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Convergence of Selections with Applications in Optimization

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We consider the problem of finding an easily implemented tie-breaking rule for a convergent set-valued algorithm, i.e., a sequence of compact, non-empty subsets of a metric space converging in the Hausdorff metric. Our tie-breaking rule is determined by nearest-point selections defined by "uniqueness" points in the space, i.e., points having a unique best approximation in the limit set of the convergent algorithm. Convergence of the algorithm is shown to be equivalent to convergence of all such nearest-point selections. Under reasonable additional hypotheses, all points in the metric space have the uniqueness property. Consequently, *all* points yield convergent nearest-point selections, i.e., tie-breaking rules, for a convergent algorithm.

We then show how to apply these results to approximate solutions for the following types of problems: infinite systems of inequalities, semi-infinite mathematical programming, non-convex optimization, and infinite horizon optimization.

1. INTRODUCTION

Suppose (X, d) is an arbitrary compact metric space. Define an *algorithm* A in X to be a mapping $n \to A_n$ of the positive integers into the set of closed, non-empty subsets $\mathscr{K}(X)$ of X (compare with [13, pp. 183–184]). Thus, an algorithm A is just a sequence $\{A_n\}$ in $\mathscr{K}(X)$. Suppose $\mathscr{K}(X)$ is

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equipped with the Hausdorff metric D derived from d (see Section 2). We will say that the algorithm A *converges* if the sequence $\{A_n\}$ converges in $(\mathscr{K}(X), D)$. If A converges, then there exists an A_∞ in $\mathscr{K}(X)$ such that $A_n \to A_\infty$ relative to D. We may think of A_∞ as the solution set to some problem and A_n as the set of approximating solutions to this problem produced by the nth iteration of the algorithm A, for $n = 1, 2, 3, \ldots$.

Given a convergent algorithm A, how do we approximate a point in A_∞, i.e., how can we construct a sequence $\{x_n\}$ such that $x_n \in A_n$, all n, and $\{x_n\}$ converges to some element of A_∞? Such a construction will be called a *tie-breaking rule*. Of course, an arbitrary choice of x_n in A_n will not do, since the x_n will not converge in general. (However, if they do converge, the limit will be an element of A_∞.) Moreover, the *theoretical* existence of such a convergent sequence is insufficient for practical purposes. One usually requires a *constructive* procedure which can be implemented to yield such a sequence. In this paper, we give such a procedure which depends on the familiar notion of best approximation or nearest-point. Specifically, given any point p in X, let x_n be a point in A_n which is nearest to p. Thus, the x_n are values at the A_n of a nearest-point selection on $\mathscr{K}(X)$ defined by p. The question we ask is the following one. If the algorithm A is convergent, under what conditions will the sequence $\{x_n\}$ converge to an element of A_∞? We show that convergence of the algorithm A is *equivalent* to the desired convergence of the sequence $\{x_n\}$ for all points p in the *uniqueness set* of A_∞, i.e., the set of points p in X having a unique best approximation in A_∞. In general, the uniqueness set may be difficult to find, thus making it difficult to choose p. In this case, our tie-breaking procedure is difficult to implement. However, in many important applications, the uniqueness set of A_∞ is *all* of X. Therefore, in such cases, *any* point in X may be used as the defining point for a nearest-point selection whose values at the A_n converge to a point in A_∞. (Note that this is in fact the case for convex subsets of a Hilbert space.) We apply these results to a variety of optimization problems.

In Section 2, we review the topological results required concerning the Hausdorff metric, the lim inf and lim sup of sets and as well as their connections.

In Section 3, we establish the selection convergence results. Specifically, we show that an algorithm is convergent if and only if selections canonically defined by continuous real-valued functions on X are convergent to an element of the algorithm's limit set, provided the function attains its minimum at a unique point of the limit set. This is equivalent to convergence of the nearest-point selections defined by the uniqueness points of the limit set. We also give a parallel set of conditions involving convergent selections which are equivalent to the non-emptiness of the lim inf of the algorithm. We complete this section by showing that if, in addition, the algorithm consists of convex subsets of a Hilbert space, then its limit set is convex and hence, its uniqueness set is the whole space.

In Section 4, we apply our main results to approximation problems of the following types: (1) solving an infinite system of inequalities via approximate solution of finite subsystems, (2) semi-infinite mathematical programming via approximation by finite subprograms, (3) constrained optimization via grid approximation of the feasible region, and (4) infinite horizon optimization via finite horizon truncations.

2. Topological Preliminaries

For each x in X, the mapping $y \to d(x, y)$ is continuous on X. Thus, for each K in $\mathscr{K}(X)$, the minimum of $d(x, y)$, for y in K, is attained and we may define

$$d(x, K) = \min_{y \in K} d(x, y), \qquad x \in X, K \in \mathscr{K}(X).$$

Moreover, for such K, the mapping $x \to d(x, K)$ is also continuous on X [9, Thm. 4.2]. Hence, for each C in $\mathscr{K}(X)$, the maximum of $d(x, K)$, for x in C, is also attained and we may therefore define

$$h(C, K) = \max_{x \in C} d(x, K), \qquad C, K \in \mathscr{K}(X).$$

Although h is not a metric on $\mathscr{K}(X)$ (it is not symmetric), we may obtain a metric D on $\mathscr{K}(X)$ if we define

$$D(C, K) = \max(h(C, K), h(K, C)), \qquad C, K \in \mathscr{K}(X).$$

This is the well-known Hausdorff metric on $\mathscr{K}(X)$ [5, 10, 11]. With this metric, $\mathscr{K}(X)$ is compact [10, 12]. Convergence in $\mathscr{K}(X)$ will be understood to be relative to D.

Now let $\{K_n\}$ be an arbitrary sequence in $\mathscr{K}(X)$. As in [5, 10, 11], define lim sum K_n and lim inf K_n as follows:

(1) $x \in \limsup K_n$ if and only if there exists a subsequence $\{K_{n_k}\}$ of $\{K_n\}$ and a corresponding sequence $\{x_{n_k}\}$ such that $x_{n_k} \in K^{nk}$, all k, and $x_{n_k} \to x$, as $k \to \infty$.

(2) $x \in \liminf K_n$ if and only if, for each n, there exists $x_n \in K_n$ such that $x_n \to x$, as $n \to \infty$.

In general, $\liminf K_n \subseteq \limsup K_n$. Also, $\liminf K_n$ and $\limsup K_n$ are closed subsets of X. In fact, $\limsup K_n$ belongs to $\mathscr{K}(X)$, since it must be non-empty. However, $\liminf K_n$ may still be empty.

The next result summarizes the connection between these limit sets and Hausdorff convergence [10, 11, 12].

THEOREM 2.1. *Let* $\{K_n\}$ *be a sequence in* $\mathscr{K}(X)$ *and* K *an element of* $\mathscr{K}(X)$. *Then the following are equivalent*:

(i) $K_n \to K$ *in* $\mathscr{K}(X)$ *relative to* D.

(ii) $K = \limsup K_n = \liminf K_n$, *i.e.*, $K \subseteq \liminf K_n$ *and* $\limsup K_n \subseteq K$.

COROLLARY 2.2. *If* $\limsup K_n = \{x\}$, *then* $\liminf\{K_n\} = \{x\}$ *also. In this case*, $x_n \to x$, *as* $n \to \infty$, *for* all *choices* x_n *in* K_n, *all* n.

Proof. Let $x_n \in K_n$, all n. If $x_n \nrightarrow x$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which is bounded away from x. Since X is compact, passing to a subsequence if necessary, we may assume there exists y in X such that $x_{n_k} \to y$, as $k \to \infty$. Consequently $y \in \limsup K_n$, i.e., $y = x$, by hypothesis. Contradiction. ∎

3. SELECTION CONVERGENCE RESULTS

We define a *selection* s on $\mathscr{K}(X)$ to be a mapping s: $\mathscr{K}(X) \to X$ satisfying $s(K) \in K$, all K in $\mathscr{K}(X)$. Note that selections are not required to be continuous. Our objective is to equate convergence of K_n to K in $(\mathscr{K}(X), D)$ with convergence of $s(K_n)$ to $s(K)$ in (X, d) for nearest-point selections s_p corresponding to appropriately chosen points p in X. Before we can do this, we need to establish some additional concepts.

Let f be a continuous real-valued function defined on X. Define m_f: $\mathscr{K}(X) \to \mathscr{R}$ by

$$m_f(K) = \min_{x \in K} f(x),$$

and $M_f : \mathscr{K}(X) \to \mathscr{K}(X)$ by

$$M_f(K) = \{x \in K : f(x) = m_f(K)\}.$$

Then $m_f(K)$ is the minimum value of f on K and $M_f(K)$ is the compact, non-empty subset of K on which this minimum is attained. (Note that $M_f(K)$ being a singleton is a generalization of K being a singleton.) We then define an *f-selection* to be any selection s_f such that $s_f(K) \in M_f(K)$, $K \in \mathscr{K}(X)$.

Now let p be any point in X. Define

$$f_p(x) = d(p, x), \qquad x \in X.$$

Then f_p is a continuous real-valued function on X. Denoting m_{f_p} by m_p and

M_{f_p} by M_p, we see that $m_p(K)$ is the distance from p to K and $M_p(K)$ is the subset of K where this distance is attained. As above, we will write s_p for s_{f_p}. Observe that $s_p(K)$ is any point in K which is nearest to p. For this reason, we call s_p a *nearest-point* selection defined by p.

Now fix K in $\mathscr{K}(X)$. If p is such that there exists a *unique* x in K such that $d(p, K) = d(p, x)$, i.e., $M_p(K)$ is a *singleton*, then we will say that p is a *uniqueness point* for K (relative to d). In this case, $d(p, x) < d(p, y)$, for all y in K different from x. Let $U(K)$ denote the uniqueness set for K, $K \in \mathscr{K}(X)$. If $p \in U(K)$ and s_p is a nearest-point selection defined by p, then $s_p(K)$ is uniquely determined in K. In general, $K \subseteq U(K) \subseteq X$, $K \in \mathscr{K}(X)$. Note also that $U(K)$ being equal to X is a generalization of K being a singleton.

The next lemma shows that, in general, m_f is continuous, while M_f is only partially continuous.

LEMMA 3.1. *If $f: X \to \mathscr{R}$ is continuous and $K_n \to K_\infty$ in $\mathscr{K}(X)$, as $n \to \infty$, then $m_f(K_n) \to m_f(K_\infty)$, as $n \to \infty$ and* $\limsup M_f(K_n) \subseteq M_f(K)$.

Proof. An application of the (minimum version of the) Maximum Theorem [5, p. 116] yields the convergence of $m_f(K_n)$ to $m_f(K_\infty)$ as well as the upper semi-continuity (in the sense of [5, p. 109]) of the set mapping $F(n) = M_f(K_n)$, $n = \infty, 1, 2, \ldots$, where $\{\infty, 1, 2, \ldots\}$ is viewed as a compact metric space under stereographic projection. By [5, p. 112, Theorem 6] it follows that the mapping F is also closed. Consequently, $\limsup M_f(K_n) \subseteq M_f(K)$ by [5, p. 111, Theorem 4]. ∎

There is an important special case where M_f and all f-selections are continuous.

THEOREM 3.2. *Let $K \in \mathscr{K}(X)$ and suppose $f: X \to \mathscr{R}$ is continuous. If $M_f(K)$ is a singleton, then*

(i) *The transformation $M_f: \mathscr{K}(X) \to \mathscr{K}(X)$ is continuous at K.*

(ii) All *f-selections $s_f: \mathscr{K}(X) \to X$ are continuous at K.*

Proof. Suppose $K_n \to K$ in $\mathscr{K}(X)$. By Lemma 3.1, $\limsup M_f(K_n) \subseteq M_f(K)$. Since $\limsup M_f(K_n)$ is necessarily non-empty, it must be a singleton by hypothesis. The theorem then follows from Theorem 2.1 and Corollary 2.2 applied to $\{M_f(K_n)\}$. ∎

COROLLARY 3.3. *Let K be an element of $\mathscr{K}(X)$ and p a point in $U(K)$. Then*

(i) *The transformation $M_p: \mathscr{K}(X) \to \mathscr{K}(X)$ is continuous at K.*

(ii) All *nearest-point selections s_p defined by p are continuous at K.*

The following are our main results. The first says that, given a convergent sequence of sets, all nearest-point selections defined by uniqueness points of the limit set converge to a point in the limit set.

THEOREM 3.4. *Let* $\{K_n\}$ *be a sequence in* $\mathscr{K}(X)$ *and* K *an element of* $\mathscr{K}(X)$. *Suppose* $\limsup K_n \subseteq K$. *Then the following are equivalent*:

(i) $K \subseteq \liminf K_n$, *i.e.*, $K_n \to K$ *relative to* D *in* $\mathscr{K}(X)$, *as* $n \to \infty$.

(ii) $s_p(K_n) \to s_p(K)$ *in* X, *as* $n \to \infty$, *for* all *nearest-point selections* s_p *defined by* any p *in* $U(K)$.

(iii) $s_f(K_n) \to s_f(K)$ *in* X, *as* $n \to \infty$, *for* all *f-selections* s_f *defined by* all *continuous functions* $f\colon X \to \mathscr{R}$ *for which* $M_f(K)$ *is a singleton.*

(iv) $s(K_n) \to s(K)$ *in* X, *as* $n \to \infty$, *for* all *selections* s *which are continuous at* K.

Proof. (i) implies (iv): Apply Theorem 2.1 (iv) implies (iii): Apply Theorem 3.2. (iii) implies (ii): If $p \in U(K)$, then $M_f(K)$ is a singleton, where $f = f_p$ is continuous. (ii) implies (i): Let p be an element of K, so that $p \in U(K)$. By (ii), $s_p(K_n) \to s_p(K)$, as $n \to \infty$, where $s_p(K_n) \in K_n$, all n, and $s_p(K) = p$. Hence, $p \in \liminf K_n$ by definition. ▮

COROLLARY 3.5. *If* $U(K) = X$, *then the following are equivalent*:

(i) $K_n \to K$ *in* $\mathscr{K}(X)$, *as* $n \to \infty$.

(ii) $s_p(K_n) \to s_p(K)$ *in* X, *as* $n \to \infty$, *for* all *nearest-point selections* s_p *defined by* any p *in* X.

Analogously, we have the following result on *existence* of continuous selections. Intuitively, it is a dual version of Theorem 3.4.

THEOREM 3.6. *Let* $\{K_n\}$ *be a sequence in* $\mathscr{K}(X)$ *and* K *an element of* $\mathscr{K}(X)$. *Suppose* $\limsup K_n \subseteq K$. *Then the following are equivalent*:

(i) $\phi \neq \liminf K_n$.

(ii) $s_p(K_n) \to s_p(K)$ *in* X, *as* $n \to \infty$, *for* all *nearest-point selections* s_p *defined by* some p *in* $U(K)$.

(iii) $s_f(K_n) \to s_f(K)$ *in* X, *as* $n \to \infty$, *for* all *f-selections* s_f *defined by* some *continuous function* $f\colon X \to \mathscr{R}$ *for which* $M_f(K)$ *is a singleton.*

(iv) $s(K_n) \to s(K)$ *in* X, *as* $n \to \infty$, *for* some *selection* s *which is continuous at* K.

Proof. (i) implies (ii): Let x be an element of $\liminf K_n$. By definition, there exists x_n in K_n, all n, such that $x_n \to x$, as $n \to \infty$. By hypothesis, $x \in K$, so that $x \in U(K)$. Let s_x be any nearest-point selection corresponding to x. Then $d(x, s_x(K_n)) \to 0$, as $n \to \infty$, since $d(x, s_x(K_n)) \leqslant d(x, x_n)$, all n. Hence, $s_x(K_n) \to x = s_x(K)$, as $n \to \infty$. (ii) implies (iii): Let p be an element of $U(K)$ and s_p a corresponding nearest-point selection satisfying $s_p(K_n) \to s_p(K)$. As before, define $f(x) = d(p, x)$, $x \in X$. Then f is continuous on X, s_p is an f-selection and $M_f(K)$ is a singleton since $p \in U(K)$. (iii) implies (iv): Let $f: X \to \mathscr{R}$ be a continuous function for which $M_f(K)$ is a singleton. Then each f-selection s_f has the desired properties (Theorem 3.2(ii)). (iv) implies (i): Let s be a selection which is continuous at K and satisfies $s(K_n) \to s(K)$, as $n \to \infty$. Then $s(K) \in \liminf K_n$ by definition. ∎

Remarks. (1) In the statement of Theorem 3.6 it suffices to assume more generally that $\limsup K_n \subseteq U(K)$. This is the case for example if $U(K) = X$. (2) Obviously, we are interested in approximating a point in $\liminf K_n$. Theorem 3.6 gives conditions under which this can be done. While this result is of theoretical interest, it does not tell us *how* to construct such an approximation.

The previous results show that in dealing with nearest-point selections, it is essential that the reference point p be a uniqueness point of the relevant limit set. In general, such p may be difficult to find. Thus, it is desirable that the uniqueness set be the *entire* space, so that *any* point can be chosen as a reference point. This will be the case, for example, if K is a convex subset of a Hilbert space. Specifically, we have:

LEMMA 3.7. *If X is a compact subset of a Hilbert space and K is a convex element of $\mathscr{K}(X)$, then $U(K) = X$.*

Proof. Follows from [2, p. 15] or [8, p. 23]. ∎

In order to apply this lemma, we need a useful sufficient condition for the limit of convex sets to be convex. The following lemma gives us this condition.

LEMMA 3.8. *Suppose X is a compact subset of a linear metric space. Let $\{K_n\}$ be a sequence in $\mathscr{K}(X)$ having the property that each K_n is convex, $n = 1, 2, \ldots$. Then* $\liminf K_n$ *is also convex. Consequently, the limits of sequences of convex elements of $\mathscr{K}(X)$ are also convex.*

Proof. Let $x, y \in \liminf K_n$ and $0 \leqslant a \leqslant 1$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n, y_n \in K_n$, all n, $x_n \to x$ and $y_n \to y$, as $n \to \infty$. Also, $ax_n + (1-a) y_n \in K_n$, all n, by hypothesis and $ax_n + (1-a) y_n \to ax + (1-a) y$, which must be an element of $\liminf K_n$. Thus, $\liminf K_n$ is convex. The remaining statement follows from Theorem 2.1. ∎

We conclude this section with several illustrative examples.

EXAMPLE 3.9. Let (X, d) be an arbitrary compact metric space with at least two distinct points. Suppose $x_i \to x$ and $y_i \to y$ in X, as $i \to \infty$, where $x \neq y$. Define $K_i = \{x_i, y_i\}$, $i = 1, 2, \ldots$, and $K = \{x, y\}$. Then it is easy to see that $K \subseteq \liminf K_i$ and $\limsup K_i \subseteq K$, so that they are equal. Hence, $K_i \to K$ in $\mathscr{K}(X)$, as $i \to \infty$. Moreover,

$$U(K) = \{z \in X: d(x, z) \neq d(y, z)\}.$$

Let p be any element of $U(K)$. Without loss of generality, assume $d(p, x) < d(p, y)$, so that $s_p(K) = x$. Then $s_p(K_i) = x_i$ eventually and $s_p(K_i) \to s_p(K)$, as $i \to \infty$.

EXAMPLE 3.10. Let X be any compact subset of $\mathscr{R}^2$ which contains the unit disk. For each n, let K_n be the ellipse given by $x^2 + n^2y^2 = 1$, $n = 1, 2, \ldots$. Then K_n is a compact subset of X, $n = 1, 2, \ldots$; i.e., $\{K_n\}$ is a sequence in $\mathscr{K}(X)$ whose limit K is easily seen to be the compact, convex interval $\{(x, 0) : |x| \leqslant 1\}$ in $\mathscr{R}^2$ [10, p. 169]. Moreover, it is also clear that the uniqueness set of K is all of X. Thus, for any p in X and any nearest-point selection s_p defined by p, $s_p(K_n) \to s_p(K)$ in $\mathscr{R}^2$, as $n \to \infty$.

EXAMPLE 3.11. Let X denote the product of countably many copies of the interval $[-1, 1]$. The compact product topology is metrizable by the metric d given by

$$d(x, y) = \sum_{i=1}^{\infty} |x_i - y_i|/2^i, \qquad x, y \in X,$$

where $x = (x_i)$, $y = (y_i)$. Let $0 < \alpha < 1$ and consider the following infinite horizon mathematical program:

$$\max \sum_{i=1}^{\infty} \alpha^i x_i^2 \tag{MP}$$

subject to

$$|x_i| \leqslant 1, \qquad i = 1, 2, \ldots.$$

Let X^* denote the set of optimal solutions to (MP). It is easy to see that

$$X^* = \{x \in X : x_i = \pm 1, \text{ all } i\},$$

so that X^* is uncountable. One interpretation of this is that for every discount factor α, there exist uncountably many infinite horizon optima for this problem.

Now let N be a positive integer and consider the following N-horizon version of (MP):

$$\max \sum_{i=1}^{N} \alpha^i x_i^2 \qquad (\mathrm{MP})_N$$

subject to

$$|x_i| \leqslant 1, \qquad i = 1, 2, ..., N.$$

If X_N^* denotes the set of optimal solutions to $(\mathrm{MP})_N$, then it is obvious that

$$X_N^* = \{x \in X : x_i = \pm 1, 1 \leqslant i \leqslant N\}, \qquad N = 1, 2, ...$$

Thus,

$$X_1^* \supset X_2^* \supset X_3^* \supset \cdots \supset X_N^* \supset \cdots \supset X^*$$

and

$$X^* = \bigcap_{N=1}^{\infty} X_N^*.$$

It then follows [11, p. 339] that

$$X^* = \liminf X_N^* = \limsup X_N^*,$$

so that $X_N^* \to X^*$ in $K(X)$, as $N \to \infty$. In fact, one may verify that

$$D(X_N^*, X^*) \leqslant 2^{1-N}, \qquad N = 1, 2, ...$$

The uniqueness set of X^* is given by

$$U(X^*) = \{x \in X : x_i \neq 0, \text{all } i\}.$$

Let p be an element of $U(X^*)$. Then a nearest-point selection s_p defined by p satisfies

$$s_p(X^*)_i = \begin{cases} 1, & \text{if } p_i > 0, \\ -1, & \text{if } p_i < 0, \end{cases}$$

and

$$s_p(X_N^*)_i = \begin{cases} 1, & \text{if } 1 \leqslant i \leqslant N \text{ and } p_i > 0, \\ -1, & \text{if } 1 \leqslant i \leqslant N \text{ and } p_i < 0, \\ p_i, & \text{if } i > N. \end{cases}$$

Note that $s_p(X_N^*)$ is the *unique* element of X_N^* having the property that $d(p, X_N^*) = d(p, s_p(X_N^*))$, $N = 1, 2, \ldots$. Of course, $s_p(X_N^*) \to s_p(X^*)$, as $n \to \infty$, as required by Theorem 3.4. In fact, we may verify that

$$d(s_p(X_N^*), s_p(X^*)) \leqslant 2^{-N}, \qquad N = 1, 2, \ldots.$$

4. Applications

We are now ready to apply our main results on selection convergence. The following result will be very useful in this section. As in Section 1, let A be an algorithm in X, where we assume that X is a compact subset of some Hilbert space.

THEOREM 4.1. *Suppose each A_n is convex, $n = 1, 2, \ldots$, and A is convergent to A_∞. Then for each p in X, we have $s_p(A_n) \to s_p(A_\infty)$ in X, as $n \to \infty$, where s_p is the* unique *nearest-point selection corresponding to p.*

Proof. Since $\{A_n\}$ converges in $\mathscr{K}(X)$, it follows that A_∞ is also convex (Lemma 3.8). Hence, for each $n = \infty, 1, 2, \ldots, A_n$ contains a unique nearest-point to any p in X (Lemma 3.7), i.e., the uniqueness set of A_n is X. Thus, for each p in X, there exists a *unique* nearest-point selection defined by p. It follows from Corollary 3.5 that $s_p(A_n) \to s_p(A_\infty)$, as $n \to \infty$. ∎

4.1. *Systems of Inequalities*

Now consider the following problem of finding a solution to an infinite system of inequalities; that is, we seek $x = (x_1, \ldots, x_n)$ in $\mathscr{R}^n$ such that $g_i(x) \leqslant b_i$, $i = 1, 2, \ldots$, where each g_i is a real-valued continuous function of a real variable, and $u_j \leqslant x_j \leqslant v_j$, $j = 1, \ldots, n$ [6, 7]. Let $X = \prod_{j=1}^n [u_j, v_j]$ so that X is a compact, convex subset of $\mathscr{R}^n$. For each $N = 1, 2, \ldots$, define

$$K_N = \{x \in X : g_i(x) \leqslant b_i, i = 1, \ldots, N\}.$$

Then K_N is the set of solutions in X to the first N inequalities. Moreover:

(i) Each K_N is a compact subset of X.

(ii) The K_N are decreasing, i.e., $K_{N+1} \subseteq K_N$, $N = 1, 2, \ldots$.

(iii) The set K_∞ of all solutions to the original problem is equal to $\bigcap_{N=1}^\infty K_N$.

THEOREM 4.2. *Suppose $K_\infty \neq \varnothing$, so that $K_N \in \mathscr{K}(X)$, all N. Suppose also that g_i is convex, $i = 1, 2, \ldots$. If p is* any *point in X, then the sequence of points $\{x_N\}$, where x_N is the solution to the first N inequalities nearest to p, converges to the solution of the system of inequalities which is nearest p.*

Proof. By hypothesis, each K_N is also convex, $N = 1, 2, \ldots$. Moreover, $K_N \to K_\infty$ in $\mathscr{K}(X)$ relative to the Hausdorff metric [11, p. 339]. Now apply Theorem 4.1 to complete the proof. ∎

Remark. Without loss of generality, we may assume the origin is in X. Thus, in particular, we may choose p to be the origin in $\mathscr{R}^n$. Then the sequence of points in the K_N closest to the origin converges to the solution of the original problem which is closest to the origin (i.e., of minimum norm).

4.2. *Semi-Infinite Programming*

Consider the following semi-infinite, convex mathematical program:

$$\max c(x_1, \ldots, x_n) \tag{P}$$

subject to

$$\sum_{j=1}^{n} a_{ij}(x_j) \leqslant b_i, \qquad i = 1, 2, \ldots,$$

$$u_j \leqslant x_j \leqslant v_j, \qquad j = 1, \ldots, n,$$

where $-c$ and each a_{ij} are continuous and convex, $j = 1, \ldots, n$, $i = 1, 2, \ldots$ [1, p. 66]. Define X as in the previous problem and let K denote the feasible region to (P). As above, K is a compact, convex subset of X. Assume $K \neq \varnothing$.

For each $N = 1, 2, \ldots$, consider the finite subprogram given by

$$\max c(x_1, \ldots, x_n) \tag{P_N}$$

subject to

$$\sum_{j=1}^{n} a_{ij}(x_j) \leqslant b_i, \qquad i = 1, \ldots, N,$$

$$u_j \leqslant x_j \leqslant v_j, \qquad j = 1, \ldots, n.$$

If K_N denotes the corresponding feasible region, then the K_N are as in the previous application. In particular, $K_N \to K$ in $\mathscr{K}(X)$, as $N \to \infty$. Let

$$c_N^* = \max\{c(x) \mid x \in K_N\}, \qquad N = 1, 2, \ldots$$

and

$$c^* = \max\{c(x) \mid x \in K\}.$$

Then, by Lemma 3.1, we have $c_N^* \to c^*$, as $N \to \infty$, i.e., value convergence holds. Also, let

$$K_N^* = \{x \in K_N \mid c(x) = c_N^*\}, \qquad N = 1, 2, ...,$$

and

$$K^* = \{x \in K \mid c(x) = c^*\}.$$

Then K^* and each X_N^* is a compact, non-empty subset of X, for $N = 1, 2, ...$, and in addition, $\limsup K_N^* \subseteq K^*$ (Lemma 3.1), i.e., solution convergence holds partially in general. If (P) has a unique solution, i.e., K^* is a singleton, then $K_N^* \to K^*$, as $N \to \infty$, by Corollary 2.2.

THEOREM 4.3. *Suppose A is an algorithm for the (P_N) for which A_N is a non-empty, compact, convex subset of K_N^*, $N = 1, 2, ...$ If A converges, then for each point p in X, the sequence $\{x_N^*\}$, where x_N^* is the solution to (P_N) in A_N nearest to p, converges to an optimal solution to* (P). *In particular, this is true if p is the origin in $\mathscr{R}^n$.*

Proof. By hypothesis, there exists A_∞ in $\mathscr{K}(X)$ such that $\limsup A_N = \liminf A_N = A_\infty$. Since $A_N \subseteq X_N^*$, $N = 1, 2, ...,$ it follows that $A_\infty \subseteq \limsup K_N^*$ [5, p. 121], i.e., $A_\infty \subseteq K^*$. Now apply Theorem 4.1 and the succeeding remark. ∎

Remark. If (P) has a *unique* solution, then the theorem is valid for x_N^* *any* element of A_N, $N = 1, 2, ...$

4.3. *Non-Convex Optimization*

Consider the optimization problem

$$\max_{x \in K} f(x), \tag{P}$$

where K is a non-empty, compact subset of m-dimensional Euclidean space $\mathscr{R}^m$ and f is a continuous function of m real variables [3]. Suppose we try to solve this problem by the following grid-approximation technique.

For convenience, let X be any compact subset of $\mathscr{R}^m$ satisfying $K \subseteq X$. Assume also that K is the closure of its interior K^0. For each $n = 1, 2, ...,$ let

$$Z_n = \{k/n : k = \text{integer}\},$$
$$G_n = Z_n \times \cdots \times Z_n \qquad (m \text{ times})$$

and

$$K_n = K \cap G_n.$$

Then K_n is a finite subset of K which is eventually non-empty since K^0 is non-empty. Thus, $\{K_n\}$ is a sequence in $\mathscr{K}(X)$, for sufficiently large n.

LEMMA 4.4. *The sequence $\{K_n\}$ converges to K in $\mathscr{K}(X)$ relative to the Hausdorff metric.*

Proof. Since $K_n \subseteq K$, all n, it is clear that $\limsup K_n \subseteq K$. Suppose x is an element of K^0 and V is an arbitrary neighborhood of x contained in K^0. It is easy to see that V is eventually intersected by the K_n, i.e., $x \in \liminf K_n$ [11, p. 335]. Thus, $K^0 \subseteq \liminf K_n$, which implies that $K \subseteq \liminf K_n$, since K is the closure of K^0. The result then follows from Theorem 2.1. ∎

Let K^* denote the non-empty, compact set of optimal solutions to (P) and f^* the optimal objective value. Also let K_n^* denote the set of optimal solutions to the finite approximation problem

$$\max_{x \in K_n} f(x), \qquad (\mathrm{P}_n)$$

and f_n^* the corresponding optimal objective value, $n = 1, 2, \ldots$. (These are well defined for sufficiently large n.) Note that K_n^* is a finite, (eventually) non-empty subset of K_n, $n = 1, 2, \ldots$, i.e., $\{K_n^*\}$ is a sequence in $\mathscr{K}(X)$, for large n.

THEOREM 4.5. *The sequence $\{f_n^*\}$ converges to f^*. Also,* $\limsup K_n^* \subseteq K^*$. *Moreover, if K^* is a singleton $\{x^*\}$, then* any *selection of x_n^* in K_n^*, $n = 1, 2, \ldots$, converges to x^*.*

Proof. Follows from Lemma 4.4, Lemma 3.1, and Theorem 3.2. ∎

Remark. A sufficient (but not necessary) condition for K^* to be a singleton is that K be convex and f be strictly convex.

COROLLARY 4.6. *If K is convex, then for any p in X, any sequence of solutions to the problem* (P_n) *closest to p converges to a solution of* (P).

4.4. *Discrete Infinite Horizon Optimization*

Consider an infinite sequential decision problem where the jth decision is to be chosen from the finite set $\{0, 1, \ldots, M\}$ (see [4]). An infinite sequence of such decisions is a *strategy*. (It is assumed that all strategies extend over the infinite time horizon.) In particular, let $\theta = (0, 0, \ldots)$. The *strategy space* Y is then the product of countably many copies of the given decision set; it is a compact Hausdorff space relative to the product topology. If we fix $0<\beta<1$, then Y is also a metric space with metric given by

$$d_\beta(x, y)=\sum_{j=1}^{\infty} \beta^j |x_j-y_j|, \qquad x, y \in Y.$$

In general, not all strategies are feasible. Thus, we will assume there exists a closed, non-empty subset X of Y consisting of the *feasible* strategies. Also, let $\succ$ denote the canonical lexicographic ordering of the elements of Y. As in Ryan, Bean, and Smith [15], it can be verified that if $\beta<1/(M+1)$, then $d_\beta(\theta, x)>d_\beta(\theta, y)$ if and only if $x \succ y$. Moreover, $d_\beta(\theta, x)$ is a continuous function of x in Y. Consequently, if K is any element of $\mathscr{K}(X)$, then θ is in the uniqueness set of K and the unique element $s_\theta(K)$ of K closest to θ is the *lexicographic minimum* of K relative to $\succ$.

Suppose there is a cumulative net cost function associated with each strategy. In order to compare costs over a finite or infinite horizon, we continuously discount them to time zero relative to a suitable interest rate. Let X^* denote the subset of X consisting of those feasible strategies having minimum discounted infinite horizon cost. Assume X^* is non-empty and closed. Likewise, for $T>0$, let $X^*(T)$ denote the subset of X consisting of those feasible strategies having minimum discounted T-horizon cost. As above, assume each $X^*(T)$ is non-empty and closed. Then X^* is an element of $\mathscr{K}(X)$ and $\{X^*(T) \mid T>0\}$ is a generalized sequence in $\mathscr{K}(X)$. (Note that the results of Sections 2 and 3 are valid for sets indexed by $T>0$. We omit the details.) Let D_β be the Hausdorff metric on $\mathscr{K}(X)$ corresponding to d_β. Application of Theorem 3.4 yields:

THEOREM 4.8. *Suppose $\beta<1/(M+1)$. If $X^*(T) \to X^*$ in $\mathscr{K}(X)$, as $T \to \infty$, relative to D_β, then the generalized sequence of lexicographic minima of the $X^*(T)$ converges to the lexicographic minimum of X^*.*

Remarks. (1) In the presence of Hausdorff convergence of the finite horizon optimal solution sets, the previous theorem yields a tie-breaking algorithm for approximating an infinite horizon optimum by finite horizon optima. (2) In [16], Shapiro and Wagner considered an infinite horizon version of the knapsack problem. Ryan [14] has shown that Hausdorff convergence holds in this case. Hence, this problem provides an example where Theorem 4.8 holds.

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