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Isovariant maps and the Borsuk-Ulam theorem

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Abstract

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The classical Borsuk-Ulan theorem asserts that if a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$ commutes with the antipodal map and sends only the origin to the origin then $n \leq m$. Such a map is said to be isovariant with respect to the $\mathbb{Z}_2$ action defined by the antipodal map. In this paper it is shown that there is a wide class of compact Lie groups, BUG, with the property that if $G \in$ BUG then any G-isovariant map $f: V \to W$ between representations of G with $V^G = 0$ must raise dimension, i.e., dimension $V \leq$ dimension W. It is conjectured that every compact Lie group is in BUG.

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The Borsuk-Ulam theorem states that, if a continuous map from $\mathbb{R}^n$ to $\mathbb{R}^m$ commutes with the antipodal map and sends only the origin to the origin, then $n \leq m$. There have been many generalizations and extensions of this provocative result cf. [2, 5-8, 10-15, 18-23, 26, 28, 29]. We couch the theorem in the language of transformation groups: if there exists an isovariant map from a representation V of $\mathbb{Z}_2$ to a representation W of $\mathbb{Z}_2$ then the "dimension" of W must be greater than the "dimension" of V. (We place quotes around dimension because we must ignore the dimension of the fixed point set—see Remark 1.) We consider then the obvious generalization to Lie groups other than $\mathbb{Z}_2$.

Isovariant maps are important in their own right; they arise in the classification of G-spaces [24] and in the study of equivariant surgery [1]. If we consider G-spaces as being stratified by orbit types then isovariant maps are just the strata preserving equivariant maps [25]. Very little is known about the existence and classification of isovariant maps between general G-spaces; the special case where the spaces are representation spaces of the group is a natural place to start the study.

In this note we show that the second formulation of the Borsuk-Ulam theorem can be extended to a very wide class of groups that we dub the Borsuk-Ulam groups, BUGs. In fact, we conjecture that all compact Lie groups are BUGs.

Definitions

A map $f: X \to Y$ between G-spaces is *equivariant* if $f(gx) = gf(x)$ for all $x \in X$, $g \in G$; f is *isovariant* if, in addition, $f(gx) = f(x)$ implies $gx = x$.

For any G-space X we denote by X^G the fixed point set of X; $X^G = \{x \in X \mid gx = x$ for all $g \in G\}$. For any point $x \in X$ we denote by G_x the isotopy subgroups of G at x; $G_x = \{g \in G \mid gx = x\}$.

If X is a real (respectively complex) vector space and the action of G on X is via real (respectively complex) linear transformations then we say that X is a real (respectively complex) representation (space) of G.

Remark 1. If X is any G-space and Z is a trivial G-space, i.e., $gz = z$ for all $g \in G$, $z \in Z$, then the projection map $\pi: X \times Z \to X$ is an isovariant map. If V is a representation of G then $V \approx V/V^G \times V^G$; hence there is an isovariant map $\pi: V \to V/V^G$. Since inclusions are isovariant we also have an isovariant map $V/V^G \to V$. Thus, there exists an isovariant map $f: V \to W$ if and only if there exists an isovariant map $f': V/V^G \to W/W^G$.

Definition. A compact Lie group G is said to be a *Borsuk-Ulam group, BUG*, if, whenever we have an isovariant map $f: V \to W$ between representations of the group G, dimension $V/V^G \leq$ dimension W/W^G.

We recall that a real (respectively complex) representation is determined up to a real (respectively complex) linear equivariant isomorphism by the character of X, $\chi_X: G \to \mathbb{R}$, (respectively $\chi_X: G \to \mathbb{C}$); $\chi_X(g) =$ trace of the linear transformation defined by g. Note also that $\chi_X(e) =$ the dimension of X. If X is a representation of G then the dimension of X^G can be easily computed from χ_X; for finite groups $\dim X^G = \sum \chi_X(g)/|G|$ where the sum is over all $g \in G$ and $|G|$ denotes the order of G—more generally, we replace the sum by the integral over G (with respect to Haar measure) of χ_X. Thus, for finite groups,

$$\text{dimension } X/X^G = \chi_X(e) - (\sum \chi_X(g))/|G| = \sum (\chi_X(e) - \chi_X(g))/|G|. \quad (1)$$

Remark 2. Note that if $f: V \to W$ is isovariant then $f \times f: V \times V \to W \times W$ is also isovariant and both $V \times V$ and $W \times W$ may be considered complex representations of G. Thus, in view of (1) we may restate the dimension conclusion as

$$(*) \qquad \sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g))/|G| \geq 0.$$

Proposition 3. *If H is a closed normal subgroup of the BUG G, then G/H is a BUG.*

Proof. Any representation of G/H may be pulled back to the group G via the prjection π and any G/H-isovariant map is then seen to be G-isovariant. □

Proposition 4. *If $1 \to H \to G \to K \to 1$ is an exact sequence of compact Lie groups and H and K are BUGS, then G is a BUG.*

Proof. Let V and W be representations of G and let $f: V \to W$ be an isovariant map. Since f is also an H-isovariant map and H is a BUG we have $\dim V/V^H \leq \dim W/W^H$ or

$$\dim V - \dim V^H \leq \dim W - \dim W^H. \tag{2}$$

Now the spaces V^H and W^H are representation spaces for the group $K \approx G/H$ since H is normal in G; moreover, $f|_{V^H}: V^H \to W^H$ is a K-isovariant map. Thus, since K is a BUG we have that $\dim V^H/(V^H)^K \leq \dim W^H/(W^H)^K$. However, $(V^H)^K \approx V^G$ and $(W^H)^K \approx W^G$; thus $\dim V^H/V^G \leq \dim W^H/W^G$ or

$$\dim V^H - \dim V^G \leq \dim W^H - \dim W^G. \tag{3}$$

Combining (2) and (3) yields $\dim V - \dim V^G \leq \dim W - \dim W^G$; thus G is a BUG. □

Remark 5. The above shows, in fact, that

$$\begin{aligned}
&(\dim W/W^G) - (\dim V/V^G) \\
&\quad = (\dim W/W^H) - (\dim V/V^H) + (\dim W^H/W^G) - (\dim V^H/V^G) \\
&\quad \geq (\dim W/W^H) - (\dim V/V^H)
\end{aligned}$$

whenever H is a normal subgroup of G. Restating this in terms of characters

$$\begin{aligned}
&\sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g))/|G| \\
&\quad \geq \sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g))/|H|
\end{aligned}$$

where the sum on the left is over the group G and the sum on the right is over H.

Corollary 6. *If G is a compact Lie group and the identity component of G, G_0, is a BUG and the factor group, G/G_0, is a BUG, then G is a BUG.*

Recall that a composition series for a finite group G is a collection of subgroups, G_j, $0 \leq j \leq r$, such that $G_0 = e$, $G_r = G$, and G_j is a maximal normal subgroup of G_{j+1} for $0 \leq j \leq r-1$. The factor groups, G_{j+1}/G_j, are finite simple groups and are called the composition factors of G; they are independent of the choice of the composition series.

Proposition 7. *If all the composition factors of the finite group G are BUGs, then G is a BUG.*

Proof. If G has only one factor, i.e., $G = G_1$, then $G_1/G_0 \approx G$ and hence, G is a BUG. Assume inductively that the proposition is true for groups with n factors and let $G = G_{n+1}$. Consider the sequence $1 \to G_n \to G_{n+1} \to G_{n+1}/G_n \to 1$; G_n is a BUG by our inductive hypothesis and G_{n+1}/G_n is a composition factor, hence $G = G_{n+1}$ is a BUG by Proposition 4. □

In view of Proposition 7 it behooves us to find finite simple BUGs.

Proposition 8. *If p is a prime, then $\mathbb{Z}_p$ is a BUG.*

The case $p = 2$ is the classical Borsuk–Ulam Theorem, cf. [4, 9, 27]. Proofs for p an odd prime can be found in [14, 21].

An immediate consequence of Propositions 4 and 8 is that any finite Abelian group is a BUG. Almost as obvious is the following:

Proposition 9. *The n-torus, T^n, is a BUG.*

Proof. Using the exact sequence $1 \to T^{n-1} \to T^n \to S^1 \to 1$ and Proposition 4 we see that it is enough to prove the proposition for $G = S^1$. We now suppose that V and W are representations of S^1 and that $f: V \to W$ is an S^1-isovariant map. There are only a finite number of subgroups of S^1 that occur as isotropy subgroups in V or W, say $\mathbb{Z}_{n_1}, \mathbb{Z}_{n_2}, \ldots, \mathbb{Z}_{n_r}$, with $n_i < n_{i+1}$ for all i and possibly also S^1. Choose a prime, p, such that $p > n_r$. Considering the map f as a $\mathbb{Z}_p$-isovariant map and using Proposition 8 we have that the dimension $V/V^{\mathbb{Z}_p} \leq$ dimension $W/W^{\mathbb{Z}_p}$. Moreover, $V^{\mathbb{Z}_p} = V^{S^1}$ and $W^{\mathbb{Z}_p} = W^{S^1}$ and thus dimension $V/V^{S^1} \leq$ dimension W/W^{S^1}. □

Corollary 10. *If G_0 is a toral group and G/G_0 is a BUG, then G is a BUG.*

Remark 11. If G is a finite group and $g_1 \in G$, $g_2 \in G$, we say that g_1 is algebraically conjugate to g_2, $g_1 \sim g_2$, if $g_1 = g_2^r$ for some r prime to the order of G; equivalently, $g_1 \sim g_2$ if $\langle g_1 \rangle = \langle g_2 \rangle$ where $\langle g \rangle$ denotes the group generated by g. Algebraic conjugacy is clearly an equivalence relation; thus a finite group is the disjoint union of its conjugacy classes.

Definition. An *integer n* is said to *satisfy the prime condition* if we have $\sum_{i=1}^{s} 1/p_i \leq 1$, where $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$. p_i prime and $1 \leq r_i$ for $1 \leq i \leq s$.

If G is a finite group and $g \in G$ we denote by $|g|$ the order of g and by $|G|$ the order of G.

Definition. A *finite simple group* G is said to satisfy the prime condition if, for each $g \in G$, the integer $|g|$ satisfies the prime condition. A *finite group* G is said to satisfy the prime condition if each composition factor of G satisfies the prime condition.

There are many simple groups that satisfy the prime condition. For example, among the 26 sporadic simple groups we have that the Mathieu groups, M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, the Janko groups, J_1, J_2, J_3 (but not J_4), the Suzuki group, Suz, the Higman, Sims group, HS, the Held/Higman, McKay group, He, the O'Nan/Sims group, O'N, and the Rudvalis group, Ru, all satisfy the prime condition; the other 13 sporadic groups do not. See [3]. The alternating groups, A_n, for $n \leqslant 11$, satisfy the prime condition but for $n \geqslant 12$ they do not.

Our main theorem is:

Theorem 12. *If G satisfies the prime condition, then G is a BUG.*

The proof of Theorem 12 will require a lemma.

Lemma 13. *If $f: V \to W$ is a C-isovariant map, where C is a cyclic group and $|C|$ satisfies the prime condition, then*

$$\sum_{\text{gen } C} (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g)) \geqslant 0.$$

Remark 14. Note that any cyclic *group*, C, will certainly satisfy the prime condition; rather, we require that the *integer*, $|C|$, satisfy the prime condition. Note also that the set of generators of C is just the algebraic conjucagy class of any generator of C.

Proof of Theorem 12. By Proposition 4 it is sufficient to consider the case when G is simple. Let $f: V \to W$ be a G-isovariant map; by Remark 2 we must show

$$\sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g))/|G| \geqslant 0$$

where the sum is over all $g \in G$. By Remark 11 it suffices to show that for each conjugacy class we have

$$\sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g)) \geqslant 0$$

where the sum is taken over the algebraic conjugacy class.

Let $[g]$ be an algebraic conjugacy class; then $[g]$ is the set of generators of the cyclic group $\langle g \rangle = C$. Since G is a finite simple group satisfying the prime condition, $|g| = |C|$ satisfies the prime condition; furthermore, the map f is also C-isovariant and hence, by Lemma 13 we have that $\sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g)) \geqslant 0$ where the sum is taken over the algebraic conjugacy class. □

Proof of Lemma 13. We will prove a slightly stronger statement. Let $h(g)=\chi_W(e)-\chi_W(g)-\chi_V(e)+\chi_V(g)$; we claim that $0\leq\sum_{\mathrm{gen}\,C} h(g)\leq\sum_C h(g)$. We proceed by induction on the order of C. For $|C|=1$ the claim is trivial. Now, $\sum_C h(g)=\sum_{C\supset C'}\sum_{\mathrm{gen}\,C'} h(g)+\sum_{\mathrm{gen}\,C} h(g)$. Furthermore, for C' a proper subgroup of C we have by induction that $0\leq\sum_{\mathrm{gen}\,C'} h(g)$ and hence that

$$\sum_{\mathrm{gen}\,C} h(g)=\sum_C h(g)-\sum_{C\supset C'}\sum_{\mathrm{gen}\,C'} h(g)\leq\sum_C h(g).$$

We now prove the other half of the inequality. Let $|C|=p_1^{r_1}p_2^{r_2}\cdots p_s^{r_s}$, p_i prime and $1\leq r_i$ for $1\leq i\geq s$; $C=\langle g\rangle$. Then there are subgroups, C_i, of index p_i in C, $C_i=\langle g^{p_i}\rangle$, such that $C=\mathrm{gen}\,C\cup C_1\cup C_2\cup\cdots\cup C_S$. Note that although this union is not disjoint each summand is the union of algebraic conjugacy classes. Thus we may write $\sum_C h(g)=\sum_{\mathrm{gen}\,C} h(g)+\sum_{i=1}^{s}\sum_{C_i} h(g)-$(the sum over those algebraic conjugacy classes contained in more than one C_i). (Precisely, we must subtract the sum over an algebraic conjugacy class $r-1$ times if the class is contained in r of the C_i's.) Now each such algebraic conjugacy class is of the form gen C' for some $C'\subset C$ and thus by our inductive hypothesis the sum over such an algebraic conjugacy class is nonnegative. Hence, $\sum_{\mathrm{gen}\,C} h(g)\geq\sum_C h(g)-\sum_{i=1}^{s}\sum_{C_i} h(g)$. We now recall that for any $C_i\subset C$, $\sum_C h(g)/|C|\geq\sum_{C_i} h(g)/|C_i|$ by Remark 5; thus,

$$\sum_{\mathrm{gen}\,C} h(g)\geq\sum_C h(g)-\sum_{i=1}^{s}\frac{|C|}{|C_i|}\sum_C h(g)=\sum_C h(g)\left(1-\sum_{i=1}^{s}\frac{1}{p_i}\right)\geq 0$$

since $|C|$ satisfies the prime condition. □

Remark 15. It is reasonable to conjecture that every finite group is a BUG; the prime condition apparently required in Theorem 12 might be eliminated by a better argument. However, Lemma 13 is definitely false if $|C|$ does not satisfy the prime condition as the following example shows.

Example. Let $C=\mathbb{Z}_{30}$ and let $g\in C$ be a generator; note that $30=|C|$ does not satisfy the prime condition. Define one-dimensional complex representations, V_j, of C by $gz=\zeta^j z$ where V_j is a copy of the complex numbers, $z\in V_j$ and $\zeta=e^{\pi i/15}$. Let $V=V_1\oplus V_1$ and let $W=V_2\oplus V_3\oplus V_5$. Let $f: V\to W$ be given by $f(z,w)=(z^2, z^3+w^3, w^5)$. One quckly verifies that f is isovariant. We have that $\sum_C h(g)=|C|\cdot(\dim W/W^C-\dim V/V^C)=30$; simiiarly, $\sum_{C_i} h(g)=15$ for $C_i=\mathbb{Z}_{15}$, $\sum_{C_i} h(g)=10$ for $C_i=\mathbb{Z}_{10}$, $\sum_{C_i} h(g)=6$ for $C_i=\mathbb{Z}_6$, and $\sum_{C_i} h(g)=0$ for $C_i=\mathbb{Z}_2$ or $\mathbb{Z}_3$ or $\mathbb{Z}_5$. Putting these facts together yields $\sum_{\mathrm{gen}\,C} h(g)=-1<0$.

We close with some open questions.

(1) Is a subgroup of a BUG a BUG?

(2) Which connected groups are BUGs?

(3) Which finite groups are BUGs?

(4) Does there exist a group G, representations V, W of G and an *equivariant* map $f: S(V) \to S(W)$ such that $\dim V > \dim W$ and $W^G = 0$? ($S(V)$ denotes the unit sphere in V.)

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