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Simplified Description of Slow-in-the-Average Markov Walks

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The averaging theory for the slow Markov walks is extended to the so called slow-in-the-average Markov processes where the jump vector takes arbitrarily large values with sufficiently small probabilities. The results obtained are important for applications, in particular, communication networks and manufacturing systems.

I. INTRODUCTION

A slow Markov walk process is defined as

$$x(n+1) = x(n) + \varepsilon\Phi(x(n), \xi(n)),$$
$$x \in R^N,\ \xi \in R^W,\ \Phi\colon R^N \times R^W \to R^N,\ 0 < \varepsilon \ll 1,$$

where $\xi(n)$, $n = 0, 1, \ldots$, is a sequence of conditionally independent random variables and $\Phi(x(n), \xi(n))$ takes values of order 1. There have been many results obtained [1–6], concerning deterministic approximations of such processes. Applications of these results have been reported in [7–12]. In some applications, however, the jump vector, Φ, takes values of the order $1/\varepsilon$ and therefore the results of [1–6] are not applicable. The purpose of this paper is to extend the method of [1] to the so called slow-in-the-average Markov walks which admit Φ arbitrarily large but with sufficiently small probabilities.

The structure of this paper is as follows: In Section II, the notion of the slow-in-the-average Markov walk is introduced; in Section III, the main theorems are formulated; the proofs are given in the Appendices.

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II. Slow-in-the-Average Markov Walks

Consider the Markov process defined by

$$x(n+1) = x(n) + \Phi(x(n), ..., x(n-r), \xi(n)), \tag{1}$$
$$x \in R^N,\ \xi \in R^W,\ \Phi\colon \underbrace{R^N \times R^N \times \cdots \times R^N}_{(r+1)} \times R^W \to R^N,$$

where $\xi(n)$, $n = 0, 1, ...$, is a sequence of conditionally independent random variables with the following conditional probability distribution:

$$f(\xi(n) \mid x(n), ..., x(n-r))$$
$$= \{f_1(\xi_1(n) \mid x(n), ..., (n-r)), ..., f_W(\xi_W(n) \mid x(n), ..., x(n-r))\}. \tag{2}$$

Assume that

$$\begin{aligned} &E\{\Phi_i(x(n), ..., x(n-r), \xi(n)) \mid x(n), ..., x(n-r)\} \\ &\quad = \varepsilon\phi_i(x(n), ..., x(n-r)), \\ &\mathrm{Var}\{\Phi_i(x(n), ..., x(n-r), \xi(n)) \mid x(n), ..., x(n-r)\} \\ &\quad = \varepsilon^2 k_i(x(n), ..., x(n-r)), \\ &\qquad i = 1, ..., N, \qquad 0 < \varepsilon \ll 1, \end{aligned} \tag{3}$$

where $\phi_i(\cdot)$, $k_i(\cdot)$ are functions of order 1, and, in addition,

$$|\phi_i(\cdot)| \leqslant R, \qquad |k_i(\cdot)| \leqslant S, \qquad \text{for all } i \in \{1, ..., N\}, \text{ for all } x \in Q \subset R^N, \tag{4}$$

and R and S are independent of ε. Assume also that both $\phi(\cdot) = [\phi_1(\cdot), ..., \phi_N(\cdot)]^T$ and $k(\cdot) = [k_1(\cdot), ..., k_N(\cdot)]^T$ are Lipschitz in $Q \subset R^N$,

$$\begin{aligned} &\|\phi(x(n), ..., x(n-r)) - \phi(x(m), ..., x(m-r))\| \\ &\quad \leqslant \lambda_1\{\|x(n) - x(m)\| + \cdots + \|x(n-r) - x(m-r)\|\} \\ &\|k(x(n), ..., x(n-r)) - k(x(m), ..., x(m-r))\| \\ &\quad \leqslant \lambda_2\{\|x(n) - x(m)\| + \cdots + \|x(n-r) - x(m-r)\|\}, \end{aligned} \tag{5}$$

where $\|Z\| = \sum_{i=1}^N |Z_i|$.

The process defined by (1)–(5) will be referred to as *slow-in-the-average* Markov walk. As it follows from (1), (3), the jump vector, Φ, can take arbitrarily large values but with sufficiently small probabilities.

III. Main Theorems

Along with (1) consider the following deterministic equation:

$$y(n+1)=y(n)+\varepsilon\phi(y(n),\ldots,y(n-r)),$$
$$y\in R^N,\qquad y^0=[y(n_0),\ldots,y(n_0-r)]^T=[x(n_0),\ldots,x(n_0-r)]^T=x^0. \tag{6}$$

Theorem 1. *Under the assumptions* (3)–(5), *for any* $\sigma>0$ *and* $\tau>0$ *there exists* $\varepsilon_0=\varepsilon_0(\sigma)$ *and* $F=F(\tau)$ *such that for all* $0<\varepsilon\leqslant\varepsilon_0$

$$P\{\|x(n,x^0,n_0)-y(n,x^0,n_0)\|<\sigma\}\geqslant 1-\sigma F(\tau),\qquad n\in[n_0,n_0+\tau/\varepsilon], \tag{7}$$

where $x(n,x^0,n_0)$, *and* $y(n,x^0,n_0)$, *are solutions of* (1) *and* (6) *respectively which belong, together with their* σ-*vicinity, to* Q.

Theorem 2. *Assume that all trajectories of* (1) *are bounded a.s. and the equilibrium of* (6) *is globally asymptotically stable. Then, under the assumptions* (3)–(5), *for any* $\delta>0$, *there exists* $\varepsilon_0>0$ *such that for all* $0<\varepsilon\leqslant\varepsilon_0$

$$P\{\|x(n,x^0,n_0)-y(n,x^0,n_0)\|<\delta\}\geqslant 1-\delta,\qquad n\in[n_0,\infty). \tag{8}$$

The proofs of these theorems are given in the Appendices.

Appendix I

The proof of Theorem 1 requires several auxiliary results:

Lemma 1. *Under the assumptions of Theorem* 1, *there exist positive constants* T_1 *and* T_2 *such that the following inequality is true*:

$$E\|x(l)-x(n)\|\leqslant\frac{NT_1}{T_2}\{(1+\varepsilon T_2)^{l-n}-1\},\qquad \text{for all}\quad l>n. \tag{9}$$

Proof. From (3)–(5), we obtain

$$\begin{aligned}
&E\|x(l)-x(n)\|\\
&\quad\leqslant\sum_{s=n}^{l-1}E\|\Phi(x(s),\ldots,x(s-r),\xi(s))\|\\
&\quad\leqslant\sum_{i=1}^{N}\sum_{s=n}^{l-1}E\left\{\frac{\varepsilon^2+|\Phi_i(x(s),\ldots,x(s-r),\xi(s))|^2}{2\varepsilon}\right\}\\
&\quad=\sum_{i=1}^{N}\sum_{i=1}^{l-1}E\left\{\left(\frac{\varepsilon}{2}\right)[1+k_i(x(s),\ldots,x(s-r))+\phi_i^2(x(s),\ldots,x(n-r))]\right\}
\end{aligned}$$

$$\leqslant \frac{\varepsilon}{2} N(l-n) + \frac{\varepsilon}{2} \sum_{i=1}^{N} \sum_{s=n}^{l-1} E|k_i(x(n), ..., x(n-r)) + \phi_i^2(x(n), ..., x(n-r))|$$

$$+ \frac{\varepsilon}{2} \sum_{i=1}^{N} \sum_{s=n}^{l-1} E|k_i(x(s), ..., x(s-r)) - k_i(x(n), ..., x(n-r))|$$

$$+ \frac{\varepsilon}{2} \sum_{i=1}^{N} \sum_{s=n}^{l-1} E|\phi_i^2(x(s), ..., x(s-r)) - \phi_i^2(x(n), ..., x(n-r))|$$

$$\leqslant \varepsilon N(l-n) \frac{(S+R^2+1)}{2}$$

$$+ \frac{\varepsilon}{2} \sum_{s=n}^{l-1} \{\lambda_1 E\{\|x(s)-x(n)\| + \|x(s-1)-x(n)\| + \cdots$$

$$+ \|x(s-r)-x(n)\| + \|x(n)-x(n-1)\|$$

$$+ \cdots + \|x(n)-x(n-r)\|\}$$

$$+ 2R\lambda_2 E\{\|x(s)-x(n)\| + \|x(s-1)-x(n)\|$$

$$+ \cdots + \|x(s-r)-x(n)\|$$

$$+ \|x(n)-x(n-1)\| + \cdots + \|x(n)-x(n-r)\|\}\}$$

$$\leqslant \varepsilon N(l-n) \frac{(S+R^2+1)}{2} + \frac{\varepsilon^2 \lambda_1}{2} \sum_{i=1}^{r} \frac{i(i+1)NR}{2}$$

$$+ \frac{\varepsilon^2}{2} \lambda_1 NR \frac{r(r+1)}{2} (l-n)$$

$$+ \varepsilon^2 R\lambda_2 \sum_{i=1}^{r} \frac{i(i+1)NR}{2} + \varepsilon^2 R\lambda_2 NR \frac{r(r+1)}{2} (l-n)$$

$$+ \frac{\varepsilon}{2} \sum_{s=n}^{l-1} [(r+1)\lambda_1 + 2R\lambda_2(r+1)] E\|x(s)-x(n)\|$$

$$\leqslant \frac{NT_1}{T_2} \{(1+\varepsilon T_2)^{l-n} - 1\}, \qquad \text{for all} \quad l > n,$$

where

$$T_1 = N \frac{(S+R^2+1)}{2} + \frac{\varepsilon NR}{2} (r^2+r) \frac{(\lambda_1 + 2R\lambda_2)}{2}$$

$$+ \frac{\varepsilon NRr(r+1)(r+2)}{6} \frac{(\lambda_1 + 2R\lambda_2)}{2},$$

$$T_2 = \frac{(r+1)(\lambda_1 + 2R\lambda_2)}{2}.$$

The last inequality follows from Lemma 2 in [1]. Q.E.D.

To formulate the second lemma, introduce a random variable

$$\bar{x}(n) = \frac{1}{[\delta/\varepsilon]} \sum_{i=1}^{n+[\delta/\varepsilon]-1} x(i), \tag{10}$$

where δ is positive real to be chosen below.

LEMMA 2. *Under the assumptions of Theorem 1, there exists a positive real S_1 such that the following inequality is true:*

$$E\{\|\bar{x}(n) - x(n)\|\} < \delta S_1, \qquad n \in [n_0, \infty). \tag{11}$$

Proof. From Lemma 1,

$$\begin{aligned}
E\|\bar{x}(n) - x(n)\| &\leqslant \frac{1}{[\delta/\varepsilon]} \sum_{i=n}^{n+[\delta/\varepsilon]-1} E\|x(i) - x(n)\| \\
&\leqslant \frac{1}{[\delta/\varepsilon]} \sum_{i=n}^{n+[\delta/\varepsilon]-1} \frac{NT_1}{T_2} \{(1+\varepsilon T_2)^{[\delta/\varepsilon]} - 1\} \\
&\leqslant \frac{NT_1}{T_2} \{e^{\varepsilon T_2 [\delta/\varepsilon]} - 1\} \\
&< \delta S_1,
\end{aligned}$$

where $S_1 = NT_1 e^{T_2 \delta}$. Q.E.D.

LEMMA 3. *Let*

$$\begin{aligned}
\Delta u(n+1) &= \varepsilon \phi(u(n), ..., u(n-r)), \\
\Delta v(n+1) &= \varepsilon \phi(v(n), ..., v(n-r)) + \varepsilon^{3/2} \xi(n) + \varepsilon^{5/4} \eta(n),
\end{aligned} \tag{12}$$

where $u, v, \xi, \eta \in R^N$, $u^0 = [u(n_0), ..., u(n_0 - r)]^T = [v(n_0), ..., v(n_0 - r)]^T = v^0$, and $E|\xi_i(n)| \leqslant S_2$, $E|\eta_i(n)| \leqslant S_3$, for all $i \in \{1, ..., N\}$. Then there exists a positive real F_1 such that

$$E\|u(n) - v(n)\| \leqslant \varepsilon^{1/4} F_1 \{e^{\varepsilon \lambda_1 (r+1)(n-n_0)} - 1\}. \tag{13}$$

Proof. From (12), we obtain

$$\begin{aligned}
&E\|u(n) - v(n)\| \\
&\quad \leqslant \varepsilon \sum_{s=n_0}^{n-1} E\|\phi(u(s), ..., u(s-r)) - \phi(v(s), ..., v(s-r))\| \\
&\qquad + \varepsilon^{3/2} \sum_{s=n_0}^{n-1} E\|\xi(s)\| + \varepsilon^{5/4} \sum_{n=n_0}^{n-1} E\|\eta(s)\|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \varepsilon\lambda_1 \sum_{s=n_0}^{n-1} E\{\|u(s)-v(s)\| + \|u(s-1)-v(s-1)\| \\
&\quad + \cdots + \|u(s-r)-v(s-r)\|\} \\
&\quad + \varepsilon^{3/2} \sum_{s=n_0}^{n-1} E\|\xi(s)\| + \varepsilon^{5/4} \sum_{s=n_0}^{n-1} E\|\eta(s)\| \\
&\leqslant \varepsilon\lambda_1(r+1) \sum_{s=n_0}^{n-1} E\|u(s)-v(s)\| \\
&\quad + \varepsilon^{3/2} \sum_{s=n_0}^{n-1} E\|\xi(s)\| + \varepsilon^{5/4} \sum_{s=n_0}^{n-1} E\|\eta(s)\| \\
&\leqslant \varepsilon\lambda_1(r+1) \sum_{s=n_0}^{n-1} E\|u(s)-v(s)\| \\
&\quad + \varepsilon N(\varepsilon^{1/2}S_2 + \varepsilon^{1/4}S_3)(n-n_0).
\end{aligned}$$

By [1, Lemma 2] we find that

$$E\|u(n)-v(n)\| \leqslant \varepsilon^{1/4}F_1\{e^{\varepsilon\lambda_1(r+1)(n-n_0)} - 1\},$$

where $F_1 = N/\lambda_1(r+1)(\varepsilon^{1/4}S_2 + S_3)$. Q.E.D.

Proof of Theorem 1. From (10),

$$\begin{aligned}
\Delta\bar{x}_i(n+1) &= \frac{1}{[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} \Phi_i(x(j), ..., x(j-r), \xi(j)) \\
&= \varepsilon\phi_i(x(n), ..., x(n-r)) \\
&\quad + \frac{1}{[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} [\varepsilon\phi_i(x(j), ..., x(j-r)) \\
&\quad - \varepsilon\phi_i(x(n), ..., x(n-r))] \\
&\quad + \frac{1}{[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} [\Phi_i(x(j), ..., x(j-r), \xi(j)) \\
&\quad - \varepsilon\phi_i(x(j), ..., x(j-r))] \\
&= \varepsilon\phi_i(x(n), ..., x(n-r)) + \varepsilon^{3/2}\xi_i(n) + \varepsilon^{5/4}\eta_i(n), \qquad (14)
\end{aligned}$$

where

$$\xi_i = \frac{1}{\varepsilon^{1/2}[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} [\phi_i(x(j), ..., x(j-r)) - \phi_i(x(n), ..., x(n-r))],$$

$$\eta_i(n) = \frac{1}{\varepsilon^{5/4}[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} [\Phi_i(x(j), ..., x(j-r), \xi(j)) - \varepsilon\phi_i(x(j), ..., x(j-r))].$$

Then

$$\begin{aligned} E|\xi_i(n)| &\leqslant \frac{1}{\varepsilon^{1/2}[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} E|\phi_i(x(j), ..., x(j-r)) - \phi_i(x(n), ..., x(n-r))| \\ &\leqslant \frac{\lambda_1}{\varepsilon^{1/2}[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} E\|\phi(x(j), ..., x(j-r)) - \phi(x(n), ..., x(n-r))\| \\ &\leqslant \frac{\lambda_1}{\varepsilon^{1/2}[\delta/\varepsilon]} \sum_{j=n}^{n+[\delta/\varepsilon]-1} E\{\|x(j)-x(n)\| + \cdots + \|x(j-r)-x(n-r)\|\} \\ &\leqslant \frac{\lambda_1}{\varepsilon^{1/2}[\delta/\varepsilon]} \left\{ \sum_{j=n}^{n+[\delta/\varepsilon]-1} (r+1)\, E\|x(j)-x(n)\| + \varepsilon NR\frac{r(r+1)}{2}[\delta/\varepsilon] + \varepsilon NR\frac{r(r+1)(r+2)}{6} \right\} \\ &\leqslant \frac{\lambda_1(r+1)}{\varepsilon^{1/2}}\delta S_1 + \frac{\lambda_1\varepsilon^{1/2}NRr(r+1)}{2} + \frac{\lambda_1\varepsilon^{1/2}NRr(r+1)(r+2)[\varepsilon/\delta]}{6} \\ &\leqslant \frac{\lambda_1(r+1)}{\varepsilon^{1/2}}\delta S_1 + \frac{\lambda_1\varepsilon^{1/2}NRr(r+1)}{2}\left(1+\frac{(r+2)[\varepsilon/\delta]}{3}\right). \end{aligned}$$

Let $\delta = \varepsilon^{1/2}$, then

$$E|\xi_i(n)| \leqslant S_2',$$

where $S_2' = \lambda_1 S_1(r+1) + \lambda_1\varepsilon^{1/2}NRr(r+1)((3+(r+2)\,\varepsilon^{1/2})/6)$.

From (14), we obtain the inequality

$$E|\eta_i(n)|^2 = \frac{1}{\varepsilon^{5/2}[\delta/\varepsilon]^2} E\left| \sum_{j=n}^{n+[\delta/\varepsilon]-1} [\Phi_i(x(j), ..., x(j-r), \xi(j)) - \varepsilon\phi_i(x(j), ..., x(j-r))]\right|^2$$

$$=\frac{1}{\varepsilon^{5/2}[\delta/\varepsilon]^2}\sum_{j=n}^{n+[\delta/\varepsilon]-1} E|\Phi_i(x(j), ..., x(j-r), \xi(j))$$
$$-\varepsilon\phi_i(x(j), ..., x(j-r))|^2$$
$$\leqslant\frac{1}{\varepsilon^{5/2}[\delta/\varepsilon]^2}\sum_{j=n}^{n+[\delta/\varepsilon]-1} \varepsilon^2 S.$$

Since $\delta=\varepsilon^{1/2}$,

$$E|\eta_i(n)|^2\leqslant S.$$

Then,

$$E|\eta_i(n)|\leqslant\frac{1+E|\eta_i(n)|^2}{2}\leqslant\frac{1+S}{2}=S_3,$$

where $S_3=(1+S)/2$.

In order to represent (14) in a closed form, we rewrite it as

$$\Delta\bar{x}_i(n+1)=\varepsilon\phi_i(x(n), ..., x(n-r))+\varepsilon^{3/2}\xi_i'(n)+\varepsilon^{5/4}\eta_i(n), \tag{15}$$

where $\xi_i'(n)=(1/\varepsilon^{1/2})[\phi_i(x(n), ..., x(n-r))-\phi_i(\bar{x}(n), ..., \bar{x}(n-r))]+\xi_i(n)$. Then,

$$E|\xi_i'|\leqslant\frac{1}{\varepsilon^{1/2}}E|\phi_i(x(n), ..., x(n-r))$$
$$-\phi_i(\bar{x}(n), ..., \bar{x}(n-r))|+E|\xi_i(n)|$$
$$\leqslant\frac{1}{\varepsilon^{1/2}}\lambda_1 E\{\|x(n)-\bar{x}(n)\|$$
$$+\cdots+\|x(n-r)-\bar{x}(n-r)\|\}+S_2'$$
$$\leqslant\lambda_1(r+1)\,S_1+S_2'=S_2.$$

Consider now a system of the form

$$\Delta y_i(n+1)=\varepsilon\phi_i(y(n), ..., y(n-r)),$$
$$y_i^0=[y_i(n_0), ..., y_i(n_0-r)]^T=[x_i(n_0), ..., x_i(n_0-r)]^T=x_i^0.$$

Since the conditions of Lemma 3 are met, we have

$$E\{\|\bar{x}(n)-y(n)\|\}\leqslant\varepsilon^{1/4}F_1\{e^{\varepsilon\lambda_1(r+1)(n-n_0)}-1\}.$$

Therefore,

$$\begin{aligned} E\{\|x(n)-y(n)\|\} &\leqslant E\{\|x(n)-\bar{x}(n)\|\}+E\{\|\bar{x}(n)-y(n)\|\} \\ &< \varepsilon^{1/2}S_1+\varepsilon^{1/4}F_1\{e^{\varepsilon\lambda_1(r+1)(n-n_0)}-1\} \\ &= \varepsilon^{1/4}F(\tau), \qquad \text{for all} \quad n\in[n_0, n_0+\tau/\varepsilon], \end{aligned}$$

where $F(\tau)=\varepsilon^{1/4}S_1+F_1\{e^{\lambda_1(r+1)\tau}-1\}$. Hence

$$P\{\|x(n, x^0, n_0)-y(n, x^0, n_0)\|<\varepsilon^{1/8}\}\geqslant 1-\varepsilon^{1/8}F(\tau).$$

Let $\sigma=\varepsilon^{1/8}$, then

$$P\{\|x(n, x^0, n_0)-y(n, x^0, n_0)\|<\sigma\}\geqslant 1-\sigma F(\tau). \qquad \text{Q.E.D.}$$

Appendix II

Proof of Theorem 2. Without loss of generality, assume that the origin is the equilibrium point of (6). Due to the global asymptotic stability of (6), for any $\mu>0$ there exists $\tau>0$, such that

$$\|y(n)\|\leqslant\mu/2, \qquad \text{for all} \quad n\geqslant n_0+\tau/\varepsilon.$$

Then, from the following inequality

$$\begin{aligned} &E\|x(n)\|\leqslant E\|x(n)-y(n)\|+E\|y(n)\|, \\ &\quad y^0=[y(n_0), ..., y(n_0-r)]^T=[x(n_0), ..., x(n_0-r)]^T=x^0 \end{aligned}$$

and the assumption that all trajectories of (1) are bounded a.s., when $n=n_0+N_0=n_0+\tau/\varepsilon$ we have

$$\begin{aligned} E\|x(n_0+N_0)\| &\leqslant \varepsilon^{1/4}F(\tau)+\|y(n_0+N_0)\| \\ &\leqslant \varepsilon^{1/4}F(\tau)+\mu/2. \end{aligned} \tag{16}$$

Choose number ε_0 such that, for all $\varepsilon\leqslant\varepsilon_0$,

$$\varepsilon^{1/4}F(\tau)<\mu/2.$$

Then we have

$$E\|x(n_0+N_0)\|<\mu, \qquad N_0=\tau/\varepsilon. \tag{17}$$

Let $\zeta=x(x(n_0), n_0+N_0)$. For each realization of trajectory $y(\zeta, n_0+N+n)$, the following inequality holds:

$$E\|x(\zeta, n_0+N_0+n)\|\leqslant\varepsilon^{1/4}F(\tau)+\|y(\zeta, n_0+N_0+n)\|, \qquad n\in[0, \tau/\varepsilon].$$

Averaging this inequality over the distribution of ζ and taking into account that $E(E\|x(\zeta, n_0+N_0+n) \mid \zeta\|) = E\|x(x(n_0), n_0+N_0+n)\|$, we obtain

$$E\|(x(x(n_0), n)\| < \frac{\mu}{2} + E\|y(\zeta, n)\|, \qquad n \in [n_0+N_0, n_0+2N_0]. \quad (18)$$

As follows from (17), $P\{\|\zeta\| > \mu^{1/2}\} \leqslant \mu^{1/2}$. Hence, taking into account that all trajectories of system (1) are bounded a.s., we find that

$$E\|y(\zeta, n)\| \leqslant \mu/2 + g\mu^{1/2}, \qquad n \in [n_0+N_0, n_0+2N_0],$$

where g is some constant. Thus from (18), we have

$$E\|x(x(n_0), n)\| < \mu + g\mu^{1/2}, \qquad n \in [n_0+N_0, n_0+2N_0]. \quad (19)$$

Moreover, since, by construction, $E\|y(\zeta, n_0+2N_0)\| < \mu/2$, we find from (18),

$$E\|x(x(n_0), n_0+2N)\| < \mu. \quad (20)$$

We now prove by induction that (19) holds for all $n \in [n_0, \infty)$. Let (19) hold on the interval $n \in [n_0+(l-1)N_0, n_0+lN_0]$. Then, when $n = n_0+lN_0$, (20) is true, and consequently (19) holds on the interval $[n_0+lN_0, n_0+(l+1)N_0]$. Thus (19) is valid for all $n \in [n_0+\tau/\varepsilon, \infty)$. On the other hand

$$\|y(x(n_0), n)\| \leqslant \mu/2, \qquad n \in [n_0+\tau/\varepsilon, \infty).$$

Consequently, $E\|x(n, x^0, n_0) - y(n, x^0, n_0)\| < \frac{3}{2}\mu + g\mu^{1/2}$, $n \in [n_0+\tau/\varepsilon, \infty)$. Taking into account the inequality $\varepsilon^{1/4}F(\tau) < \mu/2$, we have

$$E\|x(n, x^0, n_0) - y(n, x^0, n_0)\| < \tfrac{3}{2}\mu + g\mu^{1/2}, \qquad n \in [n_0, \infty).$$

Let $\delta^2 = \frac{3}{2}\mu + g\mu^{1/2}$, then

$$P\{\|x(n, x^0, n_0) - y(n, x^0, n_0)\| < \delta\} \geqslant 1 - \delta. \qquad \text{Q.E.D.}$$

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