A Generalization of Webb's Theorem
to Auslander-Reiten Systems

ODILE GAROTTA

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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INTRODUCTION

Let $G$ be a finite group and $k$ be a field whose characteristic $p$ divides the order of $G$. In [3] we introduce the notion of Auslander-Reiten system of $G$ on a symmetric interior $G$-algebra $A$ (that is a symmetric $k$-algebra together with a homomorphism $\phi: G \to A^\times$), as a generalization of the notion of almost split sequences of $kG$-modules. The language of interior $G$-algebras proves to be useful to study their restriction to certain subgroups such as the defect group (vertex) of their extremity. Furthermore we show in [3] that each non projective primitive idempotent $i$ of $A^G$ (see Section 1) is the extremity of a unique Auslander-Reiten system, up to embedding of $A$ into other symmetric interior $G$-algebras and to conjugacy (cf. [3, VI], recall embeddings are those one-to-one homomorphisms $f: A \to B$ which satisfy $\text{Im } f = f(1)A f(1)$). Thus in particular the middle term of "the" Auslander-Reiten system terminating in $i$ is well defined (up to embedding and conjugacy), and we may look at its decomposition into primitive idempotents. Now if we set $A[G] = A \otimes kG$, we have a symmetric algebra again, which is similar to $kG$ in many ways, and which enables us to view our systems of $G$ over $A$ ([3, I]) as short exact sequences of modules: to any idempotent $i$ of $A^G$, we associate the $A$-projective $A[G]$-module $iA$ (with $(b \otimes g) \cdot i a = \phi(g) i ab$). This way any system $\mathcal{S} = (i, i^\circ, i', d, d')$ of $G$ over $A$ determines an exact sequence of $A[G]$-modules

$$0 \longrightarrow i' A \xrightarrow{d'} i^\circ A \xrightarrow{d} iA \longrightarrow 0,$$

which is almost split if and only if the system $\mathcal{S}$ is an Auslander-Reiten system (cf. [3]). Moreover if $\mathcal{R}: 0 \to \Omega^2(iA) \to R \to iA \to 0$ is almost split on $A[G]$, then all its terms are $A$-projective and we can embed $A$ into the symmetric interior $G$-algebra $B = \text{End}_A(A \oplus \Omega^2(iA) \oplus R)$, where the
A GENERALIZATION OF WEBB'S THEOREM

A sequence $R$ gives rise to an Auslander-Reiten system of $G$ over $B$. Therefore the study of the Auslander-Reiten quiver of the algebra $A[G]$ (actually we may restrict ourselves to the components of $A$-projective modules), will provide us with information about the decomposition of the middle terms of Auslander-Reiten systems terminating, first in an idempotent of $A^G$, and then also in those idempotents which appear in such a middle term after several steps of "associating to each primitive non projective idempotent in the middle term, the Auslander-Reiten system terminating in it" (each such step may require embedding into a larger symmetric interior $G$-algebra). In this paper we generalize P. Webb's main theorem in [5] to the algebra $A[G]$. The proof depends on the construction of a subadditive function on a connected component of the stable Auslander-Reiten quiver: we show how the simple (non cohomological) construction given by T. Okuyama in [4] may be generalized to our setting.

**THEOREM 1.** Let $A$ be a connected component of the stable Auslander-Reiten quiver of $A[G]$. If $A$ contains $A$-projective $A[G]$-modules, then the tree class and the reduced graph of $A$ are both either a Dynkin diagram (finite or infinite) or a Euclidian diagram.

As a consequence we obtain information on the middle terms of Auslander-Reiten systems of $G$ over $A$:

**THEOREM 2.** Let $n$ (resp. $n_0$) be the number of idempotents (resp. of projective idempotents) in a primitive decomposition of the middle term of an Auslander-Reiten system of $G$ over $A$. Then $n \leq 5$, $n_0 \leq 1$, and $n - n_0 \leq 4$. Moreover if $k$ is algebraically closed we have $n - n_0 \leq 2$.

Theorem 2 follows from Theorem 1 via an analogous statement (which we omit here) about the maximum number of indecomposable direct summands in the middle term of an almost split sequence of $A$-projective $A[G]$-modules (see [1, 2.31.3, 4, and the comments above]).

1. **Notations**

Throughout the paper we let $A$ be a symmetric interior $G$-algebra and denote by $A^+$ the group of units of $A$. The action of $G$ on $A$ by $g \cdot a = \phi(g^{-1}) a \phi(g) = a^g$ makes it into a $G$-algebra. If $H$ is any subgroup of $G$, we denote by $A^H$ the algebra of $H$-fixed elements of $A$, and by $\text{Tr}_H^A : A \to A^H$ the relative trace map defined by $\text{Tr}_H^A(a) = \sum a^x$, where $x$ runs over $H$; its image is the two-sided ideal $A^H$ of $A^H$. Regarding our comments about Auslander-Reiten systems in the introduction, we refer the reader to [3] and recall that any idempotent $i$ of a $A^G$ may be written as a finite sum of
mutually orthogonal primitive idempotents of $A^G$, and that the corresponding set of primitive idempotents is unique up to $(A^G)^\times$-conjugacy. We call it a primitive decomposition $I$ of $i$. The subsets $I \cap A_1^G$ and $I \setminus A_1^G$ are then also unique up to $(A^G)^\times$-conjugacy. We say that $i$ is projective (over $G$) if $i \in A_1^G$. See also Section 2.1.

All our modules and algebras are finite dimensional $k$-spaces, and our modules are right modules. We let $\mathcal{M}(A)$ be the Green ring of $A$: as a group it is the free $\mathbb{Z}$-module generated by the isomorphism classes of indecomposable $A$-modules. We denote respectively by $\Pi M$, $\Omega M$, and $\Theta M$ a projective cover, a Heller translate and an injective hull of the module $M$, and if $N = \Omega M$, we write $M = \Omega^{-1} N$ and $\Omega N = \Omega^2 M$. The space of all projective homomorphisms from an $A$-module $M$ to an $A$-module $N$ (i.e., of those homomorphisms $M \to N$ which factor through a projective module) is denoted by $\text{Proj}_A(M, N)$. We use the notation $k_G$ for the trivial $kG$-module, and the symbol $\otimes$ without precision for tensor products over $k$.

We refer the reader to [1] for the terminology on quivers and subadditive functions.

2. BACKGROUND ON $A[G]$-MODULES

Let $M$ be an $A$-projective $A[G]$-module. The following are elementary facts:

1. The algebra $\text{End}_A(M)$ is a symmetric interior $G$-algebra and we have $\text{End}_{A[G]}(M) = \text{End}_A(M)^G$, $\text{Proj}_{A[G]}(M, M) = \text{End}_A(M)^G$.

Let $H$ be a subgroup of $G$. For any $A[G]$-module $M$, we denote by $\text{Res}_H^G(M)$ the $A[H]$-module obtained by restriction through the injection $A[H] \to A[G]$. We then extend this by linearity and consider the functor $\text{Res}_H^G$ from $\mathcal{M}(A[G])$ to $\mathcal{M}(A[H])$.

**Induction.** Just in the same way as with $kH$-modules, one may induce any $A[H]$-module $N$ to $A[G]$: we define the $A[G]$-module $\text{Ind}_H^G(N)$ by

$$\text{Ind}_H^G(N) = N \otimes_{A[H]} A[G],$$

with $A[G]$ acting on the right of the term $A[G]$. We then extend our definition linearly to a functor $\text{Ind}_H^G$ from $\mathcal{M}(A[H])$ to $\mathcal{M}(A[G])$.

2. If the $A[H]$-module $N$ is projective, then so is the $A[G]$-module $\text{Ind}_H^G(N)$. Moreover we have for all $N$: $\text{Ind}_H^G(\Pi N - \Omega N) = \Pi(\text{Ind}_H^G(N)) - \Omega(\text{Ind}_H^G(N))$.

Note that $N \otimes_{kH} kG$ also is an $A[G]$-module, with $A$ acting on $N$ only, so that the actions of $A$ and $kG$ are compatible. It is easy to show that

3. We have $\text{Ind}_H^G(N) \cong N \otimes_{kH} kG$, as $A[G]$-modules.
Now let \( M \) be an \( A[G] \)-module. For all \( kG \)-modules \( U \), the tensor product \( M \otimes U \) is an \( A[G] \)-module, with \( A \) acting on \( M \) and \( G \) acting on both \( M \) and \( U \).

4. One has \( M \otimes \text{Ind}_H^G(k_H) \cong \text{Ind}_H^G(\text{Res}_H^G(M)) \), as \( A[G] \)-modules.

**Proof.** We first view \( M \) as a \( kG \)-module only. We have the following sequence of \( kG \)-modules identities

\[
M \otimes \text{Ind}_H^G(k_H) \cong (\text{Res}_H^G(M) \otimes k_H) \otimes_{kH} kG \cong \text{Res}_H^G(M) \otimes_{kH} kG,
\]

and we may substitute the last expression with \( \text{Ind}_H^G(\text{Res}_H^G(M)) \) by 3; we then check that the corresponding actions of \( A \) are consistent (\( A \) acts on \( M \) only).

We consider the inner product \((\ , \ , )_G\) on \( \mathcal{M}(A[G]) \) obtained by extending the form \( \dim_k \text{Hom}_{A[G]}(\ , \ , ) \) bilinearly:

5. We have \( (\text{Ind}_H^G(N), M)_G = (N, \text{Res}_H^G(M))_H \), for any \( N \) in \( \mathcal{M}(A[H]) \) and any \( M \) in \( \mathcal{M}(A[G]) \).

6. For all \( A[G] \)-modules \( L \) and \( M \), we have \( (L, M)_G = (\Pi L - \Omega L, \Pi M - \Omega M)_G \).

**Proof.** This follows by applying successively statements (ii) and (i) of the elementary

**LEMMA** (Notations of 6). (i) We have \( (L, \Pi M - \Omega M)_G = \dim \text{Proj}_{A[G]}(L, M) \).

(ii) We have \( (L, M)_G = \dim \text{Proj}_{A[G]}(\Pi L, M) - \dim \text{Proj}_{A[G]}(\Omega L, M) \).

**Proof.** (i) We apply the left exact functor \( \text{Hom}_{A[G]}(L, \ ) \) to the projective cover of \( M \): the range of the second morphism is precisely \( \text{Proj}_{A[G]}(L, M) \).

(ii) Projective and injective modules coincide on the symmetric algebra \( A[G] \). Thus the space \( \text{Proj}_{A[G]}(\Omega L, M) \) is precisely the range of the second morphism in the exact sequence \( 0 \rightarrow \text{Hom}_{A[G]}(L, M) \rightarrow \text{Hom}_{A[G]}(\Pi L, M) \rightarrow \text{Hom}_{A[G]}(\Omega L, M) \).

### 3. Periodic Modules

In the following we fix an \( A \)-projective \( A[G] \)-module \( X \) which is indecomposable and non projective, and denote by \( \Delta \) the connected component of the stable Auslander-Reiten quiver of \( A[G] \) which contains \( X \). Since \( X \) is not projective, there exists a minimal \( p \)-subgroup \( P \) of \( G \) \( (P \neq 1) \), such that \( \text{Res}_P^G(X) \) is not projective (cf. Section 2.1). We choose an
indecomposable direct summand $Y$ of $\text{Res}^G_P(X)$ which is not projective, and let $Q$ be a maximal subgroup of $P$. So the module $\text{Res}^G_P(Y)$ is projective. Following Okuyama, we turn to a lemma of Carlson [2, 2.5] to conclude that the module $Y$ is periodic of period at most two; actually we need to adjust the lemma to consider $A[P]$-modules, using the remarks of Section 2.

**Proposition 1.** We have $Y \cong \Omega^2 Y$.

*Proof.* The conditions $Q \triangleleft P$ and $P/Q$ cyclic ensure the existence of an exact sequence of $kP$-modules of the type

$$0 \to k_P \to \text{Ind}^P_Q(k_Q) \to \text{Ind}^P_Q(k_Q) \to k_P \to 0,$$

(see [2, 2.5]). Following Carlson, we apply to it the exact functor $Y \otimes -$ thus obtaining an exact sequence of $A[P]$-modules (cf. Section 2). Now we use statement 2.4 to rewrite $Y \otimes k_P$ as $Y$, and $Y \otimes \text{Ind}^P_Q(k_Q)$ as $\text{Ind}^P_G(\text{Res}^G_P(Y))$:

$$0 \to Y \to \text{Ind}^P_G(\text{Res}^G_P(Y)) \to \text{Ind}^P_G(\text{Res}^G_P(Y)) \to Y \to 0.$$ 

The module $\text{Ind}^P_G(\text{Res}^G_P(Y))$ is projective since $\text{Res}^G_P(Y)$ is (Section 2.2), so we conclude.

Taking an injective hull of $Y$, let us consider the element

$$s = \text{Ind}^G_P(Y - IY + \Omega^{-1}Y)$$

of $\mathcal{M}(A[G])$, and set $d(M) = (s, M)_G$, for all $M$ in $\mathcal{M}(A[G])$ (cf. Section 2).

**4. Subadditive Functions**

Following Okuyama's steps in [4], we prove that the map $d$ above satisfies three basic properties:

**Proposition 2.** (1) We have $d(\Delta) \subset \mathbb{N}$, and $d(X) > 0$.

(2) Setting $\Sigma M = \Omega^2 M + R - M$, where $M \in \Delta$ and the sequence $0 \to \Omega^2 M \to R \to M \to 0$ is almost split, we have $d(\Sigma M) \geq 0$, and if $d(\Sigma M) > 0$, then $M$ is periodic.

(3) For any $A[G]$-module $M$, we have $d(M) = d(\Omega^2 M)$.

*Proof.* (1) The first part follows by applying the left exact functor $\text{Hom}_{A[G]}(M, M)$ ($M$ in $\Delta$) to the exact sequence $0 \to \text{Ind}^G_P(Y) \to \text{Ind}^G_P(IY) \to \text{Ind}^G_P(\Omega^{-1}Y) \to 0$, and taking dimensions. Now suppose $d(X) = 0$. It
follows from (2.5) that $(Y - IY + \Omega^{-1}Y, \text{Res}_F^G(X)) = 0$. Therefore all homomorphisms from $Y$ to $\text{Res}_F^G(X)$ are projective, and so are in particular all endomorphisms of $Y$, since $Y$ is a direct summand of $\text{Res}_F^G(X)$. Thus $Y$ is projective, contradiction.

(2) We have $d(\Sigma M) = (\text{Ind}_F^G(Y \oplus \Omega^{-1}Y), \Sigma M)_G - (\text{Ind}_F^G(IY), \Sigma M)_G$. The definition of almost split sequences shows that the map $(, \Sigma M)_G$ takes on non negative values on $A[G]$-modules, and takes on the value $0$ on all modules of which $M$ is not a direct summand. Since this is the case for the projective module $\text{Ind}_F^G(IY)$ (Section 2.2), we conclude that $d(\Sigma M) \geq 0$. Moreover if $d(\Sigma M) > 0$, then $M$ is a direct summand of either $\text{Ind}_F^G(Y)$ or $\text{Ind}_F^G(\Omega^{-1}Y)$. But those two modules are periodic by Proposition 1. So $M$ itself is periodic.

(3) We apply Assertion 2.6 twice, starting from $d(M) = (s, M)_G$ and using bilinearity. The right-hand term becomes $\Pi M - \Pi(\Omega M) + \Omega^2 M$. On the left we use the identity of Section 2.2, and obtain $\text{Ind}_F^G(\Pi Y - \Pi(\Omega Y) + \Omega^2 Y - IY + \Pi(\Omega^{-1} Y) - IY + \Omega Y)$. This simplifies to the element $s$, by Proposition 1. The exactness of the functor $\text{Hom}_{A[G]}(, I)$, for any injective module $I$, now shows that our expression for $d(M)$ reduces to $(s, \Omega^2 M)_G$. Therefore $d(M) = d(\Omega^2 M)$.

**Corollary.** The function $d$ is subadditive on $A$ and satisfies $d(M) = d(\Omega^2 M)$ (all $M$ in $A$). Furthermore, if $d$ is not additive, then $A$ contains a periodic module.

**Proof.** The labelling on $A$ [1, p. 154] is such that statement (2) of the proposition, together with (1), tells us exactly that the function $d$ is subadditive. Moreover it is additive whenever $d(\Sigma M)$ in (2) is always $0$.

**Proof of Theorem 1.** Condition $d(M) = d(\Omega^2 M)$ ($M$ in $A$), ensures that $d$ induces a function on the reduced graph of $A$. The functions induced by $d$ on both the reduced graph and the tree class of $A$ are then subadditive, like $d$. We conclude by [1, 2.30.6(i)].

**References**