Computers & Structures Vol. 45, No. 1, pp. 197–207, 1992
Printed in Great Britain.

0045-7949/92 $5.00 + 0.00

TECHNICAL NOTE

A NOTE ON AUTOMATED DETECTION OF MOBILITY† OF SKELETAL STRUCTURES

M. Chirehdast and P. Y. Papalambros
Design Laboratory, Department of Mechanical Engineering and Applied Mechanics,
The University of Michigan, Ann Arbor, Michigan, U.S.A.

(*Received* 23 *May* 1991)

Abstract—A geometry-based rigorous approach to the detection of global and internal mobility of skeletal structures (trusses and frames) is presented. The method is useful for automated design of skeletal structures at the conceptual stage where the overall topology of the structure is known, but no exact detail on geometry and size is available. Such topologies are, for example, those generated by interpreting homogenization images in topology optimization. The two-dimensional theory is illustrated utilizing two design examples, a classical two-bar truss design, and one generated in the framework of the integrated structural optimization system (ISOS), introduced in earlier publications. An extension to three-dimensional structures is also provided along with a space truss design example, solved in the literature using force-based techniques.

1. INTRODUCTION

The recent introduction of homogenization techniques for rigorous determination of proper structural topologies [1] has started to affect the research in the field of structural design. Indeed, it has worked counter to established intuition where conceptual design is performed by humans (or by some sort of 'intelligent' computer programs), the subsequent details handled by computer-based analysis. Rather, in an integrated design environment using homogenization as part of the strategy [2, 3] the conceptual design phase is performed by a rigorous analytical optimization methodology, resulting in a gray scale image of the proposed topology. It is *then* that the human (or 'intelligent' computer program) is called to interpret this image into the form of a realistic, manufacturable structure. In a three-phase process, described in the cited references, a design task is initiated by specifying only boundary conditions, type and amount of material, and generating the image by using a homogenization-based optimization procedure (phase I); image processing and interpretation, possibly including alterations due to manufacturability requirements, yield a parameterized CAD-type representation (phase II); detailed shape and size optimization that may include additional constraints puts the finishing touches on the design (phase III). An early computer implementation of this strategy is the Integrated Structural Optimization System (ISOS) [2, 3].

One class of structures resulting from processing and intepreting homogenization images is *skeletal* structures, i.e., trusses and frames. As one may expect, mathematical optimization will tend to push the topological layout to its limits of efficiency, so skeletal interpretations of homogenization images may yield kinematically mobile structures. The homogenization process itself does not generally result in mobile structures, but the skeletal interpretation may do so. To illustrate this phenomenon two examples for skeletal structures developed through ISOS are provided here. The input to homogenization, consisting of an initial design domain, and boundary and loading conditions, is depicted in Fig. 1(a), and its output for a solid/void ratio of 1/3 is shown in Fig. 1(b). Figure 1(c) shows an intermediate output of ISOS, that is a skeletal interpretation of the image shown in Fig. 1(b). Is this skeletal structure mobile or not? (In structural mechanics and kinematics the former is called a 'structure' and the latter a 'mechanism'.) For this particular design example, a truss (pin-joined) interpretation is mobile, whereas a frame (welded) interpretation is not. Another example is shown in Fig. 2 where both interpretations are immobile; this example has been extensively treated in [3].

A design engineer can use intuition and visual inspection to detect any mobility in the structure and subsequent detailed structural analysis would reveal any undetected mobility problems. Automating this task in the preliminary design stage using only geometric information requires operationally useful necessary and sufficient conditions for immobility. This paper addresses the development of such conditions and the computational implementation in the ISOS environment. The ideas are akin to those in classical kinematics, but tailored to the presnt needs. Note that the developed theory is applicable outside the framework of ISOS as well, and can be used in any automated conceptual design environment. In the remainder of this introductory section we will make some links with previous work, before we proceed with developing the needed conditions.

The well-known rule of static determinancy (indeterminacy), that twice the number of nodes minus the number of bars has to be equal to (greater than) the number of unknown external components, is mentioned in elementary texts as the necessary condition for immobility of trusses, see, e.g., [4–6]. Obviously necessary conditions do not guarantee immobility. Norris and Wilbur [4] introduce a set of heuristic rules based on engineering intuition as sufficient conditions for immobility. Weaver [5] suggests that the number and position of supports and members be adequate

† Throughout this paper the phenomenon of mobility and immobility of structures is investigated. In order to maintain consistency same terms are used in reviewing the work of other researchers, even though they have used different terms. For instance, stability and instability are avoided because they are used in the context of buckling and equilibrium bifurcation, kinematic stability is an oxymoron, and rigidity is used for structures which are not deformable.

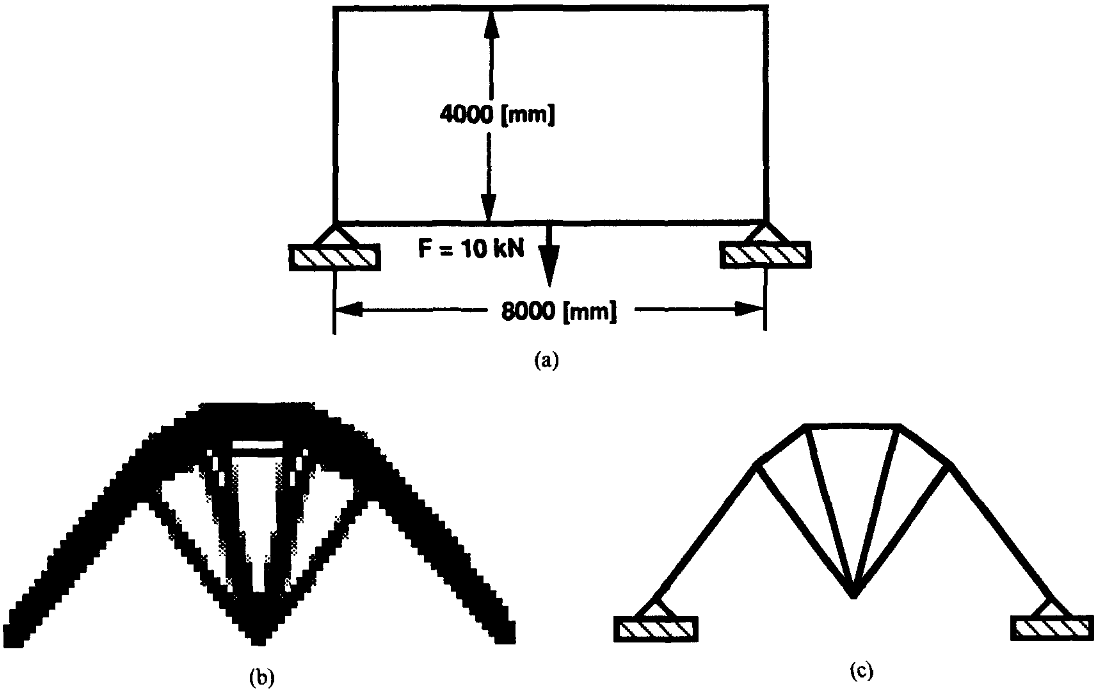

Fig. 1. (a) Input to homogenization, (b) optimal topology generated by homogenization, (c) mobile truss interpretation of image (b).

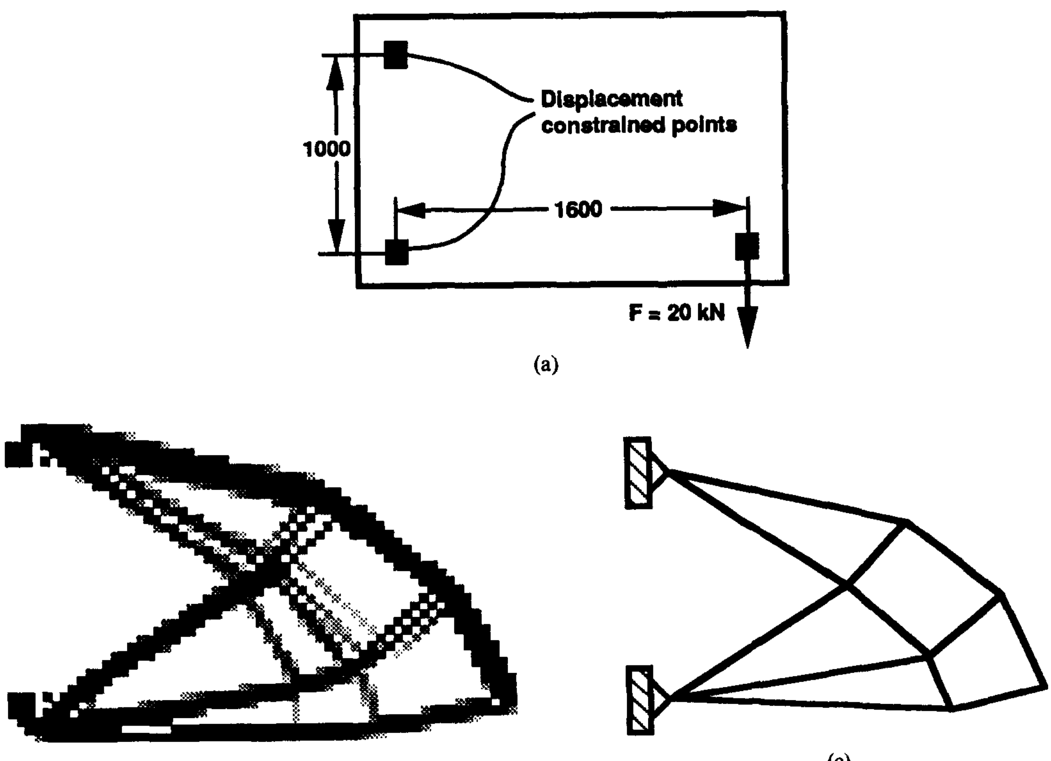

Fig. 2. Input to homogenization, (b) optimal topology generated by homogenization, (c) immobile truss interpretation of image (b).

for immobility of the structure but without giving specific means of determining such adequacy.

In early work, Southwell [7] introduced tension coefficients to determine the forces in the bars of statically determinate trusses. Generalizing the set of loads applied at the nodes can reveal a unique relation of internal forces to external loads. This uniqueness, referred to as determinacy, is equivalent to immobility of the structure. Such a force-based method was applied as recently as 1980 to detect mobility of space trusses [8]. A similar approach was taken by Timoshenko and Young [9] as a basis for deriving the *zero-load test* which is less attractive for automation than the original tension coefficient method. One may note that these procedures cannot be formally extended to statically indeterminate structures, and that no formal definition of immobility of skeletal structures is used for rigorous proofs. An additional disadvantage is that components of the structure have to be treated as external forces, resulting in a lack of differentiation between global and internal immobility, as will be discussed later in this article. Using classical mobility criteria based on potential energy of mechanical systems, such as, Dirichlet's criterion (see, e.g., [10]), has the disadvantage of global kinematic complications.

In a typical kinematics treatment, e.g., in [11], Grübler's formula is discussed as a method of determining the degree of freedom of a mechanism. A degree of freedom equal to or less than zero, although necessary for immobility of the structure, does not guarantee that the structure is not mobile. Even more sophisticated formulae for the degree of freedom, such as the one derived by Kutzbach [12], represent only a necessary condition. Hunt [13] discusses how screw theory can be applied to determine the so-called *actual* degree of freedom of mechanisms. He also indicates (without elaborating) that the theory may be applied to cases where the degree of freedom of the mechanism is less than or equal to zero, i.e., where one is dealing with structures.

As mentioned earlier, one way of detecting mobility is structural analysis, for example, using finite-element methods. For every skeletal structure a stiffness matrix is computed which, multiplied by the displacement vector, gives the loading. If the stiffness matrix of a structure is singular, without applying any loads there exist nonzero displacements. From our present viewpoint, the disadvantage of this approach is that detailed information on the structure is not available at the conceptual level of a design process.

The approach taken here is purely geometric and ignores the loading conditions. The definition of immobility is in terms of geometric quantities (see Sec. 2), the loads contributing to mobility of the structure rather than causing it. The geometric configuration of the structure is considered as the primary cause of mobility. This approach is based on ideas introduced by Grüning [14] for two-dimensional skeletal structures (both trusses and frames). The theory has been applied to two-dimensional statically-determinate topologies generated in the framework of ISOS with extensions to three-dimensional structures. As most of the homogenization results to date are two-dimensional, we will start with the two-dimensional case.

The remainder of the article is as follows. In Sec. 2 some terms are formally defined and necessary and sufficient conditions for immobility are derived. In Sec. 3 some remarks useful for application purposes are provided. Two two-dimensional examples are given in Sec. 4 for illustration. A three-dimensional extension of the theory is provided in Sec. 5, followed by a space truss example in Sec. 6.

2. NECESSARY AND SUFFICIENT CONDITIONS FOR IMMOBILITY

We begin with some definitions.

A *node* or *joint* is a point in space used to connect one or more members of a skeletal structure. A *linear member* or *element* is the linear connection between two nodes. The linear elements are divided into *bars* and *beams*. Bars can carry only axial forces whereas beams can carry axial and shear forces and also bending moments. Bars and beams can be connected at the nodes using *hinges* or *welds*. If a linear element is connected via a *hinge*, at its respective end it accepts only forces. In contrast, if a linear element is connected via to a *weld*, at its respective end it accepts both forces and moments. A weld is considered stiff about those of its axes that can transmit moments. A *clamp* or a *clamped-end* supports a structure by providing an external moment around a given axis. A (*simple*) *support* supports a structure by providing an external force component in a given direction.

Definition 1

A *skeletal structure* is defined as a set of *distinct* nodes connected by a set of linear elements and externally supported by a set of clamps and supports. Bars, beams, nodes, simple supports, and clamps are collectively referred to as the *members* of the structure.

A skeletal structure with only pin-joined bars is referred to as a *truss*, while one with only beams joined by welded joints is called a *frame*. A skeletal structure that is neither a truss nor a frame is referred to as a *hybrid structure*. If not explicitly specified, the term 'structure' is used for *hybrid structures* for the remainder of this paper.

Definition 2

A skeletal structure is *globally immobile*, if and only if changes in the positions of its nodes (i.e., node displacements) with respect to fixed points are possible only by simultaneously deflecting the bars and beams, and/or deforming the stiff joints, and/or moving the supports, and/or deforming the clamps of the structure.

Definition 3

A skeletal structure is *internally immobile*, if and only if changes in the positions of its nodes, relative to each other, are possible only by simultaneously deflecting the bars and beams and/or deforming the stiff joints of the structure.

As it will be explained later, the internal and global immobility of a structure differ only in terms of the kinematic boundary conditions used to support that structure. A skeletal structure is *mobile* (internally or globally), if and only if it is not immobile. An obvious simple theorem can be then stated as follows.

Theorem 1

A structure is mobile, if and only if displacing any combinations of its nodes is possible without deflecting or deforming members of the structure.

Proof

The proof follows trivially from the following logical equivalences and Definitions 1–3. (1) $p \Leftrightarrow q \equiv \neg p \Leftrightarrow \neg q$ and (2) $\neg(p \Leftrightarrow q) \equiv p \wedge \neg q$. In case of Equivalence (2), p corresponds to existence of nonzero nodal displacements and q corresponds to existence of nonvanishing deformations in the structure.

Theorem 1 means that a nodal displacement in a rigid structure (where no deformations take place), exists if and only if the structure is mobile. This statement can be easily verified for any mechanism.

Based on this theorem, analytical expressions for necessary and sufficient immobility conditions of two-dimensional structures can be derived. The following symbols are used. All coordinates of the nodes and their deflections are expressed with respect to a Cartesian coordinate system as shown in Fig. 3. The position of node i is expressed in terms of its coordinates x_i and y_i. The differentials Δx_i and Δy_i denote the infinitesimal displacement components of that

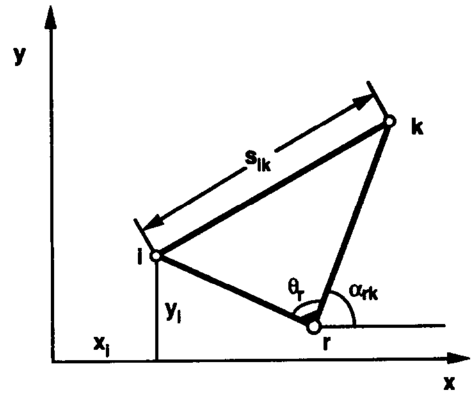

Fig. 3. Positions of nodes and dimensions and orientations of their connections.

node along the x and y axes, respectively. The length of linear element (ik) connecting nodes i and k is denoted by s_{ik}, its change by Δs_{ik}. The welded joint r connects linear elements (ri) and (rk), θ_r denotes the angle between those members and $\Delta\theta_r$ the infinitesimal change in that angle. A simple support r constrains the movement of node r along a prescribed direction for a given distance c_r, (see also Fig. 4). A clamped-end r constrains the angle of rotation of beam (rk) about an infinitesimal angle of τ_r. Note that a clamped-end r does not necessarily impose any constraints on the displacement of the corresponding node r.

With the above notation, the following corollary to Theorem 1 can be stated.

Corollary 1

A structure is mobile, if and only if the deformation quantities ($\Delta s, \Delta\theta, c$, and τ) can vanish while the displacement quantities (Δx and Δy) do not.

Note that this corollary and other theorems discussed here are not limited to two-dimensional structures.

We now proceed to derive operationally useful expressions for necessary and sufficient conditions for immobility (or mobility) of skeletal structures. As it can be readily seen from Fig. 3

$$s_{ik}^2 = (x_k - x_i)^2 + (y_k - y_i)^2 \tag{1}$$

$$(s_{ik} + \Delta s_{ik})^2 = [(x_k - x_i) + (\Delta x_k - \Delta x_i)]^2 + [(y_k - y_i) + (\Delta y_k - \Delta y_i)]^2. \tag{2}$$

The deflection Δs_{ik} of member (ik) is presumably small compared to s_{ik} so that the quantity $\Delta s_{ik}/s_{ik}$ is much smaller than one and can be neglected. The same is true for $(\Delta x_k - \Delta x_i)/s_{ik}$ and $(\Delta y_k - \Delta y_i)/s_{ik}$. Subtracting the two equations above, dividing by s_{ik}, and taking this approximation into account, we obtain

$$\Delta s_{ik} = \frac{(\Delta x_k - \Delta x_i)(x_k - x_i) + (\Delta y_k - \Delta y)(y_k - y_i)}{s_{ik}}. \tag{3}$$

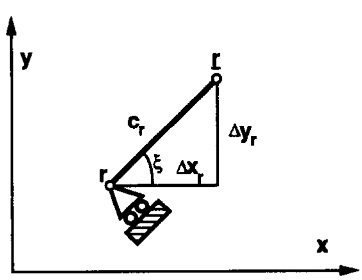

Fig. 4. Support r whose movement is constrained along a given axis.

As shown in Fig. 3, let α_{rk} be the angle between the x axis and the direction of element (rk), and let $\underline{\alpha}_{rk}$ be the same angle after the nodal positions have changed. The change in that angle is denoted by $\Delta\alpha_{rk}$; i.e.

$$\Delta\alpha_{rk} = \underline{\alpha}_{rk} - \alpha_{rk}. \tag{4}$$

For small $\Delta\alpha_{rk}$ we have

$$\Delta\alpha_{rk} \approx \tan(\Delta\alpha_{rk}) = \tan(\underline{\alpha}_{rk} - \alpha_{rk}) = \frac{\tan(\underline{\alpha}_{rk}) - \tan(\alpha_{rk})}{1 + \tan(\underline{\alpha}_{rk})\tan(\alpha_{rk})}. \tag{5}$$

The following equation can be easily derived from geometry

$$\tan(\alpha_{rk}) = \frac{y_k - y_r}{x_k - x_r}, \tag{6}$$

where $x_k \neq x_r$. Similarly for $\underline{\alpha}_{rk}$, we have

$$\tan(\underline{\alpha}_{rk}) = \frac{y_k - y_r + \Delta y_k - \Delta y_r}{x_k - x_r + \Delta x_k - \Delta x_r}. \tag{7}$$

Again assuming that the absolute values of quantities $(\Delta x_k - \Delta x_r)/s_{rk}$ and $(\Delta y_k - \Delta y_r)/s_{rk}$ are much smaller than unity the following equation is obtained from eqns (5) and (7)

$$\Delta\alpha_{rk} = \frac{(x_k - x_r)(\Delta y_k - \Delta y_r) - (y_k - y_r)(\Delta x_k - \Delta x_r)}{s_{rk}^2}. \tag{8}$$

The case where $x_k = x_r$ can be treated by the following coordinate transformation: $x^* = y$ and $y^* = -x$, where the new coordinates have an asterisk as a superscript. A back transform leads to the same final result [eqn (8)] as for the case where $x_k \neq x_r$. Noting from Fig. 3 that

$$\Delta\theta_r = \Delta\alpha_{ri} - \Delta\alpha_{rk} \tag{9}$$

and using eqn (8) we obtain that

$$\Delta\theta_r = \frac{(x_i - x_r)(\Delta y_i - \Delta y_r) - (y_i - y_r)(\Delta x_i - \Delta x_r)}{s_{ri}^2} - \frac{(x_k - x_r)(\Delta y_k - \Delta y_r) - (y_k - y_r)(\Delta x_k - \Delta x_r)}{s_{rk}^2}. \tag{10}$$

As illustrated in Fig. 4, for every simple support r which constrains the deflection of node r for a distance c_r in a direction along a line at an angle ξ with the x-axis, the following relationship is obtained

$$c_r = \Delta x_r \cos(\xi) + \Delta y_r \sin(\xi). \tag{11}$$

A relation can be derived from eqn (10) for every clamped-end r where the angle of deflection of the beam (rk) is restricted to an angle τ_r. Note that if $\Delta\alpha_{ri} = 0$, then $\tau_r = -\Delta\alpha_{rk}$. Therefore

$$\tau_r = \frac{(x_k - x_r)(\Delta y_k - \Delta y_r) - (y_k - y_r)(\Delta x_k - \Delta x_r)}{s_{rk}^2}. \tag{12}$$

Equations (3) and (10)–(12) provide the sought immobility criteria in the form of linear equations relating the displace ment quantities (Δx and Δy) to the deformation quantities ($\Delta s, \Delta\theta, c$, and τ).

The following symbols are used for the numbers of various members: k = number of nodes, r = number of linear elements, w = number of welds, a = number of support components, and e = number of clamped-ends. Now, eqns (3) and (10)–(12) can be uniformly represented in the matrix notation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = (\Delta x_1, \Delta y_1, \ldots, \Delta x_k, \Delta y_k)^T$, an n-vector,

$\mathbf{b} = (\Delta s_1, \ldots, \Delta s_r, \Delta\theta_1, \ldots, \Delta\theta_w, c_1, \ldots, c_a, \tau_1, \ldots, \tau_e)^T$, an m-vector, and $\mathbf{A}$ is an $[m \times n]$ matrix. Note that $n = 2k$ and $m = r + w + a + e$. Using Corollary 1 we conclude that a structure is globally mobile, if and only if the linear system $\mathbf{Ax} = \mathbf{0}$ has a nontrivial solution. Linear algebra theory leads to the following results (see, e.g. [15]).

Theorem 2

A structure is globally immobile, if and only if rank $(\mathbf{A}) = n$.

Corollary 2

A structure is globally mobile, if and only if rank $(\mathbf{A}) \neq n$.

Comparing Definitions 2 and 3, one can conclude that internal immobility can be studied the same way as global immobility by removing the external constraints on the structure and fixing the Cartesian coordinate system to the structure. Removing the external constraints corresponds to neglecting equations for supports and clamped-ends, i.e., eqns (11) and (12). Fixing the Cartesian coordinate system to the structure can be done by fixing one node to be the origin of the coordinate system (without loss of generality, we can assume that node to be Node 1) and constraining the motion of an additional node along one axis, such that the other axis passes through that node (again without loss of generality, we can assume that node to be Node 2). This procedure corresponds to translating and rotating the coordinate system and also setting to zero the displacement components Δx_1, Δy_1, and Δy_2. Note that the Cartesian coordinate system needs to be translated such that Node 1 is its origin and the x-axis passes through Node 2 (see [15] for details on coordinate transformations). Now the linear system of equations represented by eqns (3) and (10) remains to be studied, that is $\mathbf{A}_I\mathbf{x} = \mathbf{b}$, where $x = (\Delta x_2, \Delta x_3, \Delta y_3, \ldots, \Delta x_k, \Delta y_k)^T$, an n_i-vector, $\mathbf{b} = (\Delta s_1, \ldots, \Delta s_r, \Delta\theta_1, \ldots, \Delta\theta_w)^T$, an m_i-vector, and $\mathbf{A}_I$ is an $[m_i \times n_i]$ coefficient matrix, with $n_i = 2k - 3$, and $m_i = r + w$. Note that eqns (3) and (10) need to be represented in terms of the transformed nodal coefficients. The following result is then easily shown.

Theorem 3

A structure is internally immobile, if and only if rank $(\mathbf{A}_I) = n_i$.

This concludes the basic two-dimensional immobility results.

3. REMARKS ON APPLYING THE THEORY

Some remarks are now in order regarding the proper application of the above results in practice.

For trusses no welded joints and clamped-ends exist, therefore $e = w = 0$. No other difference exists in the study of their immobility compared to that of hybrid structures. However, note that according to our earlier definitions, trusses have certain modeling restrictions: (a) no moments can be applied; (b) forces can be applied only at the joints. As stated in [8], if a truss is mobile its equivalent frame model may be immobile. In most of such cases the nodal deflections of the frame are still considerably large, since the bending effects are dominant in order to sustain the structure which would be mobile if only axial forces were used.

For a joint with k beams welded on it, there exists only $k - 1$ linearly independent angular equations of type eqn (8), although the number of possible combinations of enclosed angles (pairs of linear elements) equals $k(k - 1)$.

The quantity a denotes the sum of the number of linearly independent constraints for each support. For example, for a pinned support that number is equal to 2, the corresponding equations being $\Delta x = \Delta y = 0$.

Half-hinges are used to model some hybrid structures. Figure 5 depicts a half-hinge. The coordinate changes of

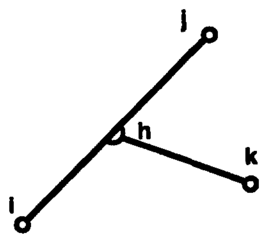

Fig. 5. A half-hinge h in the middle of linear element (ij) and at the end of (kh).

half-hinges are not treated as independent unknowns of x, the vector of the displacement quantities. This is due to the fact that the coordinates of half-hinges are linear functions of the end nodes of the beams on which they appear. However, the length of the bars or beams whose end nodes are half-hinges provide us with a constraint. Alternatively, one can introduce two independent displacement components (Δx_h and Δy_h) for each half-hinge and add two deflection equations (constraints) for every beam on which the half-hinge appears. These equations are

$$\Delta x_i + t(\Delta x_j - \Delta x_i) - \Delta x_h = 0 \tag{13}$$

$$\Delta y_i + t(\Delta y_j - \Delta y_i) - \Delta y_h = 0, \tag{14}$$

where i and j refer to the two end nodes of the beam on which the half-hinge appears. Parameter

$$t = \frac{x_h - x_i}{x_j - x_i} \quad \text{if} \quad x_i \neq x_j$$

otherwise

$$t = \frac{y_h - y_i}{y_j - y_i}.$$

Note that according to the definition of a skeletal structure (Def. 1), $x_i \neq x_j$ or $y_i \neq y_j$. Equations (13) and (14) can be derived using the fact that a half-hinge h belongs to the line passing through nodes i and j, both before and after nodal displacements take place.

A comment needs to be made on the completeness of the set of joints used to connect linear elements for both two- and three-dimensional structures. Although the joints commonly used in structural design are treated in this article, the set of joints is by no means complete. Equations for other joints, mainly used in mechanism design, can be derived similarly.

Finally, we recall that the rank of a matrix is the number of linearly independent rows or columns and the most commonly used algorithm to determine the rank of a matrix is some sort of Gaussian elimination. After performing Gaussian elimination, the matrix consists of some rows containing only zeros and some with nonzero elements, the rank of the original matrix being equal to the smaller of the two numbers of nonzero rows and columns of the transformed matrix. In practice, zero will be a number which is small relative to the others. Thus, prior to performing the Gaussian elimination, we should normalize (scale) the row vectors of the original matrix which represent comparable physical quantities. Any number in the resulting matrix whose absolute value is smaller than a certain amount, should be regarded as zero. The condition number of square matrices is a good measure of finding how singular or full rank these matrices are.

4. TWO-DIMENSIONAL EXAMPLES

In this section two truss examples are discussed to illustrate the applicability of the above results.

Figure 6 shows a statically determinate two-bar truss with two pinned supports. This configuration is immobile except

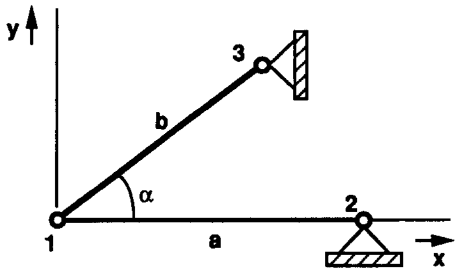

Fig. 6. Two-bar truss design example.

for the case where the angle α becomes zero or π radiants. This fact will be verified using the theory. According to the first comment in the previous section $e = w = 0$, $r = 2$, $a = 4$, $k = 3$, and therefore the matrix **A** is 6×6. The four equations for the supports [eqn (11)] are easily written by setting the x and y displacement components of Nodes 2 and 3 to zero, i.e., $\Delta x_2 = \Delta y_2 = \Delta x_3 = \Delta y_3 = 0$. Two equations are set up for the deflection of the two bars according to eqn (3): for Bar 'a' (connecting Nodes 1 and 2): $\Delta s_1 = \Delta x_2 - \Delta x_1$ and for Bar 'b' (connecting Nodes 1 and 3): $\Delta s_2 = (\Delta x_3 - \Delta x_1)\cos(\alpha) + (\Delta y_3 - \Delta y_1)\sin(\alpha)$. Thus, the homogeneous system of linear equations has the following matrix representation

$$\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ -\cos(\alpha) & -\sin(\alpha) & 0 & 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \Delta x_1 \\ \Delta y_1 \\ \Delta x_2 \\ \Delta y_2 \\ \Delta x_3 \\ \Delta y_3 \end{bmatrix} = \mathbf{0}. \tag{15}$$

Now, rank $(\mathbf{A}) = 6$, if and only if $\sin(\alpha) \neq 0$. Therefore, according to Theorem 2, the necessary and sufficient condition for the mobility of the structure is that α be equal to zero or π. One may also verify that regardless of the value of angle α, the structure is internally mobile.

The second example is a skeletal structure generated by homogenization in ISOS. Figure 7(b) shows a computer-generated skeletal image of the stiffest possible topology for a given amount of material for the design problem illustrated in Fig. 7(a). Using computer vision techniques the image of Fig. 7(b), i.e., the output of homogenization, is transformed into the truss structure shown in Fig. 7(c). The nodal coordinates and their connectivities are extracted, and support conditions are retrieved automatically and stored in data structures, serving as the input to the mobility detection algorithm. For this particular problem this information has the form shown in Table 1.

The size of matrix **A** for this problem is 18×18. The output of the algorithm shows the rank of the matrix to be equal to 18, and hence the structure is globally immobile. Note that because of the given support conditions global and internal stabilities are equivalent.

5. EXTENSION TO THREE DIMENSIONS

Definitions and theorems introduced in Sec. 2 are also valid for three-dimensional structures. The main difference is in the equations derived in Sec. 2, even though some of them can be easily extended. Some types of joints do not exist in two-dimensional structures. Equations for them will also be derived in this section. Derivatives of those equations trivially extendable from the two-dimensional to the three-dimensional case are not discussed here. The nomenclature introduced in Sec. 2 is still valid unless stated otherwise.

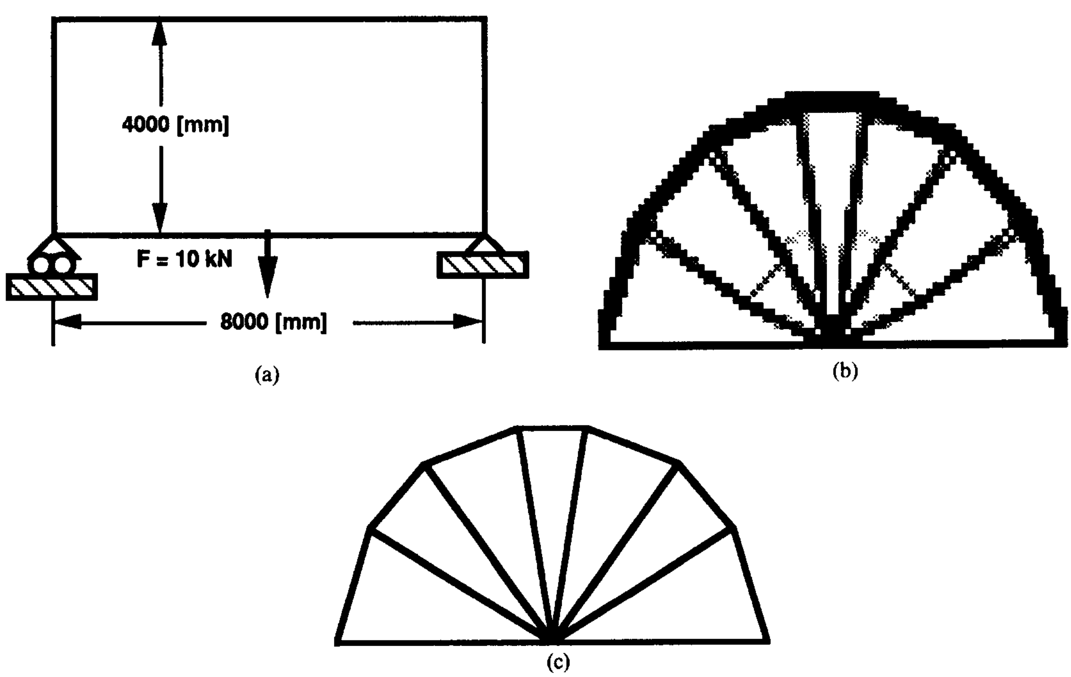

Fig. 7. (a) Initial design domain and specifications as input to ISOS, (b) computer-generated optimal topology, (c) final design interpretation of the structure as a truss.

Table 1. Input to the mobility detection algorithm.

Node	Position x	y	Connected nodes	Boundary conditions
1	40	1	2 9 3 4 6 7 5	Force applied
2	8	19	1 4 8	
3	72	19	1 5 9	
4	19	31	1 2 6	
5	62	31	1 3 7	
6	35	37	1 4 7	
7	47	36	1 5 6	
8	1	1	1 2	roller along x-axis
9	80	1	1 3	pinned joint

† This fact is not needed for the purposes of mobility detection.

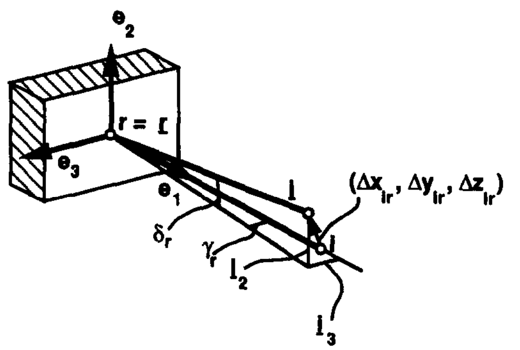

Fig. 9. A clamped-end and its characteristic dimensions and angles.

Equation (3) can be extended to eqn (16) where z_i denotes the z-coordinate of node along the z-axis

$$\Delta s_{ik} = \frac{(\Delta x_k - \Delta x_i)(x_k - x_i) + (\Delta y_k - \Delta y_i)(y_k - y_i) + (\Delta z_k - \Delta z_i)(z_k - z_i)}{s_{ik}}. \quad (16)$$

Equations for supports where the movement is restricted along a given direction are derived as follows. Two angles characterize the restricting direction of support r. These two angles are shown in Fig. 8 and are denoted by θ and φ, c_r being the amount by which the support is forced to move. The following equation can be derived for c_r as a function of the displacement components of the node and the two introduced angles

$$c_r = \Delta x_r \cos(\varphi)\cos(\theta) + \Delta y_r \sin(\varphi)\cos(\theta) + \Delta z_r \sin(\theta). \quad (17)$$

For the two-dimensional case equations for angles of clamped-ends and welds were derived similarly, whereas in the three-dimensional extension different approaches need to be taken to derive those equations. We will start with equations for clamped-ends. Figure 9 shows a clamped-end r in three-dimensional space supporting member (ri). The displaced positions of the nodes are underlined. An orthogonal coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is introduced where $\mathbf{e}_1$ is a unit vector along the initial position of member (ri), and $\mathbf{e}_2$ and $\mathbf{e}_3$ are perpendicular to $\mathbf{e}_1$ and to each other, i.e.

$$\mathbf{e}_1 = \frac{(x_i - x_r, y_i - y_r, z_i - z_r)}{s_{ir}} = \frac{(x_{ir}, y_{ir}, z_{ir})}{s_{ir}} \quad (18)$$

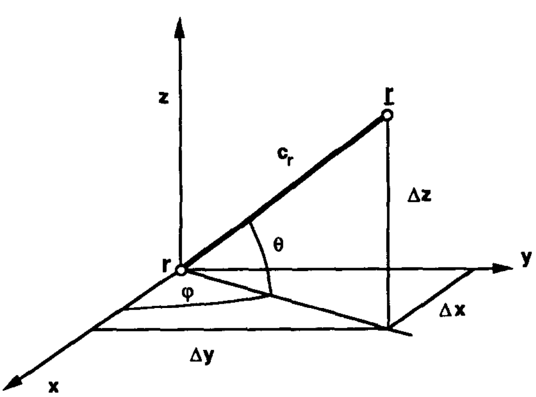

Fig. 8. Spatial position of a constrained support r.

$$\mathbf{e}_2 = \begin{cases} (1, 0, 0) & \text{if } x_{ir} = y_{ir} = 0 \\ \dfrac{(y_{ir}, -x_{ir}, 0)}{\sqrt{y_{ir}^2 + x_{ir}^2}} & \text{otherwise} \end{cases} \quad (19)$$

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2, \quad (20)$$

where × indicates a vector product. To simplify notation, we set $x_{ir} = x_i - x_r$. Similar notation is introduced for the other coordinates and nodes. The displaced position of node i is denoted by $\underline{i}$ and its coordinates can be expressed in the initial $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ and the new $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ coordinate systems as follows:

$$\underline{i} = (x_{ir} + \Delta x_{ir})\mathbf{e}_x + (y_{ir} + \Delta y_{ir})\mathbf{e}_y + (z_{ir} + \Delta z_{ir})\mathbf{e}_x = \underline{i}_1 \mathbf{e}_1 + \underline{i}_2 \mathbf{e}_2 + \underline{i}_3 \mathbf{e}_3. \quad (21)$$

Without loss of generality we can assume that the origin of both coordinate systems is $\underline{r}$, the displaced position of node r. Taking the inner product of both sides of eqn (21) with $\mathbf{e}_2$ and $\mathbf{e}_3$ gives the equations for $\underline{i}_2$ and $\underline{i}_3$, respectively

$$\underline{i}_2 = (\Delta x_{ir}, \Delta y_{ir}, \Delta z_{ir})\mathbf{e}_2 = \Delta x_{ir} e_{2x} + \Delta y_{ir} e_{2y} + \Delta z_{ir} e_{2z} \quad (22)$$

$$\underline{i}_3 = (\Delta x_{ir}, \Delta y_{ir}, \Delta z_{ir})\mathbf{e}_3 = \Delta x_{ir} e_{3x} + \Delta y_{ir} e_{3y} + \Delta z_{ir} e_{3z}. \quad (23)$$

Components of $\mathbf{e}_2$ and $\mathbf{e}_3$ can be found from eqns (19) and (20), respectively. As mentioned earlier [for the derivation of eqn (17)], two angles are needed to prescribe the position of a line with respect to a given reference frame, assuming that one point of that line is fixed. In the clamped-end case these two angles are γ_r and δ_r, shown in Fig. 10. For small angles the following approximations are valid

$$\gamma_r \approx \frac{\underline{i}_3}{s_{ir}} \quad (24)$$

$$\delta_r \approx \frac{\underline{i}_2}{s_{ir}} \quad (25)$$

so that, approximately

$$\gamma_r = \frac{(\Delta x_{ir} e_{3x} + \Delta y_{ir} e_{3y} + \Delta z_{ir} e_{3z})}{s_{ir}} \quad (26)$$

$$\delta_r = \frac{(\Delta x_{ir} e_{2x} + \Delta y_{ir} e_{2y} + \Delta z_{ir} e_{2z})}{s_{ir}}. \quad (27)$$

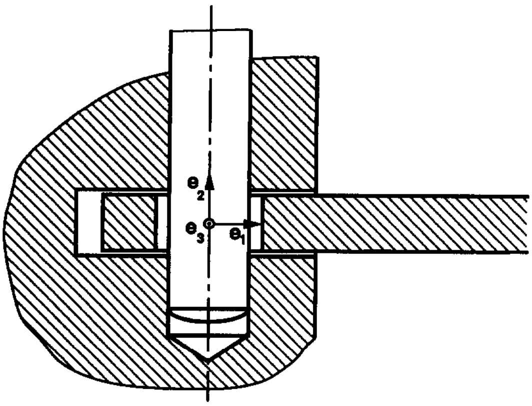

Fig. 10. Cross-sectional view of a clamped-end without in-plane angular constraints.

Figure 10 depicts the cross-section of a type of clamp for which the angular deflection of the linear element (ri) in a given plane, with the *given* normal vector $\mathbf{e}_2 = (e_{2x}, e_{2y}, e_{2z})$, is not restricted. The out-of-plane angular deflection about $\mathbf{e}_3$, denoted by v_r, is specified and can be obtained in a manner similar to δ_r in eqn (27).

A slightly more complicated situation arises when one is dealing with welded joints. Figure 11 shows a yielded joint r connecting two elements (ri) and (rk). The displaced positions of nodes are underlined. As mentioned earlier, two angles are needed to determine the relative position of two elements with respect to each other. The only way to obtain two meaningful angles is to project the nodes onto two different planes. There are infinite possibilities for two normal planes in the three-dimensional space. However, as it will be explained later, there is only one alternative pair of projections which provides us with reasonable results. First we discuss the unit vectors introduced for the purpose of the projections. The origin of all unit vectors is assumed to be in $\underline{r}$. Vectors $\mathbf{e}_i$ and $\mathbf{e}_k$ are parallel to elements (ri) and (rk) respectively and $\mathbf{e}_3$ is their cross-product. Vector $\mathbf{e}_1$ is along the vector sum of $\mathbf{e}_i$ and $\mathbf{e}_k$, and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ represents a right-handed coordinate system. Thus, the following equations can be derived for these vectors

$$\mathbf{e}_i = \frac{(x_i - x_r, y_i - y_r, z_i - z_r)}{s_{ir}} = \frac{(x_{ir}, y_{ir}, z_{ir})}{s_{ir}} \tag{28}$$

$$\mathbf{e}_k = \frac{(x_k - x_r, y_k - y_r, z_k - z_r)}{s_{kr}} = \frac{(x_{kr}, y_{kr}, z_{kr})}{s_{kr}} \tag{29}$$

$$\mathbf{e}_3 = \frac{(\mathbf{e}_i \times \mathbf{e}_k)}{\|\mathbf{e}_i \times \mathbf{e}_k\|} \tag{30}$$

$$\mathbf{e}_1 = \frac{(\mathbf{e}_i + \mathbf{e}_k)}{\|\mathbf{e}_i + \mathbf{e}_k\|} \tag{31}$$

$$\mathbf{e}_2 = \mathbf{e}_3 \times \mathbf{e}_1, \tag{32}$$

where $\|\cdot\|$ denotes the Euclidean norm of a vector.

The first projection is along the vector $\mathbf{e}_3$ (thus, onto the plane spanned by $\mathbf{e}_i$ and $\mathbf{e}_k$). The infinitesimal in-plane angle of deflection σ_r equals the difference of the two angles σ_i and σ_k. The final result for this angle is given in eqn (33) below. Replacing the components of the given vectors in that equation will lead to unnecessarily long expressions and is therefore avoided

$$\sigma_r = (\Delta x_{ir}, \Delta y_{ir}, \Delta z_{ir})(\mathbf{e}_3 \times \mathbf{e}_i) - (\Delta x_{kr}, \Delta y_{kr}, \Delta z_{kr})(\mathbf{e}_3 \times \mathbf{e}_k). \tag{33}$$

The second projection can be carried out along any vector in the plane spanned by $\mathbf{e}_i$ and $\mathbf{e}_k$. However, in order to avoid singularities and also for the linearization to be valid, the projection of either of the elements may vanish or even become small compared to the original lengths of the members. The following procedure provides a general projection fulfilling the above requirements. The projection is carried out along $\mathbf{e}_2$ (thus, onto the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_3$) as long as the angle between $\mathbf{e}_i$ and $\mathbf{e}_k$ is less than or equal to $\pi/2$; otherwise the projection is carried out along $\mathbf{e}_1$ (thus, onto the plane spanned by $\mathbf{e}_2$ and $\mathbf{e}_3$). The following equation gives a general formula for the out-of-plane angular deflection denoted by ψ_r.

$$\psi_r = \begin{cases} \dfrac{(\Delta x_{ir}, \Delta y_{ir}, \Delta z_{ir})\mathbf{e}_3}{(x_{ir}, y_{ir}, z_{ir})\mathbf{e}_1} - \dfrac{(\Delta x_{kr}, \Delta y_{kr}, \Delta z_{kr})\mathbf{e}_3}{(x_{kr}, y_{kr}, z_{kr})\mathbf{e}_1} & \text{if } \mathbf{e}_1\mathbf{e}_2 \geqslant 0 \\[2ex] \dfrac{(\Delta x_{ir}, \Delta y_{ir}, \Delta z_{ir})\mathbf{e}_3}{(\mathrm{x}_{ir}, \mathrm{y}_{ir}, \mathrm{z}_{ir})\mathbf{e}_2} - \dfrac{(\Delta x_{kr}, \Delta y_{kr}, \Delta z_{kr})\mathbf{e}_3}{(x_{kr}, y_{kr}, z_{kr})\mathbf{e}_2} & \text{if } \mathbf{e}_1\mathbf{e}_2 < 0. \end{cases} \tag{34}$$

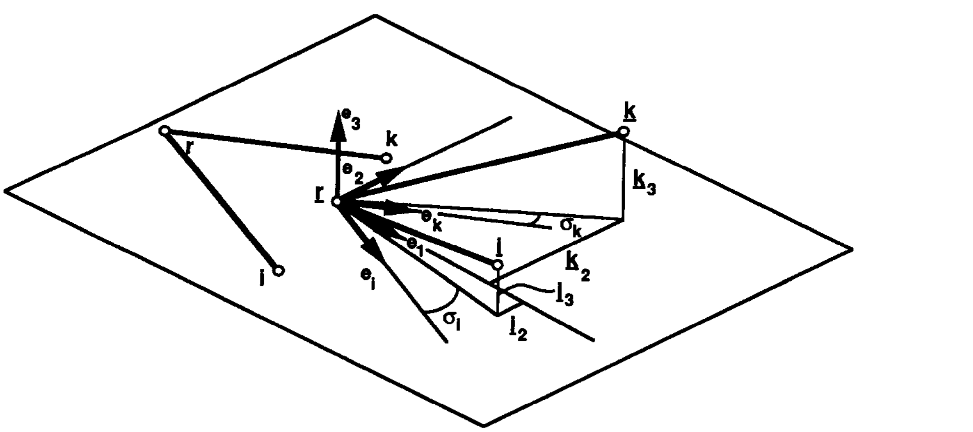

Fig. 11. A welded connection and its associated angles and dimensions.

Note that ψ_r is a smooth function of $\mathbf{e}_1\mathbf{e}_2$, since at $\mathbf{e}_1\mathbf{e}_2 = 0$ projection of the vector (x_{ir}, y_{ir}, z_{ir}) onto $\mathbf{e}_1$ and $\mathbf{e}_2$ are equal and the same is true for the vector (x_{kr}, y_{kr}, z_{kr}). Equations (33) and (34) are derived for the case where nodes r, i, and k span a unique plane. They can be easily extended to include the singular case where the three nodes are aligned. The vector $\mathbf{e}_1$ is then a unit vector parallel to the line (rik), and vectors $\mathbf{e}_2$ and $\mathbf{e}_3$ are any pair of orthogonal unit vectors perpendicular to $\mathbf{e}_1$. Equations (19) and (20) give one way of computing these vectors.

An equation relating displacement and deformation of *revolute joints* can be derived similarly to eqn (34). Revolute joints act as a hinge in the plane spanned by the three nodes and as an out-of-plane weld. A door hinge is an example of a revolute joint. It is obvious that although eqn (33) is still valid, angle γ is no longer prescribed. The prescribed out-of-plane angle of deformation is denoted by λ and computed exactly as ψ in eqn (34) for welds.

Similar immobility arguments are valid as for the two-dimensional case and will be discussed here. Equations (16), (17), (26), (27), (33), and (34) provide the sought immobility criteria in the form of linear equations relating the displacement quantities (Δx, Δy, and Δz) to the deformation quantities ($\Delta s, c, \gamma, \delta, v, \sigma, \psi$ and λ). The following symbols are used for the numbers of various members: k = number of nodes, r = number of linear elements; a = number of support components, e = number of clamped-ends, p = number of clamped-ends where the in-plane angle is not restricted, w = number of welds, and h = number of welded (stiff) joints where the in-plane angle is not restricted. Now the equations numbered above can be uniformly represented in the matrix notation $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{x} = (\Delta x_1, \Delta y_1, \ldots, \Delta x_k, \Delta y_k)^T$, an n-vector, $\mathbf{b} = (\Delta s_1, \ldots, \Delta s_r, c_1, \ldots, c_a, \gamma_1, \ldots, \gamma_e, \delta_1, \ldots, \delta_e, v_1, \ldots, v_p, \sigma_1, \ldots, \sigma_w, \psi_1, \ldots, \psi_w, \lambda_1, \ldots, \lambda_h)^T$, an m-vector, and $\mathbf{A}$ is an $[m \times n]$ matrix. Note that $n = 3k$ and $m = r + a + 2e + p + 2w + h$, Theorem 2 and its corollary derived in Sec. 2 are valid for three-dimensional structures as well.

In a way very similar to the two-dimensional case, internal immobility is studied by ignoring the external constraints and fixing the Cartesian coordinate system to the structure. Ignoring the external constraints corresponds to discarding eqns (17), (26), and (27). Fixing the coordinate system to the structure is analogous to the two-dimensional case and requires two steps. First a secondary coordinate system attached to the structure must be introduced. Second, the coordinates of nodes must be transformed from the initial coordinate system to the newly introduced one. Since the procedures to perform the second step can be found in references on elementary analytical geometry, such as [15], we will discuss only the first step in more detail. We assume that the structure has at least three non-aligned nodes, otherwise it can be treated by means discussed in Sec. 2. The origin of the new coordinate system is Node 1, its x-axis passes through Node 2, and Node 3 is on the xy plane. In order to uniquely determine the new coordinate system we assume the y coordinate of Node 3, y_3, is positive. Therefore, the following displacement components have to vanish: $\Delta x_1 = \Delta y_1 = \Delta z_1 = \Delta y_2 = \Delta z_2 = \Delta z_3 = 0$. The new coordinate system is now uniquely fixed to the structure; its unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ can be found as follows:

$$\mathbf{e}_x = \frac{(x_2 - x_1, y_2 - y_1, z_2 - z_1)}{s_{12}} \tag{35}$$

$$\mathbf{v} = \frac{(x_3 - x_1, y_3 - y_1, z_3 - z_1)}{s_{13}} \tag{36}$$

$$\mathbf{e}_z = \mathbf{e}_x \times \mathbf{v} \tag{37}$$

$$\mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_x, \tag{38}$$

where $\mathbf{v}$ is an auxiliary vector.

As in the two-dimensional case, the system of eqns (16), (33), and (34) represented in the form $\mathbf{A}_i\mathbf{x} = \mathbf{b}$, determines the internal immobility of the structure, where $\mathbf{x} = (\Delta x_2, \Delta x_3, \Delta y_3, \Delta x_4, \Delta z_4, \ldots, \Delta x_k, \Delta y_k, \Delta z_k)^T$ is an n_i-vector, $\mathbf{b} = (\Delta s_1, \ldots, \Delta s_r, \sigma_1, \ldots, \sigma_w, \psi_1, \ldots, \psi_w, \lambda_1, \ldots, \lambda_h)^T$ an m_i-vector, $\mathbf{A}_i$ is an $[m_i \times n_i]$ coefficient matrix, $n_i = 3k - 6$, and $m_i = r + 2w + h$. Theorem 3 and its corollary can be applied here to determine the internal immobility of the structure.

6. A THREE-DIMENSIONAL TRUSS EXAMPLE

As mentioned earlier, an immobility problem of a three-dimensional structure using a force-based approach is solved in [8]. We will solve the same problem using the theory developed in this article and obtain the same results.

Figure 12 shows the front and right-side view of a truss structure which consists of two regular n-sided polygons (in this case penagons) lying in parallel planes and connected via triangles. Structures of this type have applications in aerospace engineering. The corners of the polygon lying in the xy plane are pin-supported. The structure has $2n$ nodes and $3n$ bars. Since the structure is assumed to be a truss, only equations like eqns (16) and (17) need be considered. Equation (17) for the pinned-supported nodes (on the xy plane) can be expressed as follows:

$$0 = \Delta x_i = \Delta y_i = \Delta z_i \quad \text{for } i = n + 1, \ldots, 2n. \tag{39}$$

Replacing the displacement components of the pinned nodes in equations for length deformation similar to eqn (16) provides us with the required immobility matrix $\mathbf{A}$, which in this case reduces to a $[3n \times 3n]$ matrix. One can verify that matrix $\mathbf{A}$ has the format of eqn (40) by writing eqn (16) for each of the bars in the n connecting triangles; we only state

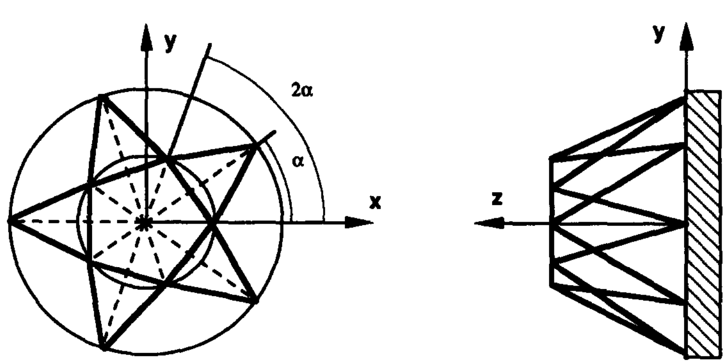

Fig. 12. A three-dimensional truss structure (n = 5).

the result, namely $\mathbf{A}$ is the matrix

$$\begin{bmatrix} r - R\cos(\alpha) & -R\sin(\alpha) & h & 0 & 0 & 0 & 0 & \cdots \\ r(1-\cos(2\alpha)) & -r\sin(2\alpha) & 0 & r(\cos(2\alpha)-1) & r\sin(2\alpha) & 0 & 0 & \cdots \\ 0 & 0 & 0 & r\cos(2\alpha) - R\cos(\alpha) & r\sin(2\alpha) - R\sin(\alpha) & h & 0 & \cdots \\ & & & \cdots & & & & \\ \cdots & r\cos(2(k-1)\alpha) - R\cos((2k-1)\alpha) & r\sin(2(k-1)\alpha) - R\sin((2k-1)\alpha) & h & 0 & 0 & \cdots & \\ \cdots & r(\cos(2(k-1)\alpha) - \cos(2k\alpha)) & r(\sin(2(k-1)\alpha) - \sin(2k\alpha)) & 0 & r(\cos(2k\alpha) - \cos(2(k-1)\alpha)) & r(\sin(2k\alpha) - \sin(2(k-1)\alpha)) & 0 & \\ \cdots & 0 & 0 & 0 & r\cos(2k\alpha) - R\cos((2k-1)\alpha) & r\sin(2k\alpha) - R\sin((2k-1)\alpha) & h & \\ & & & \cdots & & & & \\ 0 & 0 & 0 & \cdots & r\cos(2(n-1)\alpha) - R\cos((2n-1)\alpha) & r\sin(2(n-1)\alpha) - R\sin((2n-1)\alpha) & h & \\ r(1-\cos(2(n-1)\alpha)) & -r(\sin(2(n-1)\alpha)) & 0 & \cdots & r(\cos(2(n-1)\alpha) - 1) & r(\sin(2(n-1)\alpha)) & 0 & \\ r - R_c os((2n-1)\alpha) & -R\sin((2n-1)\alpha) & h & \cdots & 0 & 0 & 0 & \end{bmatrix} \quad (40)$$

According to [8], the structure is immobile if n is odd, and mobile if n is even. The Appendix provides an outline of a proof that the determinant of $\mathbf{A}$ vanishes only for even ns. Using known facts from linear algebra and Theorem 2 of this article the above result can be confirmed.

7. CONCLUSION

A rigorous procedure has been introduced to detect internal and global mobility of skeletal structures without performing any stress or force analysis. The computations require information of the nodal coordinates, their connectivity, and the support conditions of the structure. The two-dimensional analysis has been computationally integrated into the structural synthesis program ISOS.

Acknowledgements—This research has been partially supported by NSF Grant No. DDM 89-17430 at the University of Michigan. The first author has also been supported by a Horace H. Rackham Pre-doctoral Fellowship at the University of Michigan. This support is gratefully acknowledged. The authors also wish to acknowledge the useful comments by Professors Alejandro Diaz and Srindar Kota.

REFERENCES

1. M. Bendsøe and N. Kikuchi, Generating optimal topologies in structural design using a homogenization method. *Comput. Meth. Appl. Mech. Engng* **71,** 197–224 (1988).
2. P. Y. Papalambros and M. Chirehdast, An integrated environment for structural configuration design. *J. Engng Des.* **1,** 73–96 (1990).
3. B. Bremicker, M. Chirehdast, N. Kikuchi and P. Y. Papalambros, Integrated topology and shape optimization in structural. *J. Mech. Mach. Structures* **19,** 551–587 (1991).
4. C. H. Norris and J. B. Wilbur, *Elementary Structural Analysis*, 3rd Edn. McGraw-Hill, New York (1976).
5. W. Weaver, Jr, *Matrix Analysis of Framed Structures*, 3rd Edn. Van Nostrand Reinhold, New York (1990).
6. N. R. White, P. Gergely and R. G. Sexsmith, *Structural Engineering*, combined Edn. John Wiley, New York (1976).
7. R. V. Southwell, *An Introduction to Theory of Elasticity*, 2nd Edn. Oxford University Press, Oxford (1941).
8. N. J. Hoff and J. Fernandez-Sintes, Kinematically unstable space frameworks. In *Mechanics Today*, vol. 5, pp. 95–111 (Edited by S. Nemat-Nasser). Pergamon Press, Oxford (1980).
9. S. P. Timoshenko and D. H. Young, *Theory of Structures*, 2nd Edn. McGraw-Hill, New York (1965).
10. S. W. McCluskey, *An Introduction to Advanced Dynamics*. Addison-Wesley, Reading, MA (1962).
11. G. N. Sandor and A. G. Erdman, *Mechanism Design: Analysis and Synthesis*. Prentice-Hall, Englewood Cliffs, NJ (1984).
12. O. Bottema, On Grüblers formulas for mechanisms. In *Applied Scientific Research*, vol. A.2, pp. 162–164 (1950).
13. K. H. Hunt, *Kinematic Geometry of Mechanisms*. Clarendon Press, Oxford (1978).
14. M. Grüning, *Die Statik des Ebenen Tragwerkes*. Verlag von Julius Springer, Berlin (1925).
15. S. H. Friedberg, A. J. Insel and L. E. Spence, *Linear Algebra*, 2nd Edn. Prentice-Hall, Englewood Cliffs, NJ (1989).

APPENDIX

To obtain the determinant of matrix $\mathbf{A}$ in eqn (40) the following procedure should be performed. The first row is multiplied with -1 and added to the last row. The only non-vanishing member of the last row is $2R\sin(2\alpha)$, standing in the second column. Thus, the determinant is expanded with respect to that row. A $[3n-1\times 3n-1]$ matrix remains whose only non-vanishing member (h) is in the second column and the first row. Expanding this determinant with respect to its second column gives us a $[3n-2\times 3n-2]$ matrix of the form

$$\begin{bmatrix} r(1-\cos(2\alpha)) & r(\cos(2\alpha)-1) & r\sin(2\alpha) & 0 & \cdots & & & \\ 0 & & & & & & & \\ \vdots & \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 & \cdots & \mathbf{B}_{k-2} & \mathbf{B}_{k-1} & \mathbf{B}_k \\ 0 & & & & & & & \\ r(1-\cos(2\alpha)) & & \cdots & & 0 & r(\cos(2\alpha)-1) & -r(\sin(2\alpha)) & 0 \end{bmatrix},$$

where $\mathbf{B}_i$, $i=1,\ldots,3(n-1)$, are $(3n-4)$-dimensional vectors. The following equation can be verified by induction on even and odd ns

$$C = (\cos(2\alpha)-1)\det[\mathbf{B}_2,\mathbf{B}_3,\ldots,\mathbf{B}_k]$$

$$-\sin(2\alpha)\det[\mathbf{B}_1,\mathbf{B}_3,\ldots,\mathbf{B}_k]$$

$$=(\cos(2\alpha)-1)\det[\mathbf{B}_1,\ldots,\mathbf{B}_{k-3},\mathbf{B}_{k-1},\mathbf{B}_k]$$

$$+\sin(2\alpha)\det[\mathbf{B}_1,\ldots,\mathbf{B}_{k-2},\mathbf{B}_k],$$

Where C, a nonzero number, is used for later reference, and k equals to $3n-4$. The determinant of $\mathbf{A}$ (the immobility matrix) can be written in the following form

$$\det(A) = (-1)^{(3n+1)}2\,RhrC\sin(2\alpha)(1+(-1)^{(3n+1)}).$$

It can be easily verified that $\det(\mathbf{B})$ vanishes if n is even, and is nonzero otherwise.