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# Normalized coprime factorizations for linear time-varying systems *

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*Abstract:* In this paper we show that a finite dimensional linear time-varying continuous-time system admits normalized coprime factorizations if and only if it admits a stabilizable and detectable realization. We construct state-space formulas for these factorizations using the stabilizing solutions to standard Riccati differential equations. In the process, we give a simple proof that stabilizability and detectability are sufficient to ensure the existence of such solutions. Based on these results, and on recent advances in the theory of $\mathscr{H}_\infty$ optimization, we present an algorithm to compute the distance between two systems in the gap metric.

*Keywords:* Linear time-varying system; normalized coprime factorization; gap metric; Lyapunov equation; Riccati equation.

## 1. Introduction

The graph and gap metrics [1] are defined via normalized coprime factorizations. Hence, it is of interest to know when these factorizations exist for a given system and how to calculate them. A direct procedure to obtain a normalized coprime factorization of a strictly proper *linear time-invariant* (LTI) system was first reported in [11]. Later, this procedure was extended to proper plants in [18].

In this paper we study normalized coprime factorizations of finite dimensional linear time-varying (FDLTV) systems. We find that most results for the LTI case carry over to the LTV case; thus there are no surprises as far as the results are concerned. The contribution of the paper lies in the technical development. The main result is stated in Section 4, where we show that a system has a normalized coprime factorization if and only if it admits a stabilizable and detectable realization (Theorem 4.1 and Corollary 4.2). In Section 5 we use these factorizations to compute the distance between two linear time-varying systems in the gap metric. This is done by combining some recent results in $\mathscr{H}_\infty$ control theory for linear time-varying systems [16,13] with the characterization of the gap metric [5] as an $\mathscr{H}_\infty$

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[1] See [17] and [20] respectively for an introduction to these metrics and their significance in the study of stability robustness.

optimization problem. The resulting algorithm is iterative, requiring the solution of a single differential Riccati equation at each step.

Proofs of the abovementioned results require some intermediate results in Section 3 that are also of independent interest. It is shown that stabilizability and detectability are sufficient conditions for the existence of stabilizing solutions to the standard continuous-time control and filter Riccati differential equations (Lemma 3.3 and Lemma 3.4). It is well known that solutions with the stabilizing property are guaranteed to exist under the assumptions of *uniform controllability and observability* ([7,9] after the references therein). As Anderson and Moore remark [1,2], relaxing these conditions to stabilizability and detectability respectively requires nontrivial generalizations of the time-invariant results. Such generalizations seem to be more immediate in the discrete-time case ([2], page 47), and have in fact appeared explicitly in [1]. It should be noted that our approach is independent of that in [1] since we consider the continuous-time case, and we use more traditional 'closed loop' definitions of stabilizability and detectability.

The notation is standard. The symbols $\mathbb{R}$ $(=\mathbb{R}_+ \cup \mathbb{R}_-)$, $\mathbb{R}^n$, and $\mathbb{R}^{k\times m}$ denote the real line, the $n$-dimensional real Euclidean space, and the space of $k\times m$-dimensional real valued matrices respectively. The space of vector valued measurable functions on $\mathbb{R}_+$ is denoted by $\mathscr{X}(\mathbb{R}_+)$, and $\mathscr{L}_2(\mathbb{R}_+)$ represents the subspace of square integrable functions (with inner product $\langle\cdot,\cdot\rangle$, and norm $\|\cdot\|$). The symbol $\mathscr{L}_2([a,b])$ denotes the space of square integrable functions defined on the real interval $[a,b]$ (with inner product $\langle\cdot,\cdot\rangle_{[a,b]}$, and norm $\|\cdot\|_{[a,b]}$). The extended space $\mathscr{L}_{2,\mathrm{e}}(\mathbb{R}_+)$ consists of functions $f\in\mathscr{X}(\mathbb{R}_+)$ satisfying $P_t f\in\mathscr{L}_2(\mathbb{R}_+)$ for all $t>0$, where $P_t$ is the truncation operator defined as $P_t f(\tau)=f(\tau)$ if $\tau\leqslant t$, and 0 otherwise.

An operator $G:\mathscr{L}_{2,\mathrm{e}}(\mathbb{R}_+)\to\mathscr{L}_{2,\mathrm{e}}(\mathbb{R}_+)$ is said to be causal (respectively, anti-causal) if $P_tGP_t=P_tG$ (respectively, $(I-P_t)G(I-P_t)=(I-P_t)G$), for all $t\in\mathbb{R}_+$. If $G$ is simultaneously causal and anti-causal then it is called memoryless. The set of causal, linear operators on $\mathscr{L}_{2,\mathrm{e}}(\mathbb{R}_+)$ is denoted by $\mathscr{M}_{t\cdot}(\mathbb{R}_+)$. We say that $G\in\mathscr{M}_{t\cdot}(\mathbb{R}_+)$ is (finite gain) stable if

$$\|G\| := \sup_{f\in\mathscr{L}_{2,\mathrm{e}},\,P_tf\neq 0,\,t\in\mathbb{R}_+} (\|P_tGf\|/\|P_tf\|)<\infty.$$

We denote by $\mathscr{B}_{t\cdot}(\mathbb{R}_+)$ the set of all stable, linear operators, and by $\mathscr{U}_{t\cdot}(\mathbb{R}_+)$ the set of all units in $\mathscr{B}_{t\cdot}(\mathbb{R}_+)$. Note that these definitions continue to hold, with obvious modifications, when $\mathbb{R}_+$ is replaced by $\mathbb{R}_-$, or any other interval of $\mathbb{R}$. In the specific case of operators defined on $\mathscr{L}_2([a,b])$, this legitimizes the use of symbols such that $\|G\|_{[a,b]}$ and $\mathscr{B}_{t\cdot}([a,b])$. Finally, whenever the meaning is clear from the context we abbreviate $\mathscr{L}_2(\mathbb{R}_+)$ to $\mathscr{L}_2$, $\mathscr{M}_{t\cdot}(\mathbb{R}_+)$ to $\mathscr{M}_{t\cdot}$, and similarly for the other spaces and sets defined above.

## 2. Definitions and preliminary results

Throughout this paper, we will be dealing with the class of causal, linear time-varying systems that admit finite dimensional representations of the form

$$\Sigma_G := \begin{cases} \dot{x}(t)=A(t)x(t)+B(t)u(t), & x(0)=x_0,\\ y(t)=C(t)x(t)+D(t)u(t), \end{cases} \tag{1}$$

where $t\in\mathbb{R}_+$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$, and $x(t)\in\mathbb{R}^n$. We assume that $A$, $B$, $C$, and $D$ are bounded functions of time. In packed matrix notation, $\Sigma_G$ can be written as

$$\left[\begin{array}{c|c} A & B\\ \hline C & D\end{array}\right].$$

With $x(0)=0$, the system $\Sigma_G$ generates a causal operator $G \in \mathscr{M}_v$ defined by

$$y(t)=\int_0^t C(t)\Phi_G(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau + D(t)u(t), \tag{2}$$

where $\Phi_G(t,\tau)$ is the state transition matrix of the homogeneous part of (1). The operator $G$ (with realization $\Sigma_G$) is said to be strongly causal (respectively, bicausal) if $D(t)=0$ for all $t$ in $\mathbb{R}_+$ (respectively, $D(t)$) is invertible for all $t$ in $\mathbb{R}_+$ and the inverse is bounded).

**Definition.** The system $\Sigma_G$ is said to be *exponentially stable* if there exist $c_1$, $c_2>0$ such that

$$\|\Phi_G(t,\tau)\| \leqslant c_1\,\mathrm{e}^{-c_2(t-\tau)}\forall t\geqslant\tau;\quad t,\tau\in\mathbb{R}_+. \tag{3}$$

**Definition.** The system $\Sigma_G$ is said to be *stabilizable* (respectively, *detectable*) if there exists a bounded matrix function $K(t)$ (respectively, $L(t)$) such that the system $\dot{x}(t)=(A-BK)(t)x(t)$ (respectively, $\dot{x}(t)=(A-LC)(t)x(t)$) is exponentially stable.

If a system admits a stabilizable and detectable realization, internal (exponential) and external (finite-gain, input–output) stability are equivalent. We now recall some results related to the adjoint and the dual of a linear system. Given $G\in\mathscr{B}_v(\mathbb{R}_+)$ (respectively, $G\in\mathscr{B}_v([0,T])$) its adjoint $G^*$ is the unique bounded linear operator that satisfies $\langle v, Gu\rangle=\langle G^*v, u\rangle$ for all $v$, $u\in\mathscr{L}_2(\mathbb{R}_+)$ (respectively, $\langle v, Gu\rangle_{[0,T]}=\langle G^*v, u\rangle_{[0,T]}$ for all $v$, $u\in\mathscr{L}_2([0,T])$). Furthermore, $\|G\|=\|G^*\|$ (respectively, $\|G\|_{[0,T]}=\|G^*\|_{[0,T]}$). An operator $G\in\mathscr{B}_v$ is said to be *isometric* (respectively, *co-isometric*) if $G^*G=I$ (respectively, if $GG^*=I$).

We now derive a state-space realization for the adjoint $G^*$ of $G\in\mathscr{B}_v([0,T])$. Using (2) we get

$$\begin{aligned}\langle v, Gu\rangle &= \int_0^T v'(t)\left(\int_0^t C(t)\Phi_G(t,\tau)B(\tau)u(\tau)\,\mathrm{d}\tau + D(t)u(t)\right)\mathrm{d}t\\
&= \int_0^T\left(\int_\tau^T B'(\tau)\Phi_G'(t,\tau)C'(t)v(t)\,\mathrm{d}t + D'(\tau)v(\tau)\right)'u(\tau)\,\mathrm{d}\tau \qquad (4)\\
&= \langle G^*v, u\rangle, \qquad (5)\end{aligned}$$

where (4) is derived by interchanging the order of integration. If we set

$$\zeta(\tau) := \int_\tau^T \Phi_G'(t,\tau)C'(t)v(t)\,\mathrm{d}t, \tag{6}$$

and interchange $t$ and $\tau$ in both (5) and (6), we find that $G^*$ in (5) is an *anti-causal* operator that has the realization

$$\Sigma_{G^*} := \begin{cases} -\dot{\zeta}(t) = A'(t)\zeta(t) + C'(t)v(t), & \zeta(T)=0,\\ \omega(t) = B'(t)\zeta(t) + D'(t)v(t), \end{cases} \tag{7}$$

where $v(t)$ is the input and $\omega(t)$ is the output. We now make the change of variable $\hat{t}=-t$ and set $\hat{A}(\hat{t}):=A'(t)$, $\hat{B}(\hat{t}):=C'(t)$, $\hat{C}(\hat{t}):=B'(t)$ and $\hat{D}(\hat{t}):=D'(t)$. We further define the time reversed signals $\hat{x}(\hat{t}):=\zeta(t)$, $\hat{u}(\hat{t}):=v(t)$, and $\hat{y}(\hat{t}):=\omega(t)$, to get the following realization for the *dual* $\hat{G}$:

$$\Sigma_{\hat{G}} := \begin{cases} \dot{\hat{x}}(\hat{t}) = \hat{A}(\hat{t})\hat{x}(\hat{t}) + \hat{B}(\hat{t})\hat{u}(\hat{t}), & \hat{x}(-T)=0,\\ \hat{y}(\hat{t}) = \hat{C}(\hat{t})\hat{x}(\hat{t}) + \hat{D}(\hat{t})\hat{u}(\hat{t}). \end{cases} \tag{8}$$

The realization $\Sigma_{\hat{G}}$ defines a causal operator on $\mathscr{L}_2([-T, 0])$, and its state transition matrix $\Phi_{\hat{G}}(\hat{t}, \hat{\tau})$ satisfies

$$\Phi_{\hat{G}}(\hat{t}, \hat{\tau}) = \Phi'_G(\tau, t) \tag{9}$$

for all $\hat{\tau} \leqslant \hat{t}$; $\hat{t}$ and $\hat{\tau}$ in $[-T, 0]$. Moreover, it follows from (7) and (8) that $\|\hat{G}\|_{[-T,0]} = \|G^*\|_{[0,T]}$. If $G \in \mathscr{B}_c(\mathbb{R}_+)$, then the restriction of $G$ to any finite interval $[0, T]$ is a causal, bounded operator with dual $\hat{G}$ defined on $\mathscr{L}_2([-T, 0])$. It can be shown, using (9) and the definition of exponential stability, that

$$\|\Phi_{\hat{G}}(\hat{t}, \hat{\tau})\| \leqslant c_1\, \mathrm{e}^{-c_2(\hat{t}-\hat{\tau})} \quad \forall \hat{\tau} \leqslant \hat{t};\ \hat{t}, \hat{\tau} \in [-T, 0], \tag{10}$$

where the constants $c_1$, $c_2 > 0$ *are independent of* $T$.

We conclude this section with some preliminary results.

**Lemma 2.1.** *Consider the system*

$$\Sigma_S := \begin{cases} \dot{x}(t) = A(t)x(t), & x(t_0) = x_0, \\ y(t) = C(t)x(t). \end{cases} \tag{11}$$

*If* $(A, C)$ *is detectable, then there exist* $M$, $N < \infty$ *such that for every* $t_0 \in \mathbb{R}_+$ *and* $x_0 \in \mathbb{R}^n$ *we have* $y \in \mathscr{L}_2([t_0, \infty)) \Rightarrow x \in \mathscr{L}_2([t_0, \infty))$. *Furthermore,* $\|x\|_{[t_0,\infty)} \leqslant M\|x_0\| + N\|y\|_{[t_0,\infty)}$.

**Proof.** Given the conditions of the lemma, $\Sigma_S$ can be rewritten as

$$\dot{x}(t) = (A - LC)(t)x(t) + (LC)(t)x(t), \quad x(t_0) = x_0, \tag{12a}$$

$$y(t) = C(t)x(t), \tag{12b}$$

where $L(t)$ is bounded and chosen so that $\dot{x}(t) = (A - LC)(t)x(t)$ is exponentially stable. Hence, there exist constants $c_1$, $c_2 > 0$ such that the transition matrix $\Phi(t, \tau)$ of (12) satisfies (3). Using the variation of constants formula, the solution to (12) can be written as

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)L(\tau)y(\tau)\,\mathrm{d}\tau. \tag{13}$$

Let $\alpha := \sup_{t \geqslant 0}\|L(t)\|$. Routine calculations then show that

$$\|x\|^2_{[t_0,\infty)} := \int_{t_0}^{\infty} \|x(t)\|^2\,\mathrm{d}t \leqslant \frac{c_1^2}{2c_2}\|x_0\|^2 + \frac{c_1^2}{2c_2}\alpha^2\|y\|^2_{[t_0,\infty)} + \frac{c_1^2}{c_2^2}\alpha\|y\|_{[t_0,\infty)}\|x_0\|. \tag{14}$$

Setting $\gamma^2 = \max(c_1^2/(2c_2), c_1^2/c_2^2)$, it follows that

$$\|x\|_{[t_0,\infty)} \leqslant \gamma\|x_0\| + \gamma\alpha\|y\|_{[t_0,\infty)}.$$

Now we have the result with $M = \gamma$ and $N = \gamma\alpha$. □

**Lemma 2.2.** *Let the pair* $(A, B)$ *be stabilizable, and consider the related causal system*

$$\Sigma_{\hat{S}} := \begin{cases} \dot{\hat{x}}(\hat{t}) = \hat{A}(\hat{t})\hat{x}(\hat{t}), & \hat{x}(-T) = x_T, \\ \hat{y}(\hat{t}) = \hat{C}(\hat{t})\hat{x}(\hat{t}), \end{cases} \tag{15}$$

*on* $[-T, 0]$ *with* $\hat{A}(\hat{t}) := A'(t)$, $\hat{C}(\hat{t}) := B'(t)$ *and* $t := -\hat{t}$. *Then there exist* $M$, $N < \infty$ *such that for every* $T \in \mathbb{R}_+$ *and* $x_T \in \mathbb{R}^n$, *we have* $\hat{y} \in \mathscr{L}_2([-T, 0]) \Rightarrow \hat{x} \in \mathscr{L}_2([-T, 0])$. *Furthermore,* $\|\hat{x}\|_{[-T,0]} \leqslant M\|x_T\| + N\|\hat{y}\|_{[-T,0]}$.

**Proof.** Note that this is the dual of the previous result. Since the pair $(A, B)$ is stabilizable, there exists a bounded matrix function $K(t)$ such that $\dot{x}(t)=(A-BK)(t)x(t)$ is exponentially stable. Let $\hat{L}(\hat{t}):=K'(t)$. Using the duality results, it follows that for every $T>0$ the system

$$\dot{\hat{x}}(\hat{t})=(\hat{A}-\hat{L}\hat{C})(\hat{t})\hat{x}(\hat{t}) \tag{16}$$

has a state-transition matrix $\Phi_{\hat{G}}(\hat{t}, \hat{\tau})$ that satisfies (10). The result follows by reformulating the proof of Lemma 2.1 over the finite interval $[-T, 0]$, and checking that the constants $M$ and $N$ are independent of $T$. □

## 3. Stabilizing solutions to Riccati equations

In this section we show that stabilizability and detectability are sufficient to ensure the existence and uniqueness of stabilizing solutions to standard filter and control Riccati differential equations. The proof of this assertion is based on the following results on the Lyapunov stability of linear time-varying systems.

**Lemma 3.1.** *Let the pair $(A, C)$ be detectable, and suppose there exists a symmetric differentiable matrix function $P(t)$, satisfying $0 \leqslant P(t) \leqslant \beta I$ for some $\beta<\infty$, such that*

$$\dot{P}(t)+A'(t)P(t)+P(t)A(t)=-C'(t)C(t)$$

*for every $t$ in $\mathbb{R}_+$. Then the system $\dot{x}(t)=A(t)x(t)$ is exponentially stable.*

**Proof.** Consider the quadratic form $(x'Px)(t)$ and compute its time-derivative along a trajectory of (11) to get

$$\frac{\mathrm{d}(x'Px)(t)}{\mathrm{d}t}=-y'(t)y(t), \tag{17}$$

where $y(t):=C(t)x(t)$. Integrating (17) from $t_0$ to $t_1$ gives

$$(x'Px)(t_1)-(x'Px)(t_0)=-\int_{t_0}^{t_1}\|y(t)\|^2\,\mathrm{d}t.$$

By dropping the first term and changing signs on both sides, we get

$$\int_{t_0}^{t_1}\|y(t)\|^2\,\mathrm{d}t \leqslant (x'Px)(t_0) \leqslant \beta\|x_0\|^2, \tag{18}$$

where we have set $x_0:=x(t_0)$. The bound in (18) holds for all $t_1>t_0$. Hence we have

$$\|y\|^2_{[t_0,\infty)} \leqslant \beta\|x_0\|^2.$$

Using Lemma 2.1 we get $\|x\|_{[t_0,\infty)} \leqslant (M+\beta^{1/2}N)\|x_0\|$, where the bound is independent of the initial time $t_0$. It follows from a result in [3] (Theorem 3, page 190) that the system $\dot{x}(t)=A(t)x(t)$ is exponentially stable. □

**Lemma 3.2.** *Let the pair $(A, B)$ be stabilizable, and suppose there exists a symmetric differentiable matrix function $Q(t)$, satisfying $0 \leqslant Q(t) \leqslant \beta I$ for some $\beta<\infty$, such that*

$$\dot{Q}(t)-A(t)Q(t)-Q(t)A'(t)=B(t)B'(t)$$

*for every $t$ in $\mathbb{R}_+$. Then the system $\dot{x}(t)=A(t)x(t)$ is exponentially stable.*

**Proof.** Let $T$ be an arbitrary positive number, and define $\Sigma_{\hat{S}}$ over $[-T, 0]$ as in Lemma 2.2. With $\hat{Q}(\hat{t}) := Q(t)$, we obtain

$$\dot{\hat{Q}}(\hat{t}) + \hat{A}'(\hat{t})\hat{Q}(\hat{t}) + \hat{Q}(\hat{t})\hat{A}(\hat{t}) = -\hat{C}'(\hat{t})\hat{C}(\hat{t}). \tag{19}$$

The methodology used in the proof of Lemma 3.1 is then used over $[-T, 0]$ to obtain

$$\| \hat{y} \|_{[-T,0]}^2 \leqslant \beta \| x_T \|^2. \tag{20}$$

Using Lemma 2.2 it follows that $\| \hat{x} \|_{[-T,0]} \leqslant (M + \beta^{1/2}N) \| x_T \|$, where the bound is independent of $T$. Again, using the aforementioned result from [3] and equation (9), we obtain that the system $\dot{x}(t) = A(t)x(t)$ is exponentially stable. □

We now state the main results of this section.

**Lemma 3.3.** *Let $\Sigma_G$ be as in* (1). *Assume that $(A, B)$ is stabilizable and $(A, C)$ is detectable. Then, there exists a bounded symmetric differentiable matrix function $P(t) \geqslant 0$ satisfying the control Riccati equation*

$$-\dot{P}(t) = A'(t)P(t) + P(t)A(t) - P(t)B(t)B'(t)P(t) + C'(t)C(t). \tag{21}$$

*Furthermore, the system $\dot{x}(t) = (A\text{-}BB'P)(t)x(t)$ is exponentially stable.*

**Proof.** The existence of a bounded positive semidefinite function $P(t)$ that solves (21) can be shown by a simple modification of the arguments in [7]. The key idea is to relate the above Riccati equation to a certain optimal regulator problem, and use the stabilizability of $(A, B)$ to show that $P(t)$ with the properties above is well defined. The stability of $\dot{x}(t) = (A - BB'P)(t)x(t)$ now follows immediately from Lemma 3.1 because we can rewrite equation (21) as

$$\dot{P}(t) + (A - BB'P)'(t)P(t) + P(t)(A - BB'P)(t) = -P(t)B(t)B'(t)P(t) - C'(t)C(t), \tag{22}$$

which is the required Lyapunov equation. Clearly the detectability of $((A - BB'P), (PB\ \ C')')$ follows from that of $(A, C)$. □

**Lemma 3.4.** *Let $\Sigma_G$ be as in* (1). *Assume that $(A, B)$ is stabilizable and $(A, C)$ is detectable. Then, there exists a unique bounded symmetric matrix solution $Q(t) \geqslant 0$ to the filter Riccati equation*

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A'(t) - Q(t)C'(t)C(t)Q(t) + B(t)B'(t), \quad Q(0) \geqslant 0. \tag{23}$$

*Furthermore, the system $\dot{x}(t) = (A - QC'C)(t)x(t)$ is exponentially stable.*

**Proof.** The existence of a bounded, non-negative definite solution to the Riccati equation (23) follows from the classical optimal filtering theory. The key idea is to take the given data, i.e. $A(t)$, $B(t)$ and $C(t)$, and set up an optimal Kalman–Bucy filtering problem (see e.g. [9] for details). The detectability assumption on $(A, C)$ will then ensure that the optimal covariance is bounded and that a matrix function $Q$ with the properties above exists. We now rewrite (23) as

$$\begin{aligned} &\dot{Q}(t) - (A - QC'C)(t)Q(t) - Q(t)(A - QC'C)'(t) \\ &\quad = Q(t)C'(t)C(t)Q(t) + B(t)B'(t), \end{aligned} \tag{24}$$

and notice that if $(A, B)$ is stabilizable, then so is $((A - QC'C), (QC'\ \ B))$. The stability of $\dot{x}(t) = (A - QC'C)(t)x(t)$ follows from Lemma 3.2. □

## 4. Normalized coprime factorizations

In this section we derive necessary and sufficient conditions for the existence of normalized coprime factorizations of finite dimensional, linear time-varying systems.

**Definition.** Let $G$ be a finite dimensional linear time-varying system. We say that $G$ admits a stable *left-coprime* (respectively, stable *right-coprime*) *factorization* if there exist *exponentially stable* FDLTV systems $\Sigma_{\tilde{N}}$, $\Sigma_{\tilde{M}}$, $\Sigma_{\tilde{X}}$, and $\Sigma_{\tilde{Y}}$ with $\Sigma_{\tilde{M}}$ bicausal (respectively, $\Sigma_N$, $\Sigma_M$, $\Sigma_X$, and $\Sigma_Y$ with $\Sigma_M$ bicausal) such that $G=\tilde{M}^{-1}\tilde{N}$ and $\tilde{N}\tilde{X}+\tilde{M}\tilde{Y}=I$ (respectively, $G=NM^{-1}$ and $XN+YM=I$). Moreover, we say that the coprime factorization $(\tilde{N}, \tilde{M})$ (respectively, $(N, M)$) is *normalized* if $(\tilde{M}\ \tilde{N})$ is co-isometric, i.e. $\tilde{M}\tilde{M}^*=\tilde{N}\tilde{N}^*=I$ (respectively, $(M'\ N')'$ is isometric, i.e. $M^*M+N^*N=I$).

We are now ready to show that normalized coprime factorizations always exist for systems that admit stabilizable and detectable realizations. This provides an extension to the time-invariant result in [11] to the time-varying case. In our opinion the proof here is much simpler and hence it serves as an alternate (simpler) proof even for the time-invariant case. For the sake of keeping the exposition brief we give a complete derivation only for the strongly causal ($D=0$) case. The generalization to the causal case can be carried out using the ideas in [18].

**Theorem 4.1.** *Let $G$ be an* FDLTV *system with a stabilizable and detectable realization given by $\Sigma_G$ as in* (1) (*with $D=0$*). *Let $P(t)$ and $Q(t)$ be solutions to* (21) *and* (23) *respectively with the boundary condition in* (23) *set to $Q(0)=0$. Define*

$$\Sigma_{\tilde{N}}:=\left[\begin{array}{c|c} A-QC'C & B \\ \hline C & 0 \end{array}\right], \qquad \Sigma_N:=\left[\begin{array}{c|c} A-BB'P & B \\ \hline C & 0 \end{array}\right]$$
$$\Sigma_{\tilde{M}}:=\left[\begin{array}{c|c} A-QC'C & -QC' \\ \hline C & I \end{array}\right], \qquad \Sigma_M:=\left[\begin{array}{c|c} A-BB'P & B \\ \hline -B'P & I \end{array}\right]. \tag{25}$$

*Then $(\tilde{N}, \tilde{M})$ (respectively, $(N, M)$) is a normalized left-coprime (respectively, right-coprime) factorization of $G$. Moreover, any normalized left-coprime (respectively, any normalized right-coprime) factorization is unique up to multiplication on the left (respectively, right) by a memoryless, unitary operator.*

**Proof.** In this proof we restrict ourselves to the *left-coprime* case. The proof for right-coprime factorizations can be carried out in a similar manner. With $(\tilde{N}\ \tilde{M})$ defined in (25), it is easy to show that $G=\tilde{M}^{-1}\tilde{N}$. Moreover, using the results of [8,10], it follows that there exist exponentially stable systems $\Sigma_{\tilde{X}}$ and $\Sigma_{\tilde{Y}}$ such that $\tilde{N}\tilde{X}+\tilde{M}\tilde{Y}=I$. Thus, $(\tilde{N}, \tilde{M})$ is a left-coprime factorization for $G$. All that remains to be shown is that it is normalized. Let $P:=(\tilde{M}\ \tilde{N})$ denote the operator with realization

$$\Sigma_P:=\left[\begin{array}{c|cc} A-QC'C & -QC' & B \\ \hline C & I & 0 \end{array}\right]. \tag{26}$$

From Lemma 3.4 we know that $P$ defines a stable operator. We now show that $P$ is co-isometric, or equivalently that $\|P^*u\|^2=\|u\|^2$ for all $u\in\mathscr{L}_2$, where $P^*$ denotes the adjoint of $P$. Suppose, on the contrary, that there exists a $u$ such that $\|P^*u\|^2\neq\|u\|^2$, and let $|\,\|P^*u\|^2-\|u\|^2|=\varepsilon>0$. Since $u\in\mathscr{L}_2$ and $P^*$ is *anti-causal* and bounded, we can always choose $T<\infty$ such that

$$\left|\|P^*u_{\mathrm{T}}\|^2_{[0,T]}-\|u_{\mathrm{T}}\|^2_{[0,T]}\right|\geqslant\tfrac{1}{2}\varepsilon, \tag{27}$$

where $u_{\mathrm{T}}=P_{\mathrm{T}}u$ and $P_{\mathrm{T}}$ denotes the truncation operator. Having chosen a terminal time $T$ we define, over $[-T, 0]$, the dual of $P$ (restricted to $[0, T]$) as follows:

$$\Sigma_{\hat{P}}:=\begin{cases} \dot{\hat{x}}(\hat{t})=\left(\hat{A}-\hat{B}\hat{B}'\hat{Q}\right)(\hat{t})\hat{x}(\hat{t})+\hat{B}(\hat{t})\hat{u}(\hat{t}), \quad \hat{x}(-T)=0, \\ \hat{y}_1(\hat{t})=-\hat{B}'\hat{Q}(\hat{t})\hat{x}(\hat{t})+\hat{u}(\hat{t}), \\ \hat{y}_2(\hat{t})=\hat{C}(\hat{t})\hat{x}(\hat{t}), \end{cases} \tag{28}$$

where $\hat{t} := -t$, $\hat{A}(\hat{t}) := A'(t)$, $\hat{B}(\hat{t}) := C'(t)$, $\hat{C}(\hat{t}) := B'(t)$ and $\hat{Q}(\hat{t})$ satisfies the dual version of (23)

$$\dot{\hat{Q}}(\hat{t}) = \hat{A}'(\hat{t})\hat{Q}(\hat{t}) + \hat{Q}(\hat{t})\hat{A}(\hat{t}) - \hat{Q}(\hat{t})\hat{B}(\hat{t})\dot{\hat{B}}(\hat{t})\hat{Q}(\hat{t}) + \hat{C}'(\hat{t})\hat{C}(\hat{t}), \quad \hat{Q}(0) = 0. \tag{29}$$

Let $\hat{u}(\hat{t}) := u_{\mathrm{T}}(t)$ for $\hat{t}$ in $[-T, 0]$. Then, it is easily seen that the estimate (27) translates into

$$\left| \| \hat{y}_1 \|^2_{[-T,0]} + \| \hat{y}_2 \|^2_{[-T,0]} - \| \hat{u} \|^2_{[-T,0]} \right| \geqslant \tfrac{1}{2}\varepsilon. \tag{30}$$

If we consider the function $(\hat{x}'\hat{Q}\hat{x})(\hat{t})$ and compute its time derivative along the trajectory of (28), we obtain

$$\frac{\mathrm{d}\left(\hat{x}'\hat{Q}\hat{x}\right)(\hat{t})}{\mathrm{d}\hat{t}} = \| \hat{u}(\hat{t}) \|^2 - \| \hat{y}_1(\hat{t}) \|^2 - \| \hat{y}_2(\hat{t}) \|^2. \tag{31}$$

As $\hat{x}(-T) = 0$ and $\hat{Q}(0) = 0$, integrating (31) from $-T$ to 0 gives us

$$\| \hat{y}_1 \|^2_{[-T,0]} + \| \hat{y}_2 \|^2_{[-T,0]} = \| \hat{u} \|^2_{[-T,0]}, \tag{32}$$

which contradicts the estimate (30). This proves our hypothesis that $PP^* = I$ or, equivalently, that $\tilde{N}\tilde{N}^* + \tilde{M}\tilde{M}^* = I$.

Suppose now that $(\hat{N}, \hat{M})$ is another left-coprime factorization of $G$. Then, there exists a $U \in \mathscr{U}_{\mathrm{c}}$ such that $\hat{N} = U\tilde{N}$ and $\hat{M} = U\tilde{M}$. It follows by simple substitution that $UU^* = I$, and since $U$ has a stable inverse, we get that $U^* = U^{-1}$ is causal and bounded. As both $U$ and its adjoint are causal, the unitary operator $U$ must be memoryless (see [19]). □

**Remark.** For the sake of completeness we state, without proof, the form of the (normalized left-coprime) factorizations for the causal case. Let

$$\Sigma_G := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

be a stabilizable and detectable realization for $G$, and let $Q(t)$ be the unique bounded positive semidefinite stabilizing solution to the following Riccati differential equation

$$\begin{aligned}\dot{Q}(t) &= \left(A - BD'(I + DD')^{-1}C\right)(t)Q(t) + Q(t)\left(A - BD'(I + DD')^{-1}C\right)'(t) \\ &\quad - Q(t)\left(C'(I + DD')^{-1}C\right)(t)Q(t) + B(I + D'D)^{-1}B'(t), \quad Q(0) \geqslant 0.\end{aligned}$$

Then, with $E(t)E'(t) := I + D(t)D'(t)$ and $L(t) := (B(t)D'(t) + Q(t)C'(t))(I + D(t)D'(t))^{-1}$, a normalized left-coprime factorization for $G$ is given by

$$\Sigma_{\tilde{N}} := \left[\begin{array}{c|c} A - FC & B - FD \\ \hline E^{-1}C & E^{-1}D \end{array}\right], \qquad \Sigma_{\tilde{M}} := \left[\begin{array}{c|c} A - FC & -F \\ \hline E^{-1}C & E^{-1} \end{array}\right].$$

We have shown that stabilizability and detectability are *sufficient* to ensure the existence of left- and right-normalized coprime factorizations. That these conditions are also *necessary* follows immediately from Theorem 4.6 in [8]. This leads to an interesting result that relates the existence of normalized coprime factorizations for an FDLTV plant to the existence of an internally stabilizing controller. We introduce the following definition.

**Definition.** An FDLTV system $G$ is said to be *internally stabilizable via dynamic output feedback* if it admits a realization

$$\Sigma_G := \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$$

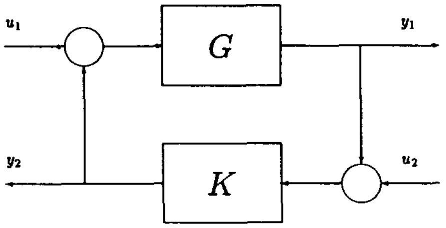


Fig. 1. Feedback interconnection.

for which there exists a controller $K$ with a realization

$$\Sigma_K := \left[\begin{array}{c|c} F & G \\ \hline H & E \end{array}\right],$$

such that $J_1(t) := (I+ED)^{-1}(t)$ (and consequently $J_2(t) := (I+DE)^{-1}(t)$) is bounded on $\mathbb{R}_+$, and the standard feedback interconnection (see Figure 1) described by

$$\left[\begin{array}{cc|c} A - BJ_1EC & -BJ_1H & \\ & & * \\ GJ_2C & F - GJ_2DH & \\ \hline & * & * \end{array}\right]$$

is exponentially stable.

Now, from Theorem 4.1 above and Theorem 4.6 in [8], we have:

**Corollary 4.2.** *Let $G \in \mathscr{M}_v$ be a causal* FDLTV *system. Then, the following statements are equivalent*:
1. *$G$ can be internally stabilized via dynamic output feedback.*
2. *$G$ admits a left-coprime factorization.*
3. *$G$ admits a right-coprime factorization.*
4. *$G$ admits a stabilizable and detectable realization.*
5. *$G$ admits a normalized left-coprime factorization.*
6. *$G$ admits a normalized right-coprime factorization.*

## 5. Calculating the gap metric

We begin with the following result that defines a metric on $\mathscr{M}_v$. A proof can be found in [5].

**Lemma 5.1.** *Let $G_1$ and $G_2$ be any two members of $\mathscr{M}_v$ with the same number of inputs and outputs. Let $G_i = \tilde{M}_i^{-1}\tilde{N}_i$ be a normalized left-coprime factorization for $G_i$ and let $P_i := (\tilde{M}_i \;\; \tilde{N}_i)$, $i = 1, 2$. Define $d(G_1, G_2)$ as follows*:

$$\delta(G_1, G_2) := \inf_{\mathscr{Q} \in \mathscr{B}_v} \| P_1 - \mathscr{Q} P_2 \|,$$

$$d(G_1, G_2) := \max(\delta(G_1, G_2), \delta(G_2, G_1)). \tag{33}$$

*Then $d(G_1, G_2)$ is a well defined metric on $\mathscr{M}_v$, taking values in the interval* [0, 1].

The gap metric was first applied to the analysis of LTI control systems by Zames and El-Sakkary [20]. However, their formulation was different from the one given above. The formula (33) was first derived for the time-invariant case by Georgiou [6] and for the time-varying case by Feintuch [5]. An equivalent metric, the *graph metric*, was introduced by Vidyasagar [17] for LTI systems. The generalization of the graph metric to the time-varying case is straightforward and can be found in [14]. The graph metric is defined as in (33) except that the optimization is carried out over the set $\{\mathcal{Q} \in \mathscr{B}_{t}: \|\mathcal{Q}\| \leqslant 1\}$. The reason for concentrating on the gap rather than the graph metric is that the optimization problem given above in (33) is tractable, whereas the restricted problem (with $\|\mathcal{Q}\| \leqslant 1$) is not.

Next, we will draw upon some recent results in $\mathscr{H}_\infty$ control of linear time-varying systems to solve the optimization problem outlined in (33). In what follows, we restrict ourselves to systems that admit finite-dimensional, stabilizable and detectable realizations. For simplicity of exposition we consider only strongly causal systems. Let $P_i$, $i = 1, 2$ have state space realizations (26)

$$\left[\begin{array}{c|cc} A_i - Q_iC_i'C_i & -Q_iC_i' & B_i \\ \hline C_i & I & 0 \end{array}\right]. \tag{34}$$

The problem is to compute

$$\inf_{\mathcal{Q} \in \mathscr{B}_t} \|P_i - \mathcal{Q}P_j\|$$

for $(i, j) = (1, 2), (2, 1)$. Let $(i, j) = (1, 2)$, and let $\gamma \leqslant 1$ be a positive real number. We will show that there exists a $\mathcal{Q} \in \mathscr{B}_t$ such that $\|P_1 - \mathcal{Q}P_2\| < \gamma$ if and only if a certain Riccati differential equation admits a stabilizing solution. Then, by iterating on $\gamma$ the infimum can be obtained to any given accuracy. The case $(i, j) = (2, 1)$ is identical. We start by defining a new 'plant'

$$P := \begin{pmatrix} P_1 & I \\ -P_2 & 0 \end{pmatrix}$$

with a realization

$$\Sigma_P := \left[\begin{array}{cc|cc} A_1 - Q_1C_1'C_1 & 0 & [-Q_1C_1' \;\; B_1] & 0 \\ 0 & A_2 - Q_2C_2'C_2 & [-Q_2C_2' \;\; B_2] & 0 \\ \hline C_1 & 0 & [I \;\; 0] & I \\ 0 & -C_2 & [-I \;\; 0] & 0 \end{array}\right].$$

Note that

$$P_1 - \mathcal{Q}P_2 = F_l(P, \mathcal{Q}) \tag{35}$$

where $F_l(\cdot, \cdot)$ is the standard notation for the lower linear fractional transformation.

Using a well known technique [15], we can bring the problem into the 'standard' form, and show that there exists a stable $\mathcal{Q}$ such that $\|F_l(P, \mathcal{Q})\| < \gamma$ iff there exists a stable $\bar{\mathcal{Q}}$ such that $\|F_l(\bar{P}, \bar{\mathcal{Q}}\| < \gamma$, where

$$\Sigma_{\bar{P}} := \left[\begin{array}{cc|cc} A_1 - Q_1C_1'C_1 & 0 & [-Q_1C_1' \;\; B_1] & 0 \\ 0 & A_2 - Q_2C_2'C_2 & [-Q_2C_2' \;\; B_2] & 0 \\ \hline C_1 & -C_2 & [0 \;\; 0] & I \\ 0 & C_2 & [I \;\; 0] & 0 \end{array}\right].$$

Now we use the appropriate generalizations of the results in [16,13] to derive the necessary and sufficient conditions for the existence of such a $\bar{\mathcal{Q}}$ (or $\mathcal{Q}$). This is stated in the following.

**Theorem 5.2.** *There exists a stable* FDLTV *system* $\mathcal{Q}$ *such that* $\|F_l(P, \mathcal{Q})\| < \gamma$ *iff the following Riccati differential equation*

$$\dot{Y} = AY + YA' + YRY - Q, \quad Y(0) = 0,$$

*has a bounded positive semidefinite stabilizing solution Y, where*

$$A := \begin{pmatrix} A_1 - Q_1 C_1' C_1 & -Q_1 C_1' C_2 \\ 0 & A_2 \end{pmatrix},$$

$$R := \frac{1}{\gamma^2} \begin{pmatrix} -\frac{1}{\gamma^2} C_1' C_1 & \frac{1}{\gamma^2} C_1' C_2 \\ \frac{1}{\gamma^2} C_2' C_1 & \left(1 - \frac{1}{\gamma^2}\right) C_2' C_2 \end{pmatrix} \quad \text{and} \quad Q := \begin{pmatrix} B_1 B_1' & B_1 B_2' \\ B_2 B_1' & B_2 B_2' \end{pmatrix}.$$

**Remark.** In the case of systems with nonzero $D$, we still have *only one* Riccati equation but, because of the complicated nature of the normalizing transformations, the equation cannot be represented in as concise a fashion as above.

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