows that if the two subspaces $\mathrm{im}B_1$ and $\mathrm{im}B_2$ are independent, then one can always find a dynamic state-feedback controller that simultaneously achieves given closed loop transfer matrices T_1 and T_2 provided they can be separately achieved using static state-feedback controllers. The simplest case for which the condition of independence of $\mathrm{im}B_1$ and $\mathrm{im}B_2$ is not satisfied is when $B_1 = B_2$. Recall from the proof of Theorem 2 that in this case Problem B has a solution if and only if $\|T_2(F)\|_\infty < 1$, where F denotes the LQR gain defined in (2.1). Therefore Problem B does not help much in solving Problem A. In this sense, further research on Problem A for the case $B_1 = B_2$ should be most useful.

The example in Section 4 illustrates that, even though the plant state is available for feedback, Problems A and B need not have a static solution. This is in significant contrast to the classical results in the LQR theory (Kalman, 1960) and the recent results in H^∞ control theory (Khargonekar *et al.*, 1988) which show that these optimal control problems always have a static state-feedback solution. This may have some implications in the output-feedback case. For instance, it might turn out that in the output-feedback case the dimension of optimal controllers in mixed H^2/H^∞ problems exceeds the plant dimension.

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APPENDIX

State-space formulae for the standard H^∞ problem

The result given in this appendix provides a solution to the standard H^∞-optimal control problem, and is a slight modification of Theorem 1 in Glover and Doyle (1988). Consider the feedback system shown in Fig. 2, where both the plant P and the controller C are real-rational and proper. Let $T(C)$ denote the closed loop transfer matrix from w to z.

Assume that P has the following realization:

$$P := \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_1 \\ C_2 & D_2 & 0 \end{array}\right],$$

along with the assumptions

(i) (A, B_2) stabilizable and (C_2, A) detectable. (A.1)
(ii) $D_1'D_1 = I$ and $D_2D_2' = I$. (A.2)
(iii) For each $\omega \in \mathbf{R}$

$$\begin{bmatrix} A - j\omega I & B_2 \\ C_1 & D_1 \end{bmatrix} \tag{A.3}$$

has full column rank.

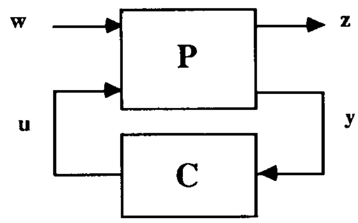

FIG. 2. Feedback system.

(iv) For each $\omega \in \mathbf{R}$

$$\begin{bmatrix} A - j\omega I & B_1 \\ C_2 & D_2 \end{bmatrix} \tag{A.4}$$

has full row rank.

Now, define the matrices

$$A_1 := A - B_2D_1'C_1 \quad A_2 := A - B_1D_2'C_2.$$

In the result given below we will make use of the following algebraic Riccati equations for X and Y, respectively.

$$A_1'X + XA_1 + X(B_1B_1' - B_2B_2')X + C_1'(I - D_1D_1')C_1 = 0 \tag{A.5}$$

$$A_2Y + YA_2' + Y(C_1'C_1 - C_2'C_2)Y + B_1(I - D_2'D_2)B_1' = 0. \tag{A.6}$$

The next result is a slightly modified version of Theorem 1 in Glover and Doyle (1988).

Theorem A.7. Consider the feedback system of Fig. 2. Then there exists an admissible controller C such that $\|T(C)\|_\infty < 1$ if and only if the following conditions are satisfied:

(i) The ARE (A.5) admits the stabilizing solution X_s, and $X_s \geq 0$.
(ii) The ARE (A.6) admits the stabilizing solution Y_s, and $Y_s \geq 0$.
(iii) $\rho(Y_sX_s) < 1$.

Moreover, when these conditions hold one such a controller is

$$C := \left[\begin{array}{c|c} A_{F_s} + (B_1 + Z_sL_sD_2)B_1'X_s + Z_sL_sC_2 & -Z_sL_s \\ \hline F_s & 0 \end{array}\right], \tag{A.7}$$

where

$$L_s := -(Y_sC_2' + B_1D_2'), \quad F_s := -(B_2'X_s + D_1'C_1),$$
$$A_{F_s} := A + B_2F_s, \quad Z_s := (I - Y_sX_s)^{-1}. \qquad \text{(A.8b)}$$