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# A local limit theorem for perturbed random walks

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*Abstract*: The main result reported here is a Stone type local limit theorem for perturbed random walks $Z_n = S_n + \xi_n$ when some slow variation conditions are imposed on $\xi_n$'s.

*Keywords*: Local (central) limit theorem; Edgeworth expansion; perturbed random walks.

## 1. Background

When specialized to one dimension and non-lattice distributions, Stone's Theorem (1965) asserts the following: Let $X_1, X_2, X_3, \ldots$ denote i.i.d. non-lattice random variables for which $\mathbb{E}[X_i] = 0$ and $\mathbb{E}[X_i^2] = 1$. Let

$$S_n = X_1 + X_2 + \cdots + X_n, \quad n = 1, 2, 3, \ldots,$$

be the sums, called 'random walks'. Then for each $L \in [0, \infty)$,

$$\varepsilon_n(L) = \sup_{c \leqslant L} \sup_{b \in \mathbb{R}} \left| \sqrt{n}\, \mathbb{P}\{b < S_n \leqslant b + c\} - c\phi(b/\sqrt{n}) \right| \underset{n \to \infty}{\longrightarrow} 0,$$

where $\phi(x)$ is the standard normal density and $\mathbb{R}$ is the real line. A consequence of this is

$$\sqrt{n}\, \mathbb{P}\{S_n \in J\} \underset{n \to \infty}{\longrightarrow} |J| / \sqrt{2\pi}$$

for any interval $J \subset \mathbb{R}$ of length $|J|$. This is the result of Shepp (1964).

There has been recent interest in sequences of random variables, called perturbed random walks. See, for example, Siegmund (1985), Woodroofe (1982) and Lalley (1984). A perturbed random walk has the form

$$Z_n = S_n + \xi_n,$$

where $S_n$ is a random walk, $\xi_n$ is independent of $X_{n+1}, X_{n+2}, \ldots$ for all $n = 1, 2, \ldots$ and the sequence $\xi_n$, $n \geqslant 1$, is slowly changing in a sense described by Woodroofe (1982, p. 41).

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## 2. A local limit theorem

The main result of this paper is a local limit theorem for perturbed random walks. The following conditions are imposed.

**Condition C** (Cramer's condition). A random variable $X_i$ is said to satisfy Condition C if the characteristic function

$$\psi(t) = \mathbb{E}[e^{itX_i}] = \int_{\mathbb{R}} e^{itx} F(dx), \quad t \in \mathbb{R},$$

has the property

$$\limsup_{t \to \infty} |\psi(t)| < 1.$$

Cramer's condition is also known as the strongly non-lattice condition.

**Condition SL** (Slow variation condition on $\xi_i$'s). A sequence of random variables, $\xi_i$, is said to satisfy Condition SL if there is a number $\alpha \in (\frac{1}{2}, 1)$, such that

$$\lim_{n \to \infty} \sqrt{n}\, \mathbb{P}\left\{ |\xi_n - \xi_{n-[n^\alpha]}| \geqslant \varepsilon \right\} = 0$$

for each $\varepsilon > 0$.

**Condition M** (Moment condition on $\xi_i$'s). A sequence of random variables, $\xi_i$, is said to satisfy Condition M if for the $\alpha$ in Condition SL,

$$\lim_{n \to \infty} \frac{\mathbb{E}[|\xi_n|]}{n^{\alpha - 1/2}} = 0.$$

For $\xi_n$ satisfying Condition SL and M, and independent of $X_{n+1}, X_{n+2}, \ldots$ for all $n = 1, 2, \ldots,$ the sequence $Z_n = S_n + \xi_n$ is called a perturbed random walk, even though the conditions imposed differ from those imposed for nonlinear renewal theory.

**Theorem 1** (Stone type local limit theorem). *If* $\mathbb{E}|X_i^3| < \infty$ *and Conditions* C, SL *and* M *hold, then*

$$\varepsilon_n(L) = \sup_{0 \leqslant c \leqslant L} \sup_{b \in \mathbb{R}} \left| \sqrt{n}\, \mathbb{P}\{b < Z_n \leqslant b + c\} - c\phi(b/\sqrt{n}) \right| \xrightarrow[n \to \infty]{} 0$$

*for any* $L < \infty$.

**Examples.** A class of examples of perturbed random walks may be constructed by letting

$$Z_n = ng(S_n/n)$$

where $g(0) = 0$, $g'(0) = 1$, and $g \in \mathscr{C}^2([-\delta, \delta])$ (i.e., $g$ is twice continuously differentiable on $[-\delta, \delta]$) for some $\delta > 0$. Assume $\mathbb{E}|X_i^3| < \infty$ and Condition C holds for the $X_i$'s. Write

$$\xi_n = ng(S_n/n) - S_n = Z_n - S_n, \quad n \in \mathbb{N}.$$

If there is a $k \geqslant 0$ and $C > 0$ such that

$$|g(x)| \leqslant C(|x| + 1)^k, \quad x \in \mathbb{R},$$

and $\mathbb{E}|X_i^p| < \infty$ for some $p > k$, then $Z_n = S_n + \xi_n$ satisfies all the conditions in the local limit theorem for perturbed random walks (Theorem 1).

The verification of these assertions will be given in the last section.

The following lemma is an important tool in the proof of the theorem.

**Lemma 2** (Edgeworth expansion, Woodroofe, 1988). *Assume* $\mathbb{E}|X_i^3| < \infty$ *and Condition* C *holds. Let* $M$ *be given*, $0 < M < \infty$. *Then for* $b \in \mathbb{R}$ *and* $c \in [0, M]$,

$$F^{*n}(b+c) - F^{*n}(b-c) = \Phi\big((b+c)/\sqrt{n}\big) - \Phi\big((b-c)/\sqrt{n}\big) + \mathrm{O}(1/n)$$

*uniformly in b and c.* □

The lemma follows directly from Theorem 1 in Section 3 of Woodroofe (1988).

**Proof of Theorem 1.** For $0 \leqslant \beta < \gamma < \infty$ and $n \geqslant 1$, let

$$\varepsilon_n(\beta, \gamma) = \sup_{\beta \leqslant c \leqslant \gamma} \sup_{b \in \mathbb{R}} \left| \sqrt{n}\, \mathbb{P}\{b < Z_n \leqslant b + c\} - c\phi(b/\sqrt{n}) \right|.$$

Then it is easily seen that

$$\varepsilon_n(L) = \varepsilon_n(0, L) \leqslant \varepsilon_n(0, \delta) + \varepsilon_n(\delta, L) \quad \forall \delta \in (0, L).$$

So it is sufficient to show that for all $\delta > 0$, $\varepsilon_n(\delta, L) \to 0$ as $n \to \infty$.

Let $\alpha \in (\frac{1}{2}, 1)$ be as in Condition SL. Let $b \in \mathbb{R}$, $c \in [\delta, L]$ and $\varepsilon \in (0, \frac{1}{2}\delta)$. For sufficiently large $n$, let $l = [n^\alpha]$ and $m = n - l$. Then

$$\begin{aligned}\mathbb{P}\{b < Z_n \leqslant b + c\} &\leqslant \mathbb{P}\{b < Z_n \leqslant b + c, |\xi_n - \xi_m| < \varepsilon\} + \mathbb{P}\{|\xi_n - \xi_m| \geqslant \varepsilon\} \\ &\leqslant \mathbb{P}\{b - \varepsilon - Z_m < S_n - S_m \leqslant b + \varepsilon - Z_m + c\} + \delta_n/\sqrt{n}\end{aligned}$$

where $\delta_n \to 0$ as $n \to \infty$ by Condition SL. Let $\mathscr{B}_m = \sigma\{X_1, X_2, \dots, X_m;\ \xi_1, \xi_2, \dots, \xi_m\}$ be the $\sigma$-field generated by $\{X_1, X_2, \dots, X_m, \xi_1, \xi_2, \dots, \xi_m\}$ and

$$\Psi_n(x, c) = F^{*l}(x + c + \varepsilon) - F^{*l}(x - \varepsilon).$$

Then

$$\begin{aligned}&\mathbb{P}\{b - \varepsilon - Z_m < S_n - S_m \leqslant b + \varepsilon - Z_m + c\} \\ &\quad = \mathbb{E}\big[\mathbb{P}\{b - \varepsilon - Z_m < S_n - S_m \leqslant b + \varepsilon - Z_m + c \mid \mathscr{B}_m\}\big] \\ &\quad = \mathbb{E}\big[\Psi_n(b - Z_m, c)\big].\end{aligned}$$

Similarly,

$$\mathbb{P}\{b - \varepsilon < S_n \leqslant b + c + \varepsilon\} = \mathbb{E}\big[\Psi_n(b - S_m, c)\big].$$

So

$$\begin{aligned}\sqrt{n}\,\mathbb{P}\{b < Z_n \leqslant b + c\} &\leqslant \sqrt{n}\,\mathbb{P}\{b - \varepsilon < S_n \leqslant b + \varepsilon + c\} \\ &\quad + \sqrt{n}\,\mathbb{E}\big[|\Psi_n(b - Z_m, c) - \Psi_n(b - S_m, c)|\big] + \delta_n. \qquad (1)\end{aligned}$$

The first term on the right-hand side will give the desired upper bound by Stone's theorem. The second term will approach zero as $n \to \infty$ by using the local Edgeworth expansion (Lemma 2). The third term goes to zero as $n \to \infty$ as above.

For the first term, by Stone's theorem,

$$\left|\sqrt{n}\,\mathbb{P}\{b-\varepsilon<S_n\leqslant b+\varepsilon+c\}-(c+2\varepsilon)\phi\left((b-\varepsilon)/\sqrt{n}\right)\right| \underset{n\to\infty}{\longrightarrow} 0 \tag{2}$$

uniformly for $b\in\mathbb{R}$ and $c\in[\delta, L]$.

By Lemma 2 and a Taylor series expansion, there are constants $C_i(L)$ depending only on $L$ for which

$$\begin{aligned}
&\sqrt{n}\,|\Psi_n(b-Z_m, c)-\Psi_n(b-S_m, c)|\\
&\quad\leqslant \sqrt{n}\left|\Phi\left(\frac{b+c+\varepsilon-Z_m}{\sqrt{l}}\right)-\Phi\left(\frac{b-\varepsilon-Z_m}{\sqrt{l}}\right)\right.\\
&\qquad\left.-\left[\Phi\left(\frac{b+c+\varepsilon-S_m}{\sqrt{l}}\right)-\Phi\left(\frac{b-\varepsilon-S_m}{\sqrt{l}}\right)\right]\right|+C_1(L)\frac{\sqrt{n}}{l}\\
&\quad\leqslant \sqrt{n}\left|\left[\phi\left(\frac{b-\varepsilon-Z_m}{\sqrt{l}}\right)-\phi\left(\frac{b-\varepsilon-S_m}{\sqrt{l}}\right)\right]\frac{c+2\varepsilon}{\sqrt{l}}\right|+\frac{\sqrt{n}}{l}\left[(\max\phi')(c+2\varepsilon)^2+C_1(L)\right]\\
&\quad\leqslant C_2(L)\frac{\sqrt{n}}{l}\,|Z_m-S_m|+C_3(L)\frac{\sqrt{n}}{l}
\end{aligned}$$

for all $b\in\mathbb{R}$ and $c\leqslant L$. Observe that the last line does not depend on $b$ or $c$. For large $n$, $l\geqslant\frac{1}{2}n^{\alpha}$. So by Condition M,

$$\mathbb{E}\left[\frac{\sqrt{n}}{l}\,|Z_m-S_m|\right]\leqslant 2\frac{\mathbb{E}[\,|\xi_m|\,]}{n^{\alpha-1/2}} \underset{n\to\infty}{\longrightarrow} 0.$$

It follows from (1) and (2) that there is a sequence $\sigma_n$, $n\geqslant 1$, for which

$$\sqrt{n}\,\mathbb{P}\{b<Z_n\leqslant b+c\}-(c+2\varepsilon)\phi\left((b-\varepsilon)/\sqrt{n}\right)\leqslant\sigma_n \underset{n\to\infty}{\longrightarrow} 0,$$

uniformly in $b\in\mathbb{R}$, $c\in[\delta, L]$. Similarly, there is a sequence $\hat{\sigma}_n$ with $\hat{\sigma}_n\to 0$ as $n\to\infty$ and

$$\sqrt{n}\,\mathbb{P}\{b-Z_n\leqslant b+c\}-(c-2\varepsilon)\phi\left((b+\varepsilon)/\sqrt{n}\right)\geqslant\hat{\sigma}_n \underset{n\to\infty}{\longrightarrow} 0$$

uniformly in $b\in\mathbb{R}$, $c\in[\delta, L]$.

The theorem follows by letting $n\to\infty$ and then $\varepsilon\downarrow 0$. □

## 3. Directly Riemann integrable functions

The following definition is taken from Feller (1966). For a bounded real valued function $h$ defined on $\mathbb{R}$, $\delta>0$ and $k\in\mathbb{Z}$, let $J_k=(k\delta, (k+1)\delta]$ and

$$m_k=\min\{h(x): x\in J_k\},\qquad M_k=\max\{h(x): x\in J_k\},$$

$$\bar{\sigma}=\bar{\sigma}_\delta(h)=\sum_{k=-\infty}^{\infty}\delta M_k,\qquad \underline{\sigma}=\underline{\sigma}_\delta(h)=\sum_{k=-\infty}^{\infty}\delta m_k.$$

Then $h$ is said to be directly Riemann integrable ($h\in\mathrm{DR}$) if: (a) $\bar{\sigma}$, $\underline{\sigma}$ converge absolutely for sufficiently small $\delta>0$; (b) $\lim_{\delta\to 0}(\bar{\sigma}_\delta-\underline{\sigma}_\delta)=0$. Then $h\in L^1$ and for all $\varepsilon>0$,

$$\underline{\sigma}_\varepsilon\leqslant\lim_{\delta\to 0}\bar{\sigma}_\delta=\int_{\mathbb{R}}h(x)\,\mathrm{d}x=\lim_{\delta\to 0}\underline{\sigma}_\delta\leqslant\bar{\sigma}_\varepsilon.$$

**Theorem 3.** *Suppose that conditions of Theorem* 1 *are satisfied. If* $h \in \mathrm{DR}$, *then*

$$\lim_{n\to\infty} \sqrt{n}\,\mathbb{E}[h(Z_n)] = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} h(x)\,\mathrm{d}x.$$

**Proof.** There is no loss of generality in supposing that $h \geqslant 0$, since otherwise positive and negative parts may be considered separately.

Given $\varepsilon > 0$, there is a $\delta > 0$ for which $\bar{\sigma}_\delta - \underline{\sigma}_\delta \leqslant \varepsilon$. With this choice of $\delta$, by Theorem 1 there is an $N_0 \in \mathbb{N}$ for which

$$\left|\sqrt{n}\,\mathbb{P}\{k\delta < Z_k \leqslant (k+1)\delta\} - \delta\phi(k\delta/\sqrt{n})\right| \leqslant \varepsilon\delta/\sqrt{2\pi}$$

for all $n \geqslant N_0$ and all $k \in \mathbb{Z}$. There is $K \in \mathbb{N}$ for which

$$\sum_{|k|>K} m_k(\delta) \leqslant \sum_{|k|>K} M_k(\delta) \leqslant \varepsilon$$

and there is an $N_1 \in \mathbb{N}$ for which

$$(1-\varepsilon)/\sqrt{2\pi} \leqslant \phi(k\delta/\sqrt{n}) \leqslant 1/\sqrt{2\pi}$$

for all $|k| < K$, $n \geqslant N_1$. Then

$$\begin{aligned}\sqrt{n}\,\mathbb{E}[h(Z_n)] &\leqslant \sum_{k\in\mathbb{Z}} M_k\sqrt{n}\,\mathbb{P}\{k\delta < Z_n \leqslant (k+1)\delta\}\\ &\leqslant \sum_{k\in\mathbb{Z}} M_k\left[\delta\phi(k\delta/\sqrt{n}) + \varepsilon\delta/\sqrt{2\pi}\right]\\ &\leqslant \sum_{k\in\mathbb{Z}} \delta M_k(1/\sqrt{2\pi} + \varepsilon/\sqrt{2\pi})\\ &\leqslant \frac{1+\varepsilon}{\sqrt{2\pi}}\left(\int_{\mathbb{R}} h(x)\,\mathrm{d}x + \varepsilon\right)\end{aligned}$$

and for all $n \geqslant N_0 \vee N_1$ (i.e., $n \geqslant \max\{N_0, N_1\}$). Similarly,

$$\sqrt{n}\,\mathbb{E}[h(Z_n)] \geqslant \frac{1-2\varepsilon}{\sqrt{2\pi}}\left[\left(\int_{\mathbb{R}} h(x)\,\mathrm{d}x - \varepsilon\right) - \delta\varepsilon\right].$$

The theorem now follows by letting $n \to \infty$, $\delta \to 0$ and then $\varepsilon \to 0$. □

The local limit theorem (Theorem 1) is the special case when $h(x) = 1_{(b,b+c]}$.

## 4. Proof of the examples

Recall the Marcinkiewicz–Zygmund Inequality (see Section 10.3 in Chow and Teicher, 1978): Assume $\mathbb{E}[|X_i|^p] < \infty$, for some $p \geqslant 3$. Then

$$\mathbb{E}[|S_n|^p] \leqslant \mathbb{E}[|X_i|^p]Cn^{p/2}$$

for some constant $C$ depending on $p$.

**Proposition 4.** *Suppose $g \in \mathscr{C}^2([-\delta, \delta])$ for some $\delta > 0$ and there exists a constant $C > 0$ such that $|g(x)| \leqslant C(1+|x|)^k$ for all $x \in \mathbb{R}$. If $\mathbb{E}[|X_i|^p] < \infty$ for some $p$ with $p > k$ and $p \geqslant 3$ where the $X_i$'s satisfy condition C, then with $\xi_n = ng(S_n/n) - S_n$, $Z_n = S_n + \xi_n$ satisfies Theorem* 1.

**Proof.** Let $A_n = \{|\bar{X}_n| \leqslant \delta\}$. Then on $A_n$, $\xi_n = g''(\Delta_n)S_n^2/n$, $|\Delta_n| \leqslant \delta$ and $|g''(\Delta_n)1_{A_n}| \leqslant C_1$ for some $C_1 \in (0, \infty)$ independent of $n$. Let $h(x) = g(x) - x$. So $h \in \mathscr{C}^2([-\delta, \delta])$, $h(0) = 0$, $h'(0) = 0$, $\xi_n = nh(S_n/n)$. It is easy to see that $|h(x)| \leqslant C_2|x|^k$ for all $|x| \geqslant \delta$ for some constant $C_2 \in (0, \infty)$, and that $|h(x)| \leqslant C_3x^2$, for $x \in [-\delta, \delta]$, for some $C_3 \in (0, \infty)$.

*Step 1.* Show that condition M is satisfied. i.e., $\mathbb{E}[|\xi_n|] = o(n^{\alpha-1/2})$ for some $\alpha \in (\frac{1}{2}, 1)$. Choose $\alpha \in (\frac{1}{2}, 1-1/p) \subseteq (\frac{1}{2}, 1)$. Now

$$\mathbb{E}[|\xi_n|] \leqslant \mathbb{E}[|\xi_n|1_{A_n}] + \mathbb{E}[|\xi_n|1_{A'_n}]$$

where $A'_n$ is the complement of $A_n$. By the boundedness of $g''$ on $A_n$ and M–Z inequality,

$$\mathbb{E}[|\xi_n|1_{A_n}] = \mathbb{E}[|g''(\Delta_n)|1_{A_n}S_n^2/n] \leqslant C_1\mathbb{E}[S_n^2]/n \leqslant C_2 = o(n^{\alpha-1/2}).$$

By the properties of $h(x)$,

$$\begin{aligned}\mathbb{E}[|\xi_n|1_{A'_n}] &= n\mathbb{E}[|h(S_n/n)|1_{A'_n}] \leqslant nC_3\mathbb{E}[|S_n/n|^k1_{A'_n}] \\ &\leqslant C_4n^{2-p} = o(n^{\alpha-1/2}) \quad (\text{since } p \geqslant 3).\end{aligned}$$

The last line is a direct corollary of M–Z inequality. So Condition M is satisfied.

*Step 2.* Show that Condition SL holds. i.e.,

$$\mathbb{P}\{|\xi_n - \xi_{n-[n^\alpha]}| > \varepsilon] = o(n^{-1/2}) \quad \forall\varepsilon > 0.$$

Write $m = n - [n^\alpha]$. Then on $A_m \cap A_n$,

$$\begin{aligned}\xi_n - \xi_m &= nh(\bar{X}_n) - mh(\bar{X}_m) \\ &= nh'(\bar{X}_m)(\bar{X}_n - \bar{X}_m) + \tfrac{1}{2}nh''(\Delta_{n,m})(\bar{X}_n - \bar{X}_m)^2 + (m-n)h(\bar{X}_m) \\ &= I_1 + I_2 + I_3.\end{aligned}$$

where $\Delta_{n,m}$ is between $\mathrm{X}_n$ and $\bar{X}_m$ and $I_i$ denotes the $i$th term. The Markov Inequality and M–Z inequality give

$$\mathbb{P}\{A'_n \cup A'_m\} \leqslant \mathbb{E}[|\bar{X}_n|^p]/\delta^p + \mathbb{E}[|\bar{X}_m|^p]/\delta^p \leqslant C_1(p)n^{-p/2} = o(n^{-1/2}).$$

So it suffices to show that

$$\mathbb{P}\{|I_1 + I_2 + I_3|1_{A_n \cap A_m} > \varepsilon\} = o(n^{-1/2}) \quad \forall\varepsilon > 0.$$

By the Markov inequality, it is enough to prove that

$$\mathbb{E}[|I_i|1_{A_n \cap A_m}] = o(n^{-1/2}) \quad \forall\varepsilon > 0, \quad \text{for } i = 1, 2, 3.$$

By the independence of the $X_i$'s,

$$\mathbb{E}[I_1 | \mathscr{B}_m] = nh'(\bar{X}_m)\mathbb{E}[\bar{X}_n - \bar{X}_m | \mathscr{B}_m] = h'(\bar{X}_m)(m-n)\bar{X}_m.$$

Since $h \in \mathscr{C}^2([-\delta, \delta])$ and $h'(0)=0$, there is a constant $C_1$ for which $|h'(x)| \leqslant C_1|x|$ for all $x \in [-\delta, \delta]$. By M–Z inequality,

$$\begin{aligned}\mathbb{E}\left[\left|\mathbb{E}[I_1 \mid \mathscr{B}_m]1_{A_m}\right|^{p/2}\right] &\leqslant C_1^{p/2}(n-m)^{p/2}\mathbb{E}\left[\left|\bar{X}_m\right|^p\right]\\ &\leqslant C_2(p)((n-m)/m)^{p/2}\\ &\leqslant C_3(p)n^{p(\alpha-1)/2} = \mathrm{o}(n^{-1/2}) \quad (\text{since } \alpha < 1-1/p).\end{aligned}$$

Let $S_{n,m} = X_{m+1} + X_{m+2} + \cdots + X_n$. Then $I_1 - \mathbb{E}[I_1 \mid \mathscr{B}_m] = h'(\bar{X}_m)S_{n,m}$. Using the independence of $\bar{X}_m$ and $S_{n,m}$ and the property of $h(x)$ in $[-\delta, \delta]$, M–Z inequality gives

$$\begin{aligned}\mathbb{E}\left[\left|I_1 - \mathbb{E}[I_1 \mid \mathscr{B}_m]1_{A_m}\right|^p\right] &\leqslant C^p\mathbb{E}\left[\left|\bar{X}_m\right|^p\right]\mathbb{E}[|S_{n,m}|^p]\\ &\leqslant C_1(p)m^{-p/2}(n-m)^{p/2}\\ &\leqslant C_2(p)n^{-p(1-\alpha)/2} = \mathrm{o}(n^{-1/2}).\end{aligned}$$

Consequently, for all $\varepsilon > 0$,

$$\mathbb{E}[|I_1|1_{A_n \cap A_m}] \leqslant \mathbb{E}\left[\left|\mathbb{E}[I_1 \mid \mathscr{B}_m]1_{A_m}\right|\right] + \mathbb{E}\left[\left|I_1 - \mathbb{E}[I_1 \mid \mathscr{B}_m]\right|1_{A_m}\right] = \mathrm{o}(n^{-1/2}).$$

To prove $\mathbb{E}[|I_2|1_{A_n \cap A_m}] = \mathrm{o}(n^{-1/2})$, note that there is a constant $C \in (0, \infty)$ for which $|h''(\Delta_{n,m})1_{A_m \cap A_n}| \leqslant C$. Then

$$\begin{aligned}|I_2|1_{A_n \cap A_m} &= \tfrac{1}{2}n|h''(\Delta_{n,m})|\left(\bar{X}_n - \bar{X}_m\right)^2 1_{A_n \cap A_m}\\ &\leqslant \tfrac{1}{2}nC\left(\bar{X}_n - \bar{X}_m\right)^2\\ &= \tfrac{1}{2}nC\left(\frac{1}{n}S_{n,m} - \frac{n-m}{nm}S_m\right)^2\\ &\leqslant C_1\left(\frac{1}{n}S_{n,m}^2 + \frac{(n-m)^2}{nm^2}S_m^2\right).\end{aligned}$$

Therefore,

$$\begin{aligned}\mathbb{E}\left[|I_2|^{p/2}1_{A_n \cap A_m}\right] &\leqslant C_2\left\{\frac{1}{n^{p/2}}\mathbb{E}[|S_{n,m}|^p] + \frac{(n-m)^p}{n^{p/2}m^p}\mathbb{E}[|S_m|^p]\right\}\\ &\leqslant C_3\left\{\left(\frac{n-m}{n}\right)^{p/2} + \left(\frac{n-m}{n}\right)^p\right\} = \mathrm{o}(n^{-1/2}).\end{aligned}$$

Since $|h(x)| \leqslant Cx^2$ on $[-\delta, \delta]$, M–Z inequality yields

$$\mathbb{E}\left[|I_3|^{p/2}1_{A_n \cap A_m}\right] \leqslant C\mathbb{E}\left[(n-m)^{p/2}|\bar{X}_m|^p 1_{A_m}\right] \leqslant C_1([n^\alpha])^{p/2}m^{-p/2} = \mathrm{o}(n^{-1/2}).$$

The proposition follows. □

## Note

The results presented here are part of the author's dissertation under the direction of M. Woodroofe.

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