Isovariant maps and the Borsuk–Ulam theorem

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Abstract


The classical Borsuk–Ulam theorem asserts that if a continuous map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) commutes with the antipodal map and sends only the origin to the origin then \( n \leq m \). Such a map is said to be isovariant with respect to the \( \mathbb{Z}_2 \) action defined by the antipodal map. In this paper it is shown that there is a wide class of compact Lie groups, \( \text{BUG} \), with the property that if \( G \in \text{BUG} \) then any \( G \)-isovariant map \( f: V \to W \) between representations of \( G \) with \( V^G = 0 \) must raise dimension, i.e., \( \text{dimension } V \leq \text{dimension } W \). It is conjectured that every compact Lie group is in \( \text{BUG} \).

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The Borsuk–Ulam theorem states that, if a continuous map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) commutes with the antipodal map and sends only the origin to the origin, then \( n \leq m \). There have been many generalizations and extensions of this provocative result cf. [2, 5–8, 10–15, 18–23, 26, 28, 29]. We couch the theorem in the language of transformation groups: if there exists an isovariant map from a representation \( V \) of \( \mathbb{Z}_2 \) to a representation \( W \) of \( \mathbb{Z}_2 \) then the “dimension” of \( W \) must be greater than the “dimension” of \( V \). (We place quotes around dimension because we must ignore the dimension of the fixed point set—see Remark 1.) We consider then the obvious generalization to Lie groups other than \( \mathbb{Z}_2 \).

Isovariant maps are important in their own right; they arise in the classification of \( G \)-spaces [24] and in the study of equivariant surgery [1]. If we consider \( G \)-spaces as being stratified by orbit types then isovariant maps are just the strata preserving equivariant maps [25]. Very little is known about the existence and classification...
of isovariant maps between general $G$-spaces; the special case where the spaces are representation spaces of the group is a natural place to start the study.

In this note we show that the second formulation of the Borsuk-Ulam theorem can be extended to a very wide class of groups that we dub the Borsuk-Ulam groups, BUGs. In fact, we conjecture that all compact Lie groups are BUGs.

**Definitions**

A map $f: X \to Y$ between $G$-spaces is **equivariant** if $f(gx) = gf(x)$ for all $x \in X$, $g \in G$; $f$ is **isovariant** if, in addition, $f(gx) = f(x)$ implies $gx = x$.

For any $G$-space $X$ we denote by $X^G$ the fixed point set of $X$; $X^G = \{x \in X | gx = x$ for all $g \in G\}$. For any point $x \in X$ we denote by $G_x$ the isotopy subgroups of $G$ at $x$; $G_x = \{g \in G | gx = x\}$.

If $X$ is a real (respectively complex) vector space and the action of $G$ on $X$ is via real (respectively complex) linear transformations then we say that $X$ is a real (respectively complex) representation (space) of $G$.

**Remark 1.** If $X$ is any $G$-space and $Z$ is a trivial $G$-space, i.e., $gz = z$ for all $g \in G$, $z \in Z$, then the projection map $\pi: X \times Z \to X$ is an isovariant map. If $V$ is a representation of $G$ then $V = V/ V^G \times V^G$; hence there is an isovariant map $\pi: V \to V/ V^G$. Since inclusions are isovariant we also have an isovariant map $V/ V^G \to V$.

Thus, there exists an isovariant map $f: V \to W$ if and only if there exists an isovariant map $f^*: V/ V^G \to W/ W^G$.

**Definition.** A compact Lie group $G$ is said to be a **Borsuk-Ulam group**, BUG, if, whenever we have an isovariant map $f: V \to W$ between representations of the group $G$, dimension $V/ V^G \leq$ dimension $W/ W^G$.

We recall that a real (respectively complex) representation is determined up to a real (respectively complex) linear equivariant isomorphism by the character of $X$, $\chi_X: G \to \mathbb{R}$ (respectively $\chi_X: G \to \mathbb{C}$); $\chi_X(g)$ = trace of the linear transformation defined by $g$. Note also that $\chi_X(e)$ = the dimension of $X$. If $X$ is a representation of $G$ then the dimension of $X^G$ can be easily computed from $\chi_X$; for finite groups $\dim X^G = \sum \chi_X(g)/|G|$ where the sum is over all $g \in G$ and $|G|$ denotes the order of $G$—more generally, we replace the sum by the integral over $G$ (with respect to Haar measure) of $\chi_X$. Thus, for finite groups,

$$\text{dimension } X/X^G = \chi_X(e) - \left(\sum \chi_X(g)\right)/|G| = \sum (\chi_X(e) - \chi_X(g))/|G|. \quad (1)$$

**Remark 2.** Note that if $f: V \to W$ is isovariant then $f\times f: V \times V \to W \times W$ is also isovariant and both $V \times V$ and $W \times W$ may be considered complex representations of $G$. Thus, in view of (1) we may restate the dimension conclusion as

$$\sum (\chi_W(e) - \chi_W(g) - \chi_V(e) + \chi_V(g))/|G| \geq 0. \quad (*)$$
Proposition 3. If $H$ is a closed normal subgroup of the BUG $G$, then $G/H$ is a BUG.

Proof. Any representation of $G/H$ may be pulled back to the group $G$ via the projection $\pi$ and any $G/H$-isovariant map is then seen to be $G$-isovariant. \(\square\)

Proposition 4. If $1 \to H \to G \to K \to 1$ is an exact sequence of compact Lie groups and $H$ and $K$ are BUGS, then $G$ is a BUG.

Proof. Let $V$ and $W$ be representations of $G$ and let $f: V \to W$ be an isovariant map. Since $f$ is also an $H$-isovariant map and $H$ is a BUG we have $\dim V/V^H \leq \dim W/W^H$ or

$$\dim V - \dim V^H \leq \dim W - \dim W^H. \quad (2)$$

Now the spaces $V^H$ and $W^H$ are representation spaces for the group $K = G/H$ since $H$ is normal in $G$; moreover, $f|_{V^H}: V^H \to W^H$ is a $K$-isovariant map. Thus, since $K$ is a BUG we have that $\dim V^H/(V^H)^K \leq \dim W^H/(W^H)^K$. However, $(V^H)^K = V^G$ and $(W^H)^K = W^G$, thus $\dim V^H/V^G \leq \dim W^H/W^G$ or

$$\dim V^H - \dim V^G \leq \dim W^H - \dim W^G. \quad (3)$$

Combining (2) and (3) yields $\dim V - \dim V^G \leq \dim W - \dim W^G$; thus $G$ is a BUG. \(\square\)

Remark 5. The above shows, in fact, that

$$(\dim W/W^G) - (\dim V/V^G)$$

$$= (\dim W/W^H) - (\dim V/V^H) + (\dim W^H/W^G) - (\dim V^H/V^G)$$

$$\geq (\dim W/W^H) - (\dim V/V^H)$$

whenever $H$ is a normal subgroup of $G$. Restating this in terms of characters

$$\sum (\chi_V(e) - \chi_V(g) - \chi_V(e) + \chi_V(g))/|G|$$

$$\geq \sum (\chi_V(e) - \chi_V(g) - \chi_V(e) + \chi_V(g))/|H|$$

where the sum on the left is over the group $G$ and the sum on the right is over $H$.

Corollary 6. If $G$ is a compact Lie group and the identity component of $G$, $G_0$, is a BUG and the factor group, $G/G_0$, is a BUG, then $G$ is a BUG.

Recall that a composition series for a finite group $G$ is a collection of subgroups, $G_j$, $0 \leq j \leq r$, such that $G_0 = e$, $G_r = G$, and $G_j$ is a maximal normal subgroup of $G_{j+1}$ for $0 \leq j \leq r - 1$. The factor groups, $G_{j+1}/G_j$, are finite simple groups and are called the composition factors of $G$; they are independent of the choice of the composition series.
Proposition 7. If all the composition factors of the finite group $G$ are BUGs, then $G$ is a BUG.

Proof. If $G$ has only one factor, i.e., $G = G_1$, then $G_1/G_0 \approx G$ and hence, $G$ is a BUG. Assume inductively that the proposition is true for groups with $n$ factors and let $G = G_{n+1}$. Consider the sequence $1 \to G_n \to G_{n+1} \to G_n/G_{n+1} \to 1$; $G_n$ is a BUG by our inductive hypothesis and $G_{n+1}/G_n$ is a composition factor, hence $G = G_{n+1}$ is a BUG by Proposition 4. □

In view of Proposition 7 it behooves us to find finite simple BUGs.

Proposition 8. If $p$ is a prime, then $\mathbb{Z}_p$ is a BUG.

The case $p = 2$ is the classical Borsuk–Ulam Theorem, cf. [4, 9, 27]. Proofs for $p$ an odd prime can be found in [14, 21].

An immediate consequence of Propositions 4 and 8 is that any finite Abelian group is a BUG. Almost as obvious is the following:

Proposition 9. The $n$-torus, $T^n$, is a BUG.

Proof. Using the exact sequence $1 \to T^{n-1} \to T^n \to S^1 \to 1$ and Proposition 4 we see that it is enough to prove the proposition for $G = S^1$. We now suppose that $V$ and $W$ are representations of $S^1$ and that $f: V \to W$ is an $S^1$-isovariant map. There are only a finite number of subgroups of $S^1$ that occur as isotropy subgroups in $V$ or $W$, say $Z_{n_1}, Z_{n_2}, \ldots, Z_{n_s}$, with $n_i < n_{i+1}$ for all $i$ and possibly also $S^1$. Choose a prime, $p$, such that $p > n_s$. Considering the map $f$ as a $\mathbb{Z}_p$-isovariant map and using Proposition 8 we have that the dimension $V/V^{2r} \leq$ dimension $W/W^{2r}$. Moreover, $V^{2r} = V^{S^1}$ and $W^{2r} = W^{S^1}$ and thus dimension $V/V^{S^1} \leq$ dimension $W/W^{S^1}$. □

Corollary 10. If $G_n$ is a toral group and $G/G_n$ is a BUG, then $G$ is a BUG.

Remark 11. If $G$ is a finite group and $g_1 \in G, g_2 \in G$, we say that $g_1$ is algebraically conjugate to $g_2, g_1 \sim g_2$, if $g_1 = g_2^r$ for some $r$ prime to the order of $G$; equivalently, $g_1 \sim g_2$ if $\langle g_1 \rangle = \langle g_2 \rangle$ where $\langle g \rangle$ denotes the group generated by $g$. Algebraic conjugacy is clearly an equivalence relation; thus a finite group is the disjoint union of its conjugacy classes.

Definition. An integer $n$ is said to satisfy the prime condition if we have $\sum_{i=1}^s 1/p_i \leq 1$, where $n = p_1^{r_1}p_2^{r_2} \cdots p_s^{r_s}$, $p_i$ prime and $1 \leq r_i$ for $1 \leq i \leq s$.

If $G$ is a finite group and $g \in G$ we denote by $|g|$ the order of $g$ and by $|G|$ the order of $G$. 
Definition. A finite simple group $G$ is said to satisfy the prime condition if, for each $g \in G$, the integer $|g|$ satisfies the prime condition. A finite group $G$ is said to satisfy the prime condition if each composition factor of $G$ satisfies the prime condition.

There are many simple groups that satisfy the prime condition. For example, among the 26 sporadic simple groups we have that the Mathieu groups, $M_{11}$, $M_{12}$, $M_{22}$, $M_{23}$, $M_{24}$, the Janko groups, $J_1$, $J_2$, $J_3$ (but not $J_4$), the Suzuki group, Suz, the Higman, Sims group, HS, the Held/Higman, McKay group, He, the O’Nan/Sims group, O’N, and the Rudvalis group, Ru, all satisfy the prime condition; the other 13 sporadic groups do not. See [3]. The alternating groups, $A_n$, for $n \leq 11$, satisfy the prime condition but for $n > 12$ they do not.

Our main theorem is:

Theorem 12. If $G$ satisfies the prime condition, then $G$ is a BUG.

The proof of Theorem 12 will require a lemma.

Lemma 13. If $f: V \to W$ is a $C$-isovariant map, where $C$ is a cyclic group and $|C|$ satisfies the prime condition, then

$$\sum_{\text{gen } C} (\chi_w(e) - \chi_w(g) - \chi_v(e) + \chi_v(g)) = 0.$$

Remark 14. Note that any cyclic group, $C$, will certainly satisfy the prime condition; rather, we require that the integer, $|C|$, satisfy the prime condition. Note also that the set of generators of $C$ is just the algebraic conjugacy class of any generator of $C$.

Proof of Theorem 12. By Proposition 4 it is sufficient to consider the case when $G$ is simple. Let $f: V \to W$ be a $G$-isovariant map; by Remark 2 we must show

$$\sum (\chi_w(e) - \chi_w(g) - \chi_v(e) + \chi_v(g))/|G| \geq 0$$

where the sum is over all $g \in G$. By Remark 11 it suffices to show that for each conjugacy class we have

$$\sum (\chi_w(e) - \chi_w(g) - \chi_v(e) + \chi_v(g)) \geq 0$$

where the sum is taken over the algebraic conjugacy class.

Let $[g]$ be an algebraic conjugacy class; then $[g]$ is the set of generators of the cyclic group $\langle g \rangle = C$. Since $G$ is a finite simple group satisfying the prime condition, $|g| = |C|$ satisfies the prime condition; furthermore, the map $f$ is also $C$-isovariant and hence, by Lemma 13 we have that $\sum (\chi_w(e) - \chi_w(g) - \chi_v(e) + \chi_v(g)) \geq 0$ where the sum is taken over the algebraic conjugacy class. □
Proof of Lemma 13. We will prove a slightly stronger statement. Let \( h(g) = \chi_w(e) - \chi_w(g) - \chi_v(e) + \chi_v(g) \); we claim that \( 0 \leq \sum_{c \in C} h(g) \leq \sum_{c \in C} h(g) \). We proceed by induction on the order of \( C \). For \(|C| = 1\) the claim is trivial. Now, \( \sum_{c \in C} h(g) = \sum_{c \in C} \sum_{c \in C} h(g) + \sum_{c \in C} h(g) \). Furthermore, for \( C' \) a proper subgroup of \( C \) we have by induction that \( 0 \leq \sum_{c \in C} h(g) \) and hence that

\[
\sum_{c \in C} h(g) = \sum_{c \in C} h(g) - \sum_{c \in C} \sum_{c \in C} h(g) \leq \sum_{c \in C} h(g).
\]

We now prove the other half of the inequality. Let \(|C| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \), \( p_i \) prime and \( 1 \leq r_i \) for \( 1 \leq i \leq s \); \( C = \langle g \rangle \). Then there are subgroups, \( C_i \), of index \( p_i \) in \( C \), \( C_i = \langle g^{p_i} \rangle \), such that \( C = \text{gen} C \cup C_1 \cup C_2 \cup \cdots \cup C_s \). Note that although this union is not disjoint each summand is the union of algebraic conjugacy classes. Thus we may write \( \sum_{c \in C} h(g) = \sum_{c \in C} h(g) + \sum_{i=1}^{s} \sum_{c \in C_i} h(g) - \text{the sum over those algebraic conjugacy classes contained in more than one } C_i \). (Precisely, we must subtract the sum over an algebraic conjugacy class \( r-1 \) times if the class is contained in \( r \) of the \( C_i \)'s.) Now each such algebraic conjugacy class is of the form \( \text{gen} C' \) for some \( C' \subset C \) and thus by our inductive hypothesis the sum over such an algebraic conjugacy class is nonnegative. Hence, \( \sum_{c \in C} h(g) \geq \sum_{c \in C} h(g) - \sum_{i=1}^{s} \sum_{c \in C_i} h(g) \). We now recall that for any \( C_i \subset C \), \( \sum_{c \in C} h(g)/|C| \geq \sum_{c \in C} h(g)/|C_i| \) by Remark 5; thus,

\[
\sum_{c \in C} h(g) \geq \sum_{c \in C} h(g) - \sum_{i=1}^{s} \frac{|C|}{|C_i|} \sum_{c \in C} h(g) = \sum_{c \in C} h(g) \left( 1 - \sum_{i=1}^{s} \frac{1}{p_i} \right) \geq 0
\]

since \(|C|\) satisfies the prime condition. \( \square \)

Remark 15. It is reasonable to conjecture that every finite group is a BUG; the prime condition apparently required in Theorem 12 might be eliminated by a better argument. However, Lemma 13 is definitely false if \(|C|\) does not satisfy the prime condition as the following example shows.

Example. Let \( C = \mathbb{Z}_{30} \) and let \( g \in C \) be a generator; note that \( 30 = |C| \) does not satisfy the prime condition. Define one-dimensional complex representations. \( V_j \), of \( C \) by \( gz = z^j \) where \( V_j \) is a copy of the complex numbers, \( z \in V_j \) and \( \zeta = e^{\pi i/15} \). Let \( V = V_1 \oplus V_2 \) and let \( W = V_1 \oplus V_2 \oplus V_3 \). Let \( f: V \to W \) be given by \( f(z, w) = (z^2, z^3 + w^3, w^5) \). One quickly verifies that \( f \) is isovariant. We have that \( \sum_{c \in C} h(g) = |C| \cdot (\dim W/ W^C - \dim V/ V^C) = 30 \); similarly, \( \sum_{c \in C} h(g) = 15 \) for \( C_i = \mathbb{Z}_{15} \), \( \sum_{c \in C} h(g) = 10 \) for \( C_i = \mathbb{Z}_{10} \), \( \sum_{c \in C} h(g) = 6 \) for \( C_i = \mathbb{Z}_6 \), and \( \sum_{c \in C} h(g) = 0 \) for \( C_i = \mathbb{Z}_2 \) or \( \mathbb{Z}_3 \) or \( \mathbb{Z}_5 \). Putting these facts together yields \( \sum_{c \in C} h(g) = -1 < 0 \).

We close with some open questions.

(1) Is a subgroup of a BUG a BUG?
(2) Which connected groups are BUGs?
(3) Which finite groups are BUGs?
(4) Does there exist a group \( G \), representations \( V \), \( W \) of \( G \) and an \textit{equivariant} map \( f : S(V) \rightarrow S(W) \) such that \( \dim V > \dim W \) and \( W^G = 0 \)? (\( S(V) \) denotes the unit sphere in \( V \).)

References


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