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A Generalization of Webb's Theorem to Auslander-Reiten Systems

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INTRODUCTION

Let G be a finite group and k be a field whose characteristic p divides the order of G. In [3] we introduce the notion of Auslander-Reiten system of G on a symmetric interior G-algebra A (that is a symmetric k-algebra together with a homomorphism ϕ: $G \to A^{\times}$), as a generalization of the notion of almost split sequences of kG-modules. The language of interior G-algebras proves to be useful to study their restriction to certain subgroups such as the defect group (vertex) of their extremity. Furthermore we show in [3] that each non projective primitive idempotent i of A^G (see Section 1) is the extremity of a unique Auslander-Reiten system, up to embedding of A into other symmetric interior G-algebras and to conjugacy (cf. [3, VI], recall *embeddings* are those one-to-one homomorphisms f: $A \to B$ which satisfy $\operatorname{Im} f = f(1)\, Af(1)$). Thus in particular the middle term of "the" Auslander-Reiten system terminating in i is well defined (up to embedding and conjugacy), and we may look at its decomposition into primitive idempotents. Now if we set $A[G] = A \otimes kG$, we have a symmetric algebra again, which is similar to kG in many ways, and which enables us to view our systems of G over A ([3, I]) as short exact sequences of modules: to any idempotent i of A^G, we associate the A-projective $A[G]$-module iA (with $(b \otimes g) \cdot ia = \phi(g)\, iab$). This way any system $\mathscr{S} = (i, i^{\circ}, i', d, d')$ of G over A determines an exact sequence of $A[G]$-modules

$$0 \longrightarrow i'A \xrightarrow{d'_{\cdot}} i^{\circ}A \xrightarrow{d_{\cdot}} iA \longrightarrow 0,$$

which is almost split if and only if the system $\mathscr{S}$ is an Auslander-Reiten system (cf. [3]). Moreover if $\mathscr{R}$: $0 \to \Omega^2(iA) \to R \to iA \to 0$ is almost split on $A[G]$, then all its terms are A-projective and we can embed A into the symmetric interior G-algebra $B = \operatorname{End}_A(A \oplus \Omega^2(iA) \oplus R)$, where the

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sequence $\mathcal{R}$ gives rise to an Auslander-Reiten system of G over B. Therefore the study of the Auslander-Reiten quiver of the algebra $A[G]$ (actually we may restrict ourselves to the components of A-projective modules), will provide us with information about the decomposition of the middle terms of Auslander-Reiten systems terminating, first in an idempotent of A^G, and then also in those idempotents which appear in such a middle term after several steps of "associating to each primitive non projective idempotent in the middle term, the Auslander-Reiten system terminating in it" (each such step may require embedding into a larger symmetric interior G-algebra). In this paper we generalize P. Webb's main theorem in [5] to the algebra $A[G]$. The proof depends on the construction of a subadditive function on a connected component of the stable Auslander-Reiten quiver: we show how the simple (non cohomological) construction given by T. Okuyama in [4] may be generalized to our setting.

THEOREM 1. *Let Δ be a connected component of the stable Auslander-Reiten quiver of $A[G]$. If Δ contains A-projective $A[G]$-modules, then the tree class and the reduced graph of Δ are both either a Dynkin diagram (finite or infinite) or a Euclidian diagram.*

As a consequence we obtain information on the middle terms of Auslander-Reiten systems of G over A:

THEOREM 2. *Let n (resp. n_0) be the number of idempotents (resp. of projective idempotents) in a primitive decomposition of the middle term of an Auslander-Reiten system of G over A. Then $n \leqslant 5$, $n_0 \leqslant 1$, and $n - n_0 \leqslant 4$. Moreover if k is algebraically closed we have $n - n_0 \leqslant 2$.*

Theorem 2 follows from Theorem 1 via an analogous statement (which we omit here) about the maximum number of indecomposable direct summands in the middle term of an almost split sequence of A-projective $A[G]$-modules (see [1, 2.31.3, 4, and the comments above]).

1. NOTATIONS

Throughout the paper we let A be a symmetric interior G-algebra and denote by $A^\times$ the group of units of A. The action of G on A by $g \cdot a = \phi(g^{-1})\, a\phi(g) = a^g$ makes it into a G-algebra. If H is any subgroup of G, we denote by A^H the algebra of H-fixed elements of A, and by $\mathrm{Tr}_1^H\colon A \to A^H$ the relative trace map defined by $\mathrm{Tr}_1^H(a) = \sum a^x$, where x runs over H; its image is the two-sided ideal A_1^H of A^H. Regarding our comments about Auslander-Reiten systems in the introduction, we refer the reader to [3] and recall that any idempotent i of a A^G may be written as a finite sum of mutually orthogonal primitive idempotents of A^G, and that the corresponding set of primitive idempotents is unique up to $(A^G)^\times$-conjugacy. We call it a *primitive decomposition* I of i. The subsets $I \cap A_1^G$ and $I \backslash A_1^G$ are then also unique up to $(A^G)^\times$-conjugacy. We say that i is *projective* (over G) if $i \in A_1^G$. See also Section 2.1.

All our modules and algebras are finite dimensional k-spaces, and our modules are right modules. We let $\mathcal{M}(A)$ be the Green ring of A: as a group it is the free $\mathbb{Z}$-module generated by the isomorphism classes of indecomposable A-modules. We denote respectively by ΠM, ΩM, and IM a projective cover, a Heller translate and an injective hull of the module M, and if $N = \Omega M$, we write $M = \Omega^{-1} N$ and $\Omega N = \Omega^2 M$. The space of all projective homomorphisms from an A-module M to an A-module N (i.e., of those homomorphisms $M \to N$ which factor through a projective module) is denoted by $\mathrm{Proj}_A(M, N)$. We use the notation k_G for the trivial kG-module, and the symbol $\otimes$ without precision for tensor products over k.

We refer the reader to [1] for the terminology on quivers and subadditive functions.

2. Background on $A[G]$-Modules

Let M be an A-projective $A[G]$-module. The following are elementary facts:

1. The algebra $\mathrm{End}_A(M)$ is a symmetric interior G-algebra and we have $\mathrm{End}_{A[G]}(M) = \mathrm{End}_A(M)^G$, $\mathrm{Proj}_{A[G]}(M, M) = \mathrm{End}_A(M)_1^G$.

Let H be a subgroup of G. For any $A[G]$-module M, we denote by $\mathrm{Res}_H^G(M)$ the $A[H]$-module obtained by restriction through the injection $A[H] \to A[G]$. We then extend this by linearity and consider the functor Res_H^G from $\mathcal{M}(A[G])$ to $\mathcal{M}(A[H])$.

Induction. Just in the same way as with kH-modules, one may induce any $A[H]$-module N to $A[G]$: we define the $A[G]$-module $\mathrm{Ind}_H^G(N)$ by

$$\mathrm{Ind}_H^G(N) = N \otimes_{A[H]} A[G],$$

with $A[G]$ acting on the right of the term $A[G]$. We then extend our definition linearly to a functor Ind_H^G from $\mathcal{M}(A[H])$ to $\mathcal{M}(A[G])$.

2. If the $A[H]$-module N is projective, then so is the $A[G]$-module $\mathrm{Ind}_H^G(N)$. Moreover we have for all N: $\mathrm{Ind}_H^G(\Pi N - \Omega N) = \Pi(\mathrm{Ind}_H^G(N)) - \Omega(\mathrm{Ind}_H^G(N))$.

Note that $N \otimes_{kH} kG$ also is an $A[G]$-module, with A acting on N only, so that the actions of A and kG are compatible. It is easy to show that

3. We have $\mathrm{Ind}_H^G(N) \simeq N \otimes_{kH} kG$, as $A[G]$-modules.

Now let M be an $A[G]$-module. For all kG-modules U, the tensor product $M \otimes U$ is an $A[G]$-module, with A acting on M and G acting on both M and U.

4. One has $M \otimes \mathrm{Ind}_H^G(k_H) \simeq \mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$, as $A[G]$-modules.

Proof. We first view M as a kG-module only. We have the following sequence of kG-modules identities

$$M \otimes \mathrm{Ind}_H^G(k_H) \simeq (\mathrm{Res}_H^G(M) \otimes k_H) \otimes_{kH} kG \simeq \mathrm{Res}_H^G(M) \otimes_{kH} kG,$$

and we may substitute the last expression with $\mathrm{Ind}_H^G(\mathrm{Res}_H^G(M))$ by 3; we then check that the corresponding actions of A are consistent (A acts on M only).

We consider the inner product $(\ , \ )_G$ on $\mathcal{M}(A[G])$ obtained by extending the form $\dim_k \mathrm{Hom}_{A[G]}(\ , \ )$ bilinearly:

5. We have $(\mathrm{Ind}_H^G(N), M)_G = (N, \mathrm{Res}_H^G(M))_H$, for any N in $\mathcal{M}(A[H])$ and any M in $\mathcal{M}(A[G])$.

6. For all $A[G]$-modules L and M, we have $(L, M)_G = (\Pi L - \Omega L, \Pi M - \Omega M)_G$.

Proof. This follows by applying successively statements (ii) and (i) of the elementary

LEMMA (Notations of 6). (i) *We have* $(L, \Pi M - \Omega M)_G = \dim \mathrm{Proj}_{A[G]}(L, M)$.

(ii) *We have* $(L, M)_G = \dim \mathrm{Proj}_{A[G]}(\Pi L, M) - \dim \mathrm{Proj}_{A[G]}(\Omega L, M)$.

Proof. (i) We apply the left exact functor $\mathrm{Hom}_{A[G]}(L, \ )$ to the projective cover of M: the range of the second morphism is precisely $\mathrm{Proj}_{A[G]}(L, M)$.

(ii) Projective and injective modules coincide on the symmetric algebra $A[G]$. Thus the space $\mathrm{Proj}_{A[G]}(\Omega L, M)$ is precisely the range of the second morphism in the exact sequence $0 \to \mathrm{Hom}_{A[G]}(L, M) \to \mathrm{Hom}_{A[G]}(\Pi L, M) \to \mathrm{Hom}_{A[G]}(\Omega L, M)$.

3. PERIODIC MODULES

In the following we fix an A-projective $A[G]$-module X which is indecomposable and non projective, and denote by Δ the connected component of the stable Auslander-Reiten quiver of $A[G]$ which contains X. Since X is not projective, there exists a minimal p-subgroup P of G ($P \neq 1$), such that $\mathrm{Res}_P^G(X)$ is not projective (cf. Section 2.1). We choose an indecomposable direct summand Y of $\mathrm{Res}_P^G(X)$ which is not projective, and let Q be a maximal subgroup of P. So the module $\mathrm{Res}_Q^P(Y)$ is projective. Following Okuyama, we turn to a lemma of Carlson [2, 2.5] to conclude that the module Y is periodic of period at most two; actually we need to adjust the lemma to consider $A[P]$-modules, using the remarks of Section 2.

PROPOSITION 1. *We have* $Y \simeq \Omega^2 Y$.

Proof. The conditions $Q \lhd P$ and P/Q cyclic ensure the existence of an exact sequence of kP-modules of the type

$$0 \to k_P \to \mathrm{Ind}_Q^P(k_Q) \to \mathrm{Ind}_Q^P(k_Q) \to k_P \to 0,$$

(see [2, 2.5]). Following Carlson, we apply to it the exact functor $Y \otimes$, thus obtaining an exact sequence of $A[P]$-modules (cf. Section 2). Now we use statement 2.4 to rewrite $Y \otimes k_P$ as Y, and $Y \otimes \mathrm{Ind}_Q^P(k_Q)$ as $\mathrm{Ind}_Q^P(\mathrm{Res}_Q^P(Y))$:

$$0 \to Y \to \mathrm{Ind}_Q^P(\mathrm{Res}_Q^P(Y)) \to \mathrm{Ind}_Q^P(\mathrm{Res}_Q^P(Y)) \to Y \to 0.$$

The module $\mathrm{Ind}_Q^P(\mathrm{Res}_Q^P(Y))$ is projective since $\mathrm{Res}_Q^P(Y)$ is (Section 2.2), so we conclude.

Taking an injective hull of Y, let us consider the element

$$s = \mathrm{Ind}_P^G(Y - IY + \Omega^{-1} Y)$$

of $\mathcal{M}(A[G])$, and set $d(M) = (s, M)_G$, for all M in $\mathcal{M}(A[G])$ (cf. Section 2).

4. SUBADDITIVE FUNCTIONS

Following Okuyama's steps in [4], we prove that the map d above satisfies three basic properties:

PROPOSITION 2. (1) *We have* $d(\Delta) \subset \mathbb{N}$, *and* $d(X) > 0$.

(2) *Setting* $\Sigma M = \Omega^2 M + R - M$, *where* $M \in \Delta$ *and the sequence* $0 \to \Omega^2 M \to R \to M \to 0$ *is almost split, we have* $d(\Sigma M) \geqslant 0$, *and if* $d(\Sigma M) > 0$, *then* M *is periodic.*

(3) *For any* $A[G]$*-module* M, *we have* $d(M) = d(\Omega^2 M)$.

Proof. (1) The first part follows by applying the left exact functor $\mathrm{Hom}_{A[G]}(\ , M)$ (M in Δ) to the exact sequence $0 \to \mathrm{Ind}_P^G(Y) \to \mathrm{Ind}_P^G(IY) \to \mathrm{Ind}_P^G(\Omega^{-1} Y) \to 0$, and taking dimensions. Now suppose $d(X) = 0$. It follows from (2.5) that $(Y - IY + \Omega^{-1}Y, \operatorname{Res}_P^G(X))_P = 0$. Therefore all homomorphisms from Y to $\operatorname{Res}_P^G(X)$ are projective, and so are in particular all endomorphisms of Y, since Y is a direct summand of $\operatorname{Res}_P^G(X)$. Thus Y is projective, contradiction.

(2) We have $d(\Sigma M) = (\operatorname{Ind}_P^G(Y \oplus \Omega^{-1}Y), \Sigma M)_G - (\operatorname{Ind}_P^G(IY), \Sigma M)_G$. The definition of almost split sequences shows that the map $(\ , \Sigma M)_G$ takes on non negative values on $A[G]$-modules, and takes on the value 0 on all modules of which M is not a direct summand. Since this is the case for the projective module $\operatorname{Ind}_P^G(IY)$ (Section 2.2), we conclude that $d(\Sigma M) \geqslant 0$. Moreover if $d(\Sigma M) > 0$, then M is a direct summand of either $\operatorname{Ind}_P^G(Y)$ or $\operatorname{Ind}_P^G(\Omega^{-1}Y)$. But those two modules are periodic by Proposition 1. So M itself is periodic.

(3) We apply Assertion 2.6 twice, starting from $d(M) = (s, M)_G$ and using bilinearity. The right-hand term becomes $\Pi M - \Pi(\Omega M) + \Omega^2 M$. On the left we use the identity of Section 2.2, and obtain $\operatorname{Ind}_P^G(\Pi Y - \Pi(\Omega Y) + \Omega^2 Y - IY + \Pi(\Omega^{-1}Y) - \Pi Y + \Omega Y)$. This simplifies to the element s, by Proposition 1. The exactness of the functor $\operatorname{Hom}_{A[G]}(\ , I)$, for any injective module I, now shows that our expression for $d(M)$ reduces to $(s, \Omega^2 M)_G$. Therefore $d(M) = d(\Omega^2 M)$.

COROLLARY. *The function d is subadditive on Δ and satisfies $d(M) = d(\Omega^2 M)$ (all M in Δ). Furthermore, if d is not additive, then Δ contains a periodic module.*

Proof. The labelling on Δ [1, p. 154] is such that statement (2) of the proposition, together with (1), tells us exactly that the function d is subadditive. Moreover it is additive whenever $d(\Sigma M)$ in (2) is always 0.

Proof of Theorem 1. Condition $d(M) = d(\Omega^2 M)$ (M in Δ), ensures that d induces a function on the reduced graph of Δ. The functions induced by d on both the reduced graph and the tree class of Δ are then subadditive, like d. We conclude by [1, 2.30.6(i)].

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