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A Dilation Theory for Polynomially Bounded Operators

SRDJAN PETROVIĆ

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109, and Department of Mathematics, Texas A & M University, College Station, Texas 77843

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In this paper we construct a special sort of dilation for an arbitrary polynomially bounded operator. This enables us to show that the problem whether every polynomially bounded operator is similar to a contraction can be reduced to a subclass of it.

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex, Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators acting on $\mathscr{H}$. An operator T in $\mathscr{L}(\mathscr{H})$ is said to be *polynomially bounded* if there exists a constant $K>0$ such that

$$\|p(T)\| \leqslant K \sup\{|p(\xi)| : |\xi| = 1\} \tag{1}$$

for every polynomial p. For simplicity of reference, we shall denote the class of all polynomially bounded operators in $\mathscr{L}(\mathscr{H})$ by $PB(\mathscr{H})$. Since, by virtue of von Neumann's inequality, every contraction T in $\mathscr{L}(\mathscr{H})$ satisfies (1) with $K=1$, one may consider the class $PB(\mathscr{H})$ as a generalization of the class of contraction operators. In fact, it is a very important and difficult problem, posed explicitly by Halmos in [8], whether every operator in $PB(\mathscr{H})$ is similar to a contraction. (That is, given an operator $T \in PB(\mathscr{H})$, does there always exist an invertible operator X in $\mathscr{L}(\mathscr{H})$ such that $\|XTX^{-1}\| \leqslant 1$.) One of the basic tools in the study of contraction operators (cf. [15]) is the old and beautiful theorem of Sz.-Nagy [14] that every contraction has a unitary dilation, i.e., if T is a contraction in $\mathscr{L}(\mathscr{H})$, then there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and a unitary operator $U \in \mathscr{L}(\mathscr{K})$ such that $T^n = PU^n | \mathscr{H}$ for every $n = 0, 1, 2, \ldots$, where P is the orthogonal projection in $\mathscr{L}(\mathscr{K})$ whose range is $\mathscr{H}$. It is obvious that only

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contractions can have *unitary* dilations, but this leaves open the question of what kind of dilation theory might be available for operators in $PB(\mathscr{H})$. The purpose of this paper is to make a start on this problem by constructing a dilation theory for polynomially bounded operators. We show, in our main theorem (Theorem 1.1), that every polynomially bounded operator has a dilation $\hat{T}$ which does have *some* good properties—namely, $\hat{T}$ is also polynomially bounded, the spectrum $\sigma(\hat{T})$ is the unit circle $\mathbb{T}$ in $\mathbb{C}$, and $\hat{T}$ satisfies

$$(\hat{T}\hat{T}^*)(\hat{T}^*\hat{T}) = (\hat{T}^*\hat{T})(\hat{T}\hat{T}^*).$$

Before stating the main results of this paper we briefly mention some notation and terminology. As usual, $\mathbb{N}$ is the set of positive integers, $\mathbb{C}$ denotes the complex plane, $\mathbb{D}$ is the open unit disk in $\mathbb{C}$, and $\mathbb{T}$ is the unit circle $\mathbb{T} = \partial\mathbb{D}$ in $\mathbb{C}$. A subspace $\mathscr{M} \subset \mathscr{H}$ is said to be invariant for an operator T in $\mathscr{L}(\mathscr{H})$ if $T\mathscr{M} \subset \mathscr{M}$, and in this situation we denote by $T|\mathscr{M}$ the restriction of T to $\mathscr{M}$. A subspace $\mathscr{M}$ is said to be semi-invariant for T if there exist invariant subspaces $\mathscr{N}_1 \supset \mathscr{N}_2$ for T such that $\mathscr{M} = \mathscr{N}_1 \ominus \mathscr{N}_2$, and in this situation we denote by $T_{\mathscr{M}}$ the compression of T to $\mathscr{M}$, i.e., $T_{\mathscr{M}} = P_{\mathscr{M}} T|\mathscr{M}$, where $P_{\mathscr{M}}$ is the (orthogonal) projection in $\mathscr{L}(\mathscr{H})$ with range $\mathscr{M}$. We shall use the notation $\operatorname{Ker} \mathscr{T}$ and $\operatorname{Ran} \mathscr{T}$ for the kernel and the range of T, respectively. By $\dim \mathscr{M}$ we denote the orthogonal dimension of a Hilbert space $\mathscr{M}$. Finally, if $\mathscr{M}$ is a subspace of $\mathscr{H}$, we denote by $\mathscr{M}^\perp$ the subspace $\mathscr{H} \ominus \mathscr{M}$.

We recall that an operator T in $\mathscr{L}(\mathscr{H})$ is called *quasinormal* if T commutes with T^*T. The structure of quasinormal operators was determined by A. Brown in [1]. Clearly a quasinormal operator T satisfies

$$(T^*T)(TT^*) = (TT^*)(T^*T). \tag{2}$$

Operators T satisfying (2), which we shall call *weakly centered* operators, have been studied in [3, 4, 5], under the name binormal operators (cf. [2]).

THEOREM 1.1. *For every polynomially bounded operator in $\mathscr{L}(\mathscr{H})$ there exists a Hilbert space $\mathscr{K}$ containing $\mathscr{H}$ and an operator $\hat{T} \in \mathscr{L}(\mathscr{K})$ such that*

(a) *$\hat{T}$ is polynomially bounded,*

(b) *$\mathscr{H}$ is a semi-invariant subspace for $\hat{T}$,*

(c) *$\sigma(\hat{T})$ is the unit circle, and*

(d) *$\hat{T}$ is weakly centered.*

For some time it was an open question whether every power bounded operator T in $\mathscr{L}(\mathscr{H})$ is similar to a contraction (notation $T \in SC(\mathscr{H})$), but this was finally negatively settled by Foguel [6] (see also [7] for a somewhat simpler proof). Several authors have addressed the problem whether every polynomially bounded operator is similar to a contraction (cf. [9, 10, 11]), and we mention in particular, some nice progress made by Paulsen [12], but as of this writing, the question remains open. The following theorem shows that it suffices to establish this fact for a subclass of $PB(\mathscr{H})$.

THEOREM 1.2. *Every polynomially bounded operator in $\mathscr{L}(\mathscr{H})$ is similar to a contraction if and only if every weakly centered polynomially bounded operator in $\mathscr{L}(\mathscr{H})$ whose spectrum is the unit circle is similar to a contraction.*

2. Some Preliminary Lemmas

The proofs of these theorems are based on some preliminary lemmas. We will omit the proofs of the first two lemmas, since they are straightforward.

LEMMA 2.1. *Suppose $T \in \mathscr{S}C(\mathscr{H})$, and let $\mathscr{M}$ be an invariant subspace for T. Then $T|\mathscr{M}$ belongs to $\mathscr{S}C(\mathscr{M})$.*

LEMMA 2.2. *Suppose $T \in \mathscr{S}C(\mathscr{H})$. If $\mathscr{M}$ is a semi-invariant subspace for T, then the compression $T_{\mathscr{M}}$ belongs to $\mathscr{S}C(\mathscr{M})$.*

LEMMA 2.3. *Let T, D, and X be operators in $\mathscr{L}(\mathscr{H})$ such that D is a unilateral weighted shift of infinite multiplicity with weight sequence $\{d_n\}_{n=1}^{\infty}$ defined as*

$$\begin{aligned} d_1 &= \frac{1}{\log 2}, \\ d_n &= \left(\frac{n-1}{n}\right)^2 \frac{\log n}{\log(n+1)}, \qquad n \geqslant 2 \end{aligned} \tag{3}$$

and Ran $X^* \subset$ Ker D^*. *Then the operator $\tilde{T}$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$, defined by*

$$\tilde{T} = \begin{pmatrix} T & X \\ 0 & D^* \end{pmatrix},$$

belongs to the class $PB(\mathscr{H} \oplus \mathscr{H})$ if and only if $T \in PB(\mathscr{H})$.

Proof. It is clear that the restriction of a polynomially bounded operator to an invariant subspace is also a polynomially bounded operator, so we confine our attention to the other half of the proof. Thus, let $T \in PB(\mathscr{H})$. It is not hard to see that for every nonnegative integer k,

$$\tilde{T}^k = \begin{pmatrix} T^k & G_k \\ 0 & D^{*k} \end{pmatrix},$$

where

$$G_0 = 0,$$

and

$$G_k = T^{k-1}X + T^{k-2}XD^* + T^{k-3}XD^{*2} + \cdots + TXD^{*k-2} + XD^{*k-1}, \qquad k \geqslant 1.$$

Thus if p is the polynomial $p(z) = \sum_{k=0}^n a_k z^k$, then

$$p(\tilde{T}) = \sum_{k=0}^n a_k \begin{pmatrix} T^k & G_k \\ 0 & D^{*k} \end{pmatrix} \begin{pmatrix} p(T) & \sum_{k=1}^n a_k G_k \\ 0 & p(D^*) \end{pmatrix}.$$

By [13, Proposition 2], $\|D^2\| = \sup\{|d_k d_{k+1}| : k \in \mathbb{N}\}$. Hence

$$\begin{aligned} \|D^2\| &= \max\left\{\sup_{k \geqslant 2}\left\{\left(\frac{k-1}{k}\right)^2 \frac{\log k}{\log(k+1)} \cdot \left(\frac{k}{k+1}\right)^2 \frac{\log(k+1)}{\log(k+2)}\right\}, d_1 d_2\right\} \\ &= \max\left\{\sup_{k \geqslant 2}\left\{\left(\frac{k-1}{k+1}\right)^2 \frac{\log k}{\log(k+2)}\right\}, \frac{1}{\log 2} \cdot \frac{1}{4} \cdot \frac{\log 2}{\log 3}\right\} \leqslant 1. \end{aligned}$$

Thus D^2 is a contraction. By a well-known argument (cf. [8]), D is similar to a contraction, hence in $PB(\mathscr{H})$, so the same is true of D^*. Therefore $\tilde{T}$ will be polynomially bounded if and only if there exists $K > 0$ (independent of P) such that $\|\sum_{k=1}^n a_k G_k\| \leqslant K \sup\{|p(\xi)| : |\xi| = 1\} = K\|p\|$. (Here we use the obvious fact that the norm of a 2×2 matrix with operator entries is less than or equal to the sum of the norms of its four entries.) To establish the existence of such a K, note that

$$\begin{aligned} \sum_{k=1}^n a_k G_k &= \sum_{k=1}^n a_k \left(\sum_{i=1}^k T^{k-i} X D^{*i-1}\right) \\ &= \sum_{i=1}^n \sum_{k=i}^n a_k T^{k-i} X D^{*i-1} \\ &= \sum_{i=1}^n \sum_{m=0}^{n-i} a_{m+i} T^m X D^{*i-1}, \end{aligned} \tag{4}$$

where we have written $m=k-i$. If we define $p_{(0)}=p$, $p_{(1)}(z)=(p(z)-p(0))/z$, and, by induction, $p_{(n)}(z)=(p_{(n-1)}(z)-p_{(n-1)}(0))/z$, then it is not hard to see that the right hand side of (4) is exactly

$$\sum_{i=1}^{n} p_{(i)}(T)\, XD^{*i-1}, \tag{5}$$

and it is well known (cf. [16, p. 418]) that for all polynomials p and for every $n\geqslant 2$, $\|p_{(n)}\|\leqslant 6\log n\|p\|$. Thus

$$\begin{aligned}\left\|\sum_{k=1}^{n} a_k G_k\right\| &= \left\|\sum_{i=1}^{n} p_{(i)}(T)\, XD^{*i-1}\right\| \\ &\leqslant \sum_{i=1}^{n} \|p_{(i)}(T)\|\ \|XD^{*i-1}\| \\ &\leqslant \sum_{i=1}^{n} M\|XD^{*i-1}\|\ \|p_{(i)}\| \\ &\leqslant 2M\|X\|\ \|p\| + \sum_{i=2}^{n} 6\log i\|p\|\ M\|XD^{*i-1}\|, \end{aligned}\tag{6}$$

where M is the polynomial bound of T. By definition of D, there exists an infinite dimensional Hilbert space $\mathscr{G}$ and a decomposition $\mathscr{H}=\bigoplus_{n=1}^{\infty}\mathscr{G}_n$, where $\mathscr{G}_n=\mathscr{G}$, $n\in\mathbb{N}$, such that relative to this decomposition, D has the matrix

$$\begin{pmatrix} 0 & & & \\ d_1 1_{\mathscr{G}} & 0 & & \\ & d_2 1_{\mathscr{G}} & 0 & \\ & & \ddots & \ddots \end{pmatrix}. \tag{7}$$

Furthermore X may thus be regarded as an operator mapping $\bigoplus_{n=1}^{\infty}\mathscr{G}_n$ into $\mathscr{H}$ and hence has a matrix $X=(X_1, X_2, ...)$, where $X_n\colon \mathscr{G}_n\to\mathscr{H}$, $n\in\mathbb{N}$. Moreover, since Ker $D^*=$ Ran X^*, it is clear that $X_n=0$ for all $n\geqslant 2$. Thus $\|XD^{*i-1}\|=\|(d_1\cdots d_{i-1})X_1\|$, and it follows trivially from this and (6) that

$$\begin{aligned}\left\|\sum_{k=1}^{n} a_k G_k\right\| &\leqslant 2M\|X_1\|\ \|p\| + \sum_{i=2}^{n} 6M\log i(d_1\cdots d_{i-1})\ \|X_1\|\ \|p\| \\ &\leqslant M\|X_1\|\ (2+\pi^2)\ \|p\|.\end{aligned}$$

Thus, $\hat{T}$ is polynomially bounded and the lemma is proved. ∎

3. Two Matricial Constructions

Before we can turn to the proof of Theorem 1.1, we need two more preliminary results. For the sake of simplicity we shall use the notation $\mathscr{M}^{(3)}$ for $\mathscr{M} \oplus \mathscr{M} \oplus \mathscr{M}$, where $\mathscr{M}$ is an arbitrary Hilbert space.

Proposition 3.1. *Suppose $T \in PB(\mathscr{H})$. Then there exists an operator $\tilde{T}$ in $\mathscr{L}(\mathscr{H}^{(3)})$ such that*

(a) $\mathscr{H} \oplus (0) \oplus (0)$ *is invariant for* $\tilde{T}$,

(b) $T = \tilde{T} \mid (\mathscr{H} \oplus (0) \oplus (0))$,

(c) $\tilde{T} \in PB(\mathscr{H}^{(3)})$,

(d) $\mathrm{Ran}(\tilde{T})$ *is closed*, $\dim(\mathrm{Ker}\, \tilde{T}) = \aleph_0$, *and* $\dim(\mathrm{Ker}\, \tilde{T}^*) = \aleph_0$.

Proof. Let M be the polynomial bound for T. Define $\tilde{T}$ to be the following 3×3 operator matrix acting on $\mathscr{H}^{(3)}$ in the usual way:

$$\tilde{T} = \begin{pmatrix} T & \sqrt{M^2 - TT^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{8}$$

It is obvious from the definition that (a) and (b) are valid, so we first show that $\tilde{T}$ is polynomially bounded. An easy computation shows that

$$p(\tilde{T}) = \begin{pmatrix} p(T) & p_{(1)}(T)\sqrt{M^2 - TT^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $p_{(1)}$ is as was defined in the proof of Lemma 2.3. Since $\|p_{(1)}\| \leqslant 2\|p\|$, clearly $\tilde{T}$ is polynomially bounded. In order to show that the range of $\tilde{T}$ is closed, it suffices to prove the same fact for $\tilde{T}^*$. We will prove that $\tilde{T}^*$ is bounded below on $(\mathrm{Ker}\, \tilde{T}^*)^\perp = \mathscr{H} \oplus (0) \oplus (0)$. So, let $x \in \mathscr{H}$. Then,

$$\|\tilde{T}^*(x \oplus 0 \oplus 0)\|^2 = \|T^*x\|^2 + \|\sqrt{M^2 - TT^*}x\|^2 = M^2\|x\|^2.$$

This shows that $\tilde{T}^*$ is bounded below on its initial space and completes the proof. ∎

Proposition 3.2. *Suppose $T \in PB(\mathscr{H})$, $\mathscr{K} = \mathscr{H}^{(3)}$, and let $\tilde{T} \in \mathscr{L}(\mathscr{K})$ be as in Proposition* 3.1. *Then there exists an operator $\hat{T} \in PB(\mathscr{K}^{(3)})$ such that*

(a) $\mathscr{M} = (0) \oplus \mathscr{K} \oplus (0)$ *is semi-invariant for* $\hat{T}$,

(b) $\hat{T}_{\mathscr{K}} = \tilde{T}$,

(c) $\hat{T} \in PB(\mathscr{H}^{(3)})$,

(d) *$\sigma(\hat{T})$ is the unit circle, and*

(e) *$\hat{T}$ is weakly centered.*

Proof. Note that, by (d) of Proposition 3.1, both the kernel and cokernel of $\tilde{T}$ are infinite dimensional. Let D be a weighted unilateral shift of infinite multiplicity in $\mathscr{L}(\mathscr{H})$ with weight sequence $\{d_n\}_{n=1}^{\infty}$ as defined in (3). Then there exist a partial isometry U in $\mathscr{L}(\mathscr{H})$ with initial space $\operatorname{Ker} \tilde{T}$ and final space $\operatorname{Ker} D^*$, and a partial isometry V in $\mathscr{L}(\mathscr{H})$ with initial space $\operatorname{Ker} D^*$ and final space $\operatorname{Ker} \tilde{T}^*$. Let a be a positive number, let M be the polynomial bound for T, and let $A = aU$, $C = MV$. We define $\hat{T}$ to be the following matrix, acting on $\mathscr{H}^{(3)}$ in the usual way:

$$\begin{pmatrix} D & A & 0 \\ 0 & \tilde{T} & C \\ 0 & 0 & D^* \end{pmatrix}. \tag{9}$$

It is obvious from this definition that conclusions (a) and (b) are valid. Since $\tilde{T}$ is polynomially bounded and $\operatorname{Ran} C^* = \operatorname{Ker} D^*$, it follows from Lemma 2.3 that the same is true for the compression $T_1 = \hat{T}_{((0) \oplus \mathscr{H} \oplus \mathscr{H})}$ and its adjoint. Furthermore, another application of the same lemma (with T_1^* replacing T) gives that $\hat{T}^*$ is polynomially bounded, and hence so is $\hat{T}$. We next show that $\hat{T}$ is invertible. To accomplish this it suffices to exhibit its inverse. Since ran $\tilde{T}$ is closed and $\tilde{T}$ maps $(\operatorname{Ker} \tilde{T})^{\perp}$ injectively onto $\operatorname{Ran} \tilde{T}$, there exists a bounded linear operator $Q \in \mathscr{L}(\mathscr{H})$ such that $\tilde{T}Q = P_{\operatorname{Ran} \tilde{T}}$ and $Q\tilde{T} = P_{(\operatorname{Ker} T)^{\perp}}$. It is obvious that in the polar decomposition $D = S(D^*D)^{1/2} = S\Delta$ of D, S is an isometry with final space $\operatorname{Ran} D$, and Δ is invertible. If we define $D_1 = S\Delta^{-1}$, then it is easy to verify that the operator

$$\begin{pmatrix} D_1^* & 0 & 0 \\ (1/a^2)\, A^* & Q & 0 \\ 0 & (1/M^2)\, C^* & D_1 \end{pmatrix},$$

is the inverse of $\hat{T}$.

Next, we prove that $\sigma(\hat{T}) \subset \mathbb{T}$. First we note that, since $\hat{T}$ is polynomially bounded, $\sigma(\hat{T}) \subset \mathbb{D}^-$. On the other hand, $0 \notin \sigma(\hat{T})$, so it clearly suffices to prove that $\partial\sigma(\hat{T}) \subset \mathbb{T}$. But, as is well known, $\partial\sigma(\hat{T}) \subset \sigma_{\mathrm{ap}}(\hat{T})$, the approximate point spectrum of $\hat{T}$, so it suffices to show that $\sigma_{\mathrm{ap}}(\hat{T}) \subset \mathbb{T}$. Suppose $\lambda_0 \in \sigma_{\mathrm{ap}}(\hat{T})$, and let $\{\tilde{x}_n\}$ be a sequence of unit vectors in $\mathscr{H}^{(3)}$ such that $\|(\hat{T} - \lambda_0)\, \tilde{x}_n\| \to 0$. Write $\tilde{x}_n = x_n \oplus y_n \oplus z_n$. Then a calculation shows that $(\hat{T}-\lambda_0)\,\tilde{x}_n = ((D-\lambda_0)\,x_n + Ay_n) \oplus ((\tilde{T}-\lambda_0)\,y_n + Cz_n) \oplus (D^*-\lambda_0)\,z_n$. Therefore

$$\|(D-\lambda_0)\,x_n + Ay_n\| \to 0, \tag{10}$$

$$\|(T-\lambda_0)\,y_n + Cz_n\| \to 0, \tag{11}$$

$$\|(D^*-\lambda_0)\,z_n\| \to 0. \tag{12}$$

Now (10) implies that $\|D^*((D-\lambda_0)\,x_n + Ay_n)\| \to 0$, and since $D^*A = 0$ by definition of A,

$$\|D^*\,Dx_n - \lambda_0\,D^*x_n\| \to 0.$$

An easy matricial calculation shows that D^*D is invertible, and hence that

$$\|((D^*\,D)^{-1}\,D^* - 1/\lambda_0)\,x_n\| \to 0.$$

Thus, if $\|x_n\| \not\to 0$, $1/\lambda_0 \in \sigma((D^*D)^{-1}\,D^*)$. But another easy matricial calculation shows that $(D^*D)^{-1}\,D^*$ is a backward weighted shift of infinite multiplicity with weight sequence $\{1/d_n\}_{n=1}^{\infty}$, and since $d_n \to 1$, one knows (cf. [13, Proposition 15]) that $\sigma((D^*D)^{-1}\,D^*) = \mathbb{D}^-$, and hence $|\lambda_0| \geqslant 1$. Thus either $\|x_n\| \to 0$ or $\lambda_0 \in \mathbb{T}$. In the former case (10) becomes $\|Ay_n\| \to 0$. In this situation write $y_n = y'_n \oplus y''_n$ relative to the decomposition $\mathscr{K} = \operatorname{Ker} A \oplus \operatorname{Ran} A^*$. Since $y'_n \in \operatorname{Ker} A = \operatorname{Ran} \tilde{T}^*$, there exists a sequence $\{v_n\} \in \operatorname{Ran} \tilde{T}$ such that $y'_n = \tilde{T}^* v_n$ for all n. On the other hand, it is easy to see that (11) implies that

$$\|\tilde{T}y_n + Cz_n\| - |\lambda_0|\;\|y_n\| \to 0.$$

Thus

$$\|\tilde{T}y_n + Cz_n\|^2 - |\lambda_0|^2\;\|y_n\|^2 \to 0,$$

and since $\operatorname{Ran} \tilde{T}$ is orthogonal to $\operatorname{Ran} C$,

$$\|\tilde{T}y_n\|^2 + \|Cz_n\|^2 - |\lambda_0|^2\;\|y_n\|^2 \to 0.$$

In particular, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for, $n \geqslant n_0$,

$$\|\tilde{T}y_n\|^2 - |\lambda_0|^2\;\|y_n\|^2 < \varepsilon. \tag{13}$$

Note that $\|Ay''_n\| = \|Ay_n\| \to 0$. But $y''_n \in \operatorname{Ran} A^*$, the initial space of the partial isometry $(1/a)\,A$, so $\|y''_n\| \to 0$. So (13) becomes

$$\|\tilde{T}y'_n + \tilde{T}y''_n\|^2 - |\lambda_0|^2\;\|y'_n + y''_n\|^2 < \varepsilon, \qquad n \geqslant n_0,$$

and since $\|y_n''\| \to 0$, we have that there exists $n_1 \in \mathbb{N}$ such that

$$\|\tilde{T} y_n'\|^2 - |\lambda_0|^2 \, \|y_n'\|^2 < 2\varepsilon, \qquad n \geqslant n_1.$$

Therefore

$$\|\tilde{T}\tilde{T}^* v_n\|^2 - |\lambda_0|^2 \, \|\tilde{T}^* v_n\|^2 < 2\varepsilon, \qquad n \geqslant n_1.$$

Using (8) we have that

$$\left\| \begin{pmatrix} M^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_n \right\|^2 - |\lambda_0|^2 \left\| \begin{pmatrix} T^* & 0 & 0 \\ \sqrt{M^2 - TT^*} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v_n \right\|^2 < 2\varepsilon, \qquad n \geqslant n_1.$$

Since $v_n \in \operatorname{Ran} \tilde{T} \subset \mathscr{K} \oplus (0) \oplus (0)$, we can write v_n as $w_n \oplus 0 \oplus 0$. So we obtain that

$$\|M^2 w_n\|^2 - |\lambda_0|^2 \left\| \begin{pmatrix} T^* w_n \\ \sqrt{M^2 - TT^*}\, w_n \\ 0 \end{pmatrix} \right\|^2 < 2\varepsilon, \qquad \text{i.e.,}$$

$$(M^4 - |\lambda_0|^2 M^2) \, \|w_n\|^2 < 2\varepsilon, \qquad n \geqslant n_1.$$

Since ε was arbitrary and $M^4 - |\lambda_0|^2 M^2 > 0$, it follows that $\|w_n\| \to 0$ and the same is true of v_n and $y_n' = \tilde{T}^* v_n$. Since y_n'' also tends to 0, we conclude that $\|y_n\| \to 0$ (under the assumption, made earlier, that $\|x_n\| \to 0$).

Next, under this same assumption, write $z_n = z_n' \oplus z_n''$ relative to the decomposition $\mathscr{K} = \operatorname{Ker} C \oplus \operatorname{Ker} D^*$. Then, $\|Cz_n''\| = \|Cz_n\| \to 0$ by (11). Since $(1/M)C$ is a partial isometry with initial space $\operatorname{Ker} D^*$, it follows that $\|z_n''\| \to 0$. On the other hand, $z_n' \in \operatorname{Ker} C = \operatorname{Ran} D$, so there exists a sequence $\{u_n\} \subset \mathscr{K}$ such that $z_n' = Du_n$. Then (12) gives that $\|D^*Du_n - \lambda_0 \, Du_n\| \to 0$. As was already observed, D^*D is invertible, so

$$\|(1/\lambda_0)\, u_n - (D^*D)^{-1} \, Du_n\| \to 0.$$

Now an easy computation shows that $(D^*D)^{-1} D$ is a forward unilateral weighted shift of infinite multiplicity with weight sequence $\{1/d_n\}$, and it has been already observed that the spectrum of this operator is equal to $\mathbb{D}^-$. Thus either $\|u_n\| \to 0$ or $\lambda_0 \in \mathbb{T}$. If $\|u_n\| \to 0$, then $\|z_n'\| \to 0$ which, together with the previous conclusions, implies that $\|\tilde{x}_n\| \to 0$. Since $\|\tilde{x}_n\| = 1$, $n \in \mathbb{N}$, this is a contradiction, and it follows that $\lambda_0 \in \mathbb{T}$. Thus, we have shown that $\sigma(\hat{T}) \subset \mathbb{T}$. To prove the opposite inclusion, we note that by [13, Proposition 15], $\sigma_{\text{ap}}(D) = \mathbb{T}$, and from (9), $\sigma_{\text{ap}}(D) \subset \sigma_{\text{ap}}(\hat{T})$, so $\sigma(\hat{T}) = \mathbb{T}$.

Finally, we show that $\hat{T}$ is weakly centered. A simple calculation shows that

$$\hat{T}\hat{T}^* = \begin{pmatrix} DD^* + AA^* & 0 & 0 \\ 0 & \tilde{T}\tilde{T}^* + CC^* & 0 \\ 0 & 0 & D^*D \end{pmatrix},$$

and

$$\hat{T}^*\hat{T} = \begin{pmatrix} D^*D & 0 & 0 \\ 0 & A^*A + \tilde{T}^*\tilde{T} & 0 \\ 0 & 0 & C^*C + DD^* \end{pmatrix}.$$

By definition of D, there exists an infinite dimensional Hilbert space $\mathscr{G}$ and a decomposition $\mathscr{K} = \bigoplus_{n=1}^{\infty} \mathscr{G}_n$, where $\mathscr{G}_n = \mathscr{G}$, $n \in \mathbb{N}$, such that, relative to this decomposition, D has the matrix (7). We observe that relative to this decomposition $DD^* + AA^*$, D^*D, and $C^*C + DD^*$ are diagonal operators with scalar multiples of $1_{\mathscr{G}}$ as diagonal entries, so $DD^* + AA^*$ commutes with D^*D and D^*D commutes with $C^*C + DD^*$. Thus it suffices to show that $\tilde{T}\tilde{T}^* + CC^*$ commutes with $A^*A + \tilde{T}^*\tilde{T}$. Using (8), we obtain that

$$\begin{aligned} \tilde{T}\tilde{T}^* + CC^* &= \tilde{T}\tilde{T}^* + M^2 P_{\operatorname{Ker} \tilde{T}^*} \\ &= \begin{pmatrix} T & \sqrt{M^2 - TT^*} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} T^* & 0 & 0 \\ \sqrt{M^2 - TT^*} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + M^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= M^2 I_{\mathscr{H}^{(3)}}, \end{aligned}$$

which obviously commutes with $A^*A + \tilde{T}^*\tilde{T}$. This completes the proof of the proposition. ∎

4. Proofs of the Theorems

Proof of Theorem 1.1. Let T be a polynomially bounded operator in $\mathscr{L}(H)$. Then applying Propositions 3.1 and 3.2 we obtain the operator $\hat{T}$ that satisfies (a), (c), and (d) of this theorem. Finally, by Proposition 3.2,

$\mathscr{K}$ is semi-invariant for $\hat{T}$, and by Proposition 3.1, $\mathscr{H}$ is invariant for $\tilde{T} = \hat{T}_{\mathscr{K}}$. Therefore, $\mathscr{H}$ is semi-invariant for $\hat{T}$. ∎

Proof of Theorem 1.2. If every polynomially bounded operator in $\mathscr{L}(\mathscr{H})$ is similar to a contraction, then the same is true, in particular, for every weakly centered polynomially bounded operator whose spectrum is the unit circle. To prove the nontrivial implication, let T be a polynomially bounded operator in $\mathscr{L}(\mathscr{H})$. By Proposition 3.1 and Lemma 2.2 there exists a certain polynomially bounded operator $\tilde{T}$ in $\mathscr{L}(\mathscr{H} \oplus \mathscr{H})$ with closed range such that if $\tilde{T}$ is similar to a contraction, the same is true for T. Thus, we can apply Proposition 3.2 and conclude that there exists a weakly centered polynomially bounded operator $\hat{T}$, whose spectrum is the unit circle, such that $\tilde{T}$ is a compression of $\hat{T}$ to a semi-invariant subspace. By hypothesis, $\hat{T}$ is similar to a contraction, so Lemma 2.2 implies that the same is true for $\tilde{T}$ and, hence, for T. ∎

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