Automatica, Vol. 28, No. 3, pp. 549–555, 1992
Printed in Great Britain.

0005-1098/92 \$5.00 + 0.00
Pergamon Press Ltd
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Aiming Control: Residence Probability and (*D*, *T*)-Stability*†

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The problem of Aiming Control is analyzed using a residence probability measure along with the associated notion of (D, T)-stability.

Key Words—Linear Systems; stochastic control; large deviations; stability criteria; approximation theory.

Abstract—In this paper, the problem of aiming control is formulated and analyzed in terms of the residence probability measure. Specifically, the notion of residence probability in a domain is introduced and its asymptotic expression is derived for linear systems with small, additive white noise. The associated notion of (D, T)-stability, which characterizes the performance of stochastic systems with no equilibrium points, is introduced and investigated. Finally, the controllability of residence probability is studied and the necessary and sufficient conditions for (D, T)-stabilizability are derived. The development is based on the asymptotic large deviations theory.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

CONSIDER A SYSTEM described by the Ito stochastic differential equation:

$$dx = (Ax + Bu)\,dt + \varepsilon C\,dw, \quad x(0) = x_0, \qquad (1.1)$$

where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^m$, $0 < \varepsilon \ll 1$ and w is a standard r-dimensional Brownian motion. Let $D \subset \mathbf{R}^n$ be an open bounded domain with 0 in its interior, $T < \infty$ a positive number and consider the following problem:

Given system (1.1) and the pair (D, T), find a feedback law

$$u = Kx, \qquad (1.2)$$

and an open set $D_0 \subset D$ such that the closed loop system (1.1), (1.2) has the following property:

$$x(t, x_0) \in D, \quad \forall t \in [0, T], \quad \forall x_0 \in [D_0],$$

where $[D_0]$ is the closure of D_0.

This problem is referred to as the problem of *aiming control*. Such a problem arises in a number of applications where the goal is to accomplish a certain task during a specified period (T) with a specified accuracy (D). Examples include the telescope pointing problem (Skelton, 1973), robot arm and laser beam pointing (Cannon and Schmitz, 1984; Katzman, 1987), missile terminal guidance (Garnell, 1977), etc.

Let ∂D be the boundary of D and define the *first passage time*

$$\tau^{\varepsilon}_{x_0} = \inf\{t \geq 0 : x(t, x_0) \in \partial D\}.$$

It is well known that $\tau^{\varepsilon}_{x_0}$ is almost surely finite (Freidlin and Wentzell, 1984). Therefore, the aiming process specifications, (D, T), cannot be met exactly and some probabilistic meaning should be attached to their interpretation. Specifically, the problem of aiming control can be re-formulated in the following two probabilistic settings.

Residence time control. Given (1.1) and a pair (D, T), find a feedback law (1.2) and an open set $D_0 \subset D$ such that

$$E[\tau^{\varepsilon}_{x_0}] \geq T, \quad \forall x_0 \in [D_0]. \qquad (1.3)$$

Residence probability control. Given (1.1), a pair (D, T) and a constant $0 < p < 1$, find a

* Received 1 November 1989; revised 13 July 1990; revised 8 January 1991; revised 19 August 1991; received in final form 3 September 1991. The original version of this paper was presented at the 11th IFAC World Congress on Automatic Control at the Service of Mankind which was held in Tallinn, Estonia during August 1990. The Published Proceedings of this IFAC Meeting may be ordered from: Pergamon Press Ltd, Headington Hill Hall, Oxford OX3 0BW, U.K. This paper was recommended for publication in revised form by Associate Editor T. Başar under the direction of Editor H. Kwakernaak.

† This work was supported by the Air Force Office of Scientific Research under Contract F49620-87-C-0079.

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feedback law (1.2) and an open set $D_0 \subset D$ such that

$$\text{Prob}\{\tau^\varepsilon_{x_0} > T\} > p, \quad \forall x_0 \in [D_0]. \quad (1.4)$$

The residence time control problem (1.4) and its generalizations have been analyzed in Meerkov and Runolfsson (1988, 1989a, 1989b, 1990) and Runolfsson (1988). In these publications, the fundamental bounds on the achievable values of $E[\tau^\varepsilon_{x_0}]$ have been investigated and the methods for controllers design, compatible with these bounds, have been developed.

Note that, since $\tau^\varepsilon_{x_0}$ is a non-negative random variable, the Markov inequality gives:

$$\text{Prob}\{\tau^\varepsilon_{x_0} \geq T\} \leq \frac{E[\tau^\varepsilon_{x_0}]}{T}.$$

Therefore, if $\text{Prob}\{\tau^\varepsilon_{x_0} \geq T\} \geq p$, the estimate for $E[\tau^\varepsilon_{x_0}]$ follows immediately:

$$E[\tau^\varepsilon_{x_0}] \geq pT.$$

This observation suggests that the residence probability control (1.5) is a stronger reformulation of the aiming control problem than the residence time control (1.4).

This paper is devoted to the investigation of the controllability properties of residence probability, i.e. to the question on when there exists a feedback law $u = Kx$ that solves the residence probability control problem. The related question of design of the residence probability controllers will be addressed in a companion paper. As it was the case in Meerkov and Runolfsson (1988, 1989a, 1989b, 1990) and Runolfsson (1988), the development is based on the large deviations theory of Freidlin and Wentzell (1984).

The idea of utilizing the residence probability as a measure of control systems performance is not new. Apparently, it was first introduced in Ruina and van Valkenburg (1960) and then analyzed in Kushner (1967), Fleming and Tsai (1981), and Dupuis and Kushner (1987).

The structure of the paper is as follows: in Section 2 an asymptotic formula for residence probability in a domain is derived. In Section 3, the notion of (D, T)-stability, that characterizes the behavior of stochastic systems with no equilibrium points, is introduced and analyzed. Section 4 presents the conditions for residence probability controllability and (D, T)-stabilizability. In Section 5, the conclusions are formulated. The proofs are given in Appendices 1–3.

2. RESIDENCE PROBABILITY IN A DOMAIN

Consider the Ito system

$$dx = Ax\,dt + \varepsilon C\,dw, \quad x(0) = x_0, \quad (2.1)$$

where, as before, $x \in \mathbf{R}^n$, $0 < \varepsilon \ll 1$, w is a standard r-dimensional Brownian motion and rank $C = r$. Let $D \subset \mathbf{R}^n$ and $D_0 \subset D$ be open bounded sets with 0 in their interior and smooth boundaries ∂D and ∂D_0, respectively. The first passage time of the trajectory originating at x_0 is

$$\tau^\varepsilon_{x_0} = \inf\{t \geq 0 : x(t) \in \partial D\}. \quad (2.2)$$

As a random variable, $\tau^\varepsilon_{x_0}$ is characterized by its probability distribution, i.e. $\text{Prob}\{\tau^\varepsilon_{x_0} \leq T\}$. Based on this distribution, the first passage probability of the trajectories originating in $[D_0]$ can be defined as follows:

$$P_{D_0}\{\tau^\varepsilon \leq T\} \triangleq \sup_{x_0 \in [D_0]} \text{Prob}\{\tau^\varepsilon_{x_0} \leq T\}. \quad (2.3)$$

Note that, due to the compactness of $[D_0]$ and the continuity of $\text{Prob}\{\tau^\varepsilon_{x_0} \leq T\}$ in x_0 (Freidlin and Wentzell, 1984), the supremum in (2.3) is actually attained. Then the *residence probability in the domain* is defined as

$$P_{D_0}\{\tau^\varepsilon > T\} \triangleq 1 - P_{D_0}\{\tau^\varepsilon \leq T\} = \min_{x_0 \in [D_0]} \text{Prob}\{\tau^\varepsilon_{x_0} > T\}. \quad (2.4)$$

These probabilities play a crucial role in the development that follows. They are characterized next.

Theorem 2.1. Assume that (A, C) is disturbable, i.e. rank $[C\,AC \cdots A^{n-1}C] = n$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln P_{D_0}\{\tau^\varepsilon \leq T\} = -\min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \leq t \leq T}} \tfrac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) = -\varphi(D_0), \quad (2.5)$$

where

$$\dot{X}(t) = AX(t) + X(t)A^T + CC^T, \quad X(0) = 0. \quad (2.6)$$

Proof. See Appendix 1.

Theorem 2.1 states, in particular, that $P_{D_0}\{\tau^\varepsilon \leq T\}$ is closely related to $e^{-\varphi(D_0)/\varepsilon^2}$, i.e. for any $\delta > 0$, there exists an $\varepsilon_0 > 0$ such that, for all $0 < \varepsilon \leq \varepsilon_0$,

$$e^{-[(\varphi(D_0)+\delta)/\varepsilon^2]} \leq P_{D_0}\{\tau^\varepsilon \leq T\} \leq e^{-[(\varphi(D_0)-\delta)/\varepsilon^2]}. \quad (2.7)$$

Note that $\varphi(D_0)$, *the logarithmic first passage probability,* is a coordinate-free characterization of system's performance. This follows from Theorem 2.1 and the fact that $\{x \in \mathbf{R}^n : x \in D\} = \{Qx \in \mathbf{R}^n : Q^{-1}x \in D\}$ for any invertible matrix Q. This property will be used in Section 4 to establish the upper bound of the achievable logarithmic first passage probability for system (1.1).

3. (D, T)-STABILITY

If a stochastic system has an equilibrium point, its stability can be characterized by the usual notion of Liapunov stability modified in an appropriate stochastic sense (Kushner, 1967; Khasminsky, 1969). If the system does not have equilibria, as is the case for (2.1), the Liapunov stability does not apply. In this situation, the notion of first passage time could be used to describe its "stability" features. One way to accomplish this is as follows.

Definition 3.1. System (2.1) is said to be (D, T)-stable with probability $0<p<1$ if there exists an open set $D_0 \subset D$ such that

$$P_{D_0}\{\tau^\varepsilon > T\} > p, \qquad (3.1)$$

or, equivalently,

$$P_{D_0}\{\tau^\varepsilon \le T\} \le 1-p, \qquad (3.2)$$

where $P_{D_0}\{\tau^\varepsilon > T\}$ and $P_{D_0}\{\tau^\varepsilon \le T\}$ are defined in (2.4) and (2.3), respectively.

In this definition, set D may be interpreted as a safe operating region, T as a desired operating time, and D_0 as an initial, "lock in", set.

Sufficient tests for (D, T)-stability and instability are given below.

Theorem 3.1. Assume that A is Hurwitz and (A, C) is disturbable. Let $M>0$ be a matrix such that

$$A^TM + MA < 0,$$

and assume there exists a positive number R_0 such that

$$\Gamma \triangleq \min_{y\in\partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} > 0,$$

where $\lambda_{\min}(M)$ is the smallest eigenvalue of M and $\|\cdot\|$ is the Euclidean norm of a vector. Then for any $\delta>0$ there exists an $\varepsilon_0>0$ such that if $0<\varepsilon\le\varepsilon_0$ system (2.1) is (D, T)-stable with probability $1-e^{-[(\alpha-\delta)/\varepsilon^2]}$ where

$$\alpha < \frac{\Gamma^2}{2\lambda_{\max}(X(T))},$$

$X(t)$ is the covariance matrix defined by (2.6) and $\lambda_{\max}(X(T))$ is the largest eigenvalue of $X(T)$. The corresponding initial set D_0 in this case is:

$$D_0 = \{x_0 \in \mathbf{R}^n : x_0^TMx_0 < R_0^2\}.$$

Proof. See Appendix 2.

Theorem 3.2. Assume (A, C) is disturbable. Then for any $\delta>0$ there exists an $\varepsilon_0>0$ such that if $0<\varepsilon\le\varepsilon_0$ system (2.1) is not (D, T)-stable with probability $p = 1-e^{-[(\alpha+\delta/\varepsilon^2]}$ if

$$\frac{\max_{y\in\partial D}\|y\|^2}{2\lambda_{\max}(X(T))} < \alpha.$$

Proof. See Appendix 2.

Theorems 3.1 and 3.2 provide the lower and upper bound for the residence probability in the domain:

$$1-\exp\left\{-\frac{\left[\min_{y\in\partial D}\|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}}\right]^2}{2\lambda_{\max}(X(T))\varepsilon^2} + \frac{\delta}{\varepsilon^2}\right\}$$
$$< P_{D_0}\{\tau^\varepsilon > T\}$$
$$< 1-\exp\left\{-\frac{\max_{y\in\partial D}\|y\|^2}{2\lambda_{\max}(X(T))\varepsilon^2} - \frac{\delta}{\varepsilon^2}\right\}. \qquad (3.3)$$

4. RESIDENCE PROBABILITY CONTROLLABILITY AND (D, T)-STABILIZABILITY

Consider again system (1.1) with control (1.2). As it follows from Theorem 2.1, if $(A+BK, C)$ is disturbable,

$$\lim_{\varepsilon\to0} \varepsilon^2 \ln P_{D_0}\{\tau^\varepsilon(K) \le T\} = -\varphi(D_0, K),$$

where $\varphi(D_0, K)$ is the logarithmic first passage probability defined by

$$\varphi(D_0, K) = \min_{x_0\in[D_0]} \min_{\substack{y\in\partial D\\ 0\le t\le T}} \tfrac{1}{2}(y - e^{(A+BK)t}x_0)^T \times X^{-1}(t, K)(y - e^{(A+BK)t}x_0), \qquad (4.1)$$

$$\dot{X}(t, K) = (A+BK)X(t, K) + X(t, K) \times (A+BK)^T + CC^T, \quad X(0, K) = 0.$$

Note that if $(A, [BC])$($[BC]$ is the matrix whose columns are the columns of B and C) is disturbable, $(A+BK, C)$ is disturbable for almost any K (Davison and Wang, 1973). We would like to choose the feedback law (1.2) so that $\varphi(D_0, K)$ is as large as desired. This may or may not be possible. To characterize the various situations we introduce

Definition 4.1. System (1.1) is said to be *strongly residence probability controllable* (srp-controllable) if for any (D, T) and $\alpha>0$ there exists $u = Kx$ and $D_0\subset D$ such that $\varphi(D_0, K) > \alpha$. Otherwise the system is *weakly residence probability controllable.*

Clearly the srp-controllability is closely related to the property of (D, T)-stabilizability:

Definition 4.2. System (1.1) is (D, T)-stabilizable if for any (D, T) and $0<p<1$ there exists $u = Kx$ and $D_0 \subset D$ such that

$$P_{D_0}\{\tau^\varepsilon > T\} > p.$$

Below, the class of srp-controllable systems is characterized.

Theorem 4.1. Under the assumption of (A, C) disturbability, (1.1) is srp-controllable if and only if $\operatorname{Im} C \subseteq \operatorname{Im} B$.

Proof. See Appendix 3.

When $\operatorname{Im} C \nsubseteq \operatorname{Im} B$, there exists an upper bound on the achievable $\varphi(D_0, K)$. This bound is analyzed next.

Consider again (1.1) and assume that B has a full rank. Then there exists a similarity transformation $x = Q\hat{x}$ that transfers (1.1) to the form

$$d\hat{x} = (\hat{A}\hat{x} + \hat{B}u)\,dt + \varepsilon\hat{C}\,dw, \tag{4.2}$$

$$Q^{-1}AQ \triangleq \hat{A} = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \quad Q^{-1}B = \hat{B} = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

$$Q^{-1}C = \hat{C} = \begin{bmatrix} \hat{C}_1 \\ \hat{C}_2 \end{bmatrix}.$$

Since $\varphi(D_0, K)$ is a coordinate free characterization of system performance, the bounds will be established in terms of the realization (4.2).

Assume (A, B) is controllable and choose

$$u = \hat{K}_\rho\hat{x}, \quad \hat{K}_\rho = -\frac{1}{\rho}\hat{B}^T P_\rho, \ \rho > 0, \tag{4.3}$$

where P_ρ is the positive definite solution of

$$\hat{A}^T P_\rho + P_\rho\hat{A} + I - \frac{1}{\rho}P_\rho\hat{B}\hat{B}^T P_\rho = 0. \tag{4.4}$$

Let $P_{22}^*(t)$ be a positive definite solution of

$$-\dot{P}_{22}^*(t) = \hat{A}_{22}^T P_{22}^*(t) + P_{22}^*(t)\hat{A}_{22} + I - P_{22}^*(t)\hat{A}_{21}\hat{A}_{21}^T P_{22}^*(t), \ P_{22}^*(T) = 0, \tag{4.5}$$

and P_{22} be the positive definite solution of

$$\hat{A}_{22}^T P_{22} + P_{22}\hat{A}_{22} + I - P_{22}\hat{A}_{21}\hat{A}_{21}^T P_{22} = 0, \tag{4.6}$$

i.e. $P_{22} = \lim_{T\to\infty} P_{22}^*(0)$.

Finally let $M > 0$ and $\rho_0 > 0$ be a matrix and a number such that

$$(\hat{A} + \hat{B}\hat{K}_\rho)^T M + M(\hat{A} + \hat{B}\hat{K}_\rho) < 0, \tag{4.7}$$

for all $0 < \rho \le \rho_0$ and let $R_0 > 0$ be number such that

$$\Gamma = \min_{y\in\partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} > 0. \tag{4.8}$$

Let $\hat{\mathcal{K}} = \{\hat{K} \in \mathbf{R}^{m\times n} : \hat{A} + \hat{B}\hat{K}$ is Hurwitz$\}$ and let $\hat{\varphi}(\hat{D}_0, \hat{K})$ be the logarithmic first passage probability of (4.2) from $\hat{D} = \{\hat{x} \in \mathbf{R}^n : Q^{-1}\hat{x} \in D\}$.

Theorem 4.2. Under the assumption of controllability of (A, B) and disturbability of (A, C) the maximal achievable logarithmic first passage probability is bounded as follows:

$$\frac{\Gamma^2}{2\lambda^{**}} \le \sup_{\hat{K}\in\hat{\mathcal{K}}} \hat{\varphi}(\hat{D}_0, \hat{K}) \le \frac{n\max_{y\in\partial\hat{D}} \|y\|^2}{2\lambda^*}, \tag{4.9}$$

where

$$\lambda^* = \operatorname{Tr} \hat{C}_2 P_{22}^*(0)\hat{C}_2, \tag{4.10}$$

$$\lambda^{**} = \operatorname{Tr} \hat{C}_2^T P_{22}\hat{C}_2. \tag{4.11}$$

Proof. See Appendix 3.

Note that in the srp-controllability case $\hat{C}_2 = 0$, $\lambda^* = \lambda^{**} = 0$ and, therefore,

$$\sup_{\hat{K}\in\hat{\mathcal{K}}} \hat{\varphi}(\hat{D}_0, \hat{K}) = \infty.$$

To illustrate the bounds of Theorem 4.3, consider an example of the roll attitude control problem in a missile disturbed by a random torque (Hotz and Skelton, 1986). The dynamics of the system are described as

$$\begin{bmatrix} \dot{\delta} \\ \dot{\omega} \\ \dot{\varphi} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 10 & -1 & 0 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} \delta \\ \omega \\ \varphi \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}u + \varepsilon\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\dot{w}, \tag{4.12}$$

where δ is the aileron deflection, ω is the roll angular velocity, φ is the roll angle, u is control of aileron actuators and $\dot{w}$ is white noise. Note that (4.12) is in the form (4.2) with $\hat{C}_2 \ne 0$, i.e. the system under consideration is wrp-controllable and $\varphi(D_0, K)$ is bounded. To evaluate this bound, assume, for simplicity, that D is a ball with radius R, $T = 3$ and calculate

$$\lambda^* \simeq 0.1, \quad \lambda^{**} = 0.1.$$

Choosing M as

$$\begin{bmatrix} 0.25 & 0.25 & 0.25 \\ 0.25 & 0.35 & 0.35 \\ 0.25 & 0.35 & 1.35 \end{bmatrix}, \tag{4.13}$$

and verifying that (4.8) holds, we finally obtain:

$$5\left(R - \frac{R_0}{\sqrt{0.042}}\right)^2 \le \sup_{K\in\mathcal{K}} \varphi(D_0, K) \le 15R^2.$$

Thus, there is no linear controller for missile (4.12) that keeps the states in the ball of radius R during interval T with probability $p > 1 - \exp\left\{-\frac{15 + \delta}{\varepsilon^2}\right\}$. On the other hand, there exists a linear state feedback that accomplishes this task with probability $p \le 1 - \exp\left\{-\frac{5}{\varepsilon^2}\left(R - \frac{R_0}{\sqrt{0.042}}\right)^2 + \frac{\delta}{\varepsilon^2}\right\}$, provided that at $t = 0$ the states are locked into the initial set $D_0 = \{x_0 : x_0^T M x_0 \le R_0^2\}$, where M is given by (4.13).

5. CONCLUSIONS

(1) The residence probability control gives a stronger reformulation of the aiming control problem than the residence time control. However, the resulting control problem is also more complex: the performance depends on the size of the initial, "lock in", domain and on the operating period.

(2) (D, T)-stability with probability p is a useful tool for characterization of the performance of stochastic systems with no equilibrium points.

(3) The logarithmic first passage probability of a controlled linear system with additive white noise can be modified in any desired manner, for instance, made as close to ∞ as desired, if and only if the range space of the noise matrix is included in the range space of the control matrix. Otherwise, the achievable logarithmic first passage probability is bounded away from ∞ and estimates of this bound are characterized herein.

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APPENDIX 1: PROOFS FOR SECTION 2

Proof of Theorem 2.1.

Consider, in conjunction with (2.1), the deterministic system

$$\dot{y} = Ay + Cu, \; y(0) = x_0, \tag{A1.1}$$

If ϕ is a trajectory of (A1.1) corresponding to an input function $u \in L_2[0, T, \mathbf{R}^m]$ define

$$S_{0T}(\phi) \triangleq S_{0T}(u) = \frac{1}{2}\int_0^T u^T(t)u(t)\,dt. \tag{A1.2}$$

For the remaining continuous functions from $[0, T]$ into $\mathbf{R}^m$ let $S_{0T}(\phi) = \infty$. Then $S_{0T}(\phi)$ is the action functional for (2.1) (Zabczyk, 1985).

Define

$$\tau_{x_0}^{\varepsilon} \triangleq \min\{t : x(t) \in \partial d\}, \tag{A1.3}$$

where $D \subset \mathbf{R}^n$ is an open bounded set in $\mathbf{R}^n$ with 0 in its interior and smooth boundary ∂D and $x(t)$ is the trajectory of (2.1) with $x_0 \in D_0$. Introduce also

$$H_D(T, x_0) \triangleq \{\phi \in C_{0T}(\mathbf{R}^n) : \phi(0) = x_0 \in D, \; \phi(s) \notin D \text{ for some } s \in [0, T]\}. \tag{A1.4}$$

$$U_D(T, x_0) \triangleq \{u \in C_{0T}(\mathbf{R}^n) : \phi \in H_D(T, x_0), \; \dot{\phi} = A\phi + Cu, \; \phi(0) = x_0\}. \tag{A1.5}$$

It follows from Theorems 1.1 and 1.2 (Chapter 4) of Freidlin and Wentzell (1984) that if

$$\inf_{u \in U_D(T, x_0)} S_{0T}(u) = \inf_{u \in [U_D(T, x_0)]} S_{0T}(u), \tag{A1.6}$$

then uniformly with respect to all $x_0 \in \mathbf{R}^n$,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln \operatorname{Prob}\{\tau_{x_0}^{\varepsilon} \le T\} = -\min_{u \in U_D(T, x_0)} S_{0T}(u). \tag{A1.7}$$

However, since $S_{0T}(u)$ is differentiable with respect to u (Kirk, 1970), it follows that (A1.6) and consequently (A1.7) is true.

To solve the minimization problem of (A1.6) we observe that, as it follows from Freidlin and Wentzell (1984),

$$\min_{u \in [U_D(T, x_0)]} S_{0T}(u) = \min_{\substack{y \in \partial D \\ 0 \le t \le T}} V(t, x_0, y), \tag{A1.8}$$

where

$$V(t, x_0, y) = \min_u \frac{1}{2}\int_0^t u^T(s)u(s)\,ds, \tag{A1.9}$$

$$\dot{\phi} = A\phi + Cu, \quad \phi(0) = x_0, \quad \phi(t) = y. \tag{A1.10}$$

Problem (A1.8)–(A1.10) can be solved using a standard variational approach. A straightforward calculation shows that

$$V(t, x_0, y) = \tfrac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0),$$

where

$$\dot{X} = AX + XA^T + CC^T, \quad X(0) = 0.$$

Thus, according to (A1.6) and (A1.8), it follows from (A1.7) that

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln \text{Prob}\,\{\tau^{\varepsilon}_{x_0} \le T\}$$

$$= \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0).$$

To complete the proof consider

$$P_{D_0}\{\tau^{\varepsilon} \le T\} = \max_{x_0 \in [D_0]} \text{Prob}\,\{\tau^{\varepsilon}_{x_0} \le T\},$$

where $D_0 \subset D$ is an open set containing 0 with a smooth boundary ∂D_0. Taking into account the continuity of ln Prob $\{\tau^{\varepsilon}_{x_0} \le T\}$, the uniformity of the limit in (A1.7), and the compactness of $[D_0]$,

$$\lim_{\varepsilon \to 0} \varepsilon^2 \ln P_{D_0}\{\tau^{\varepsilon} \le T\}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^2 \ln \max_{x_0 \in [D_0]} \text{Prob}\,\{\tau^{\varepsilon}_{x_0} \le T\}$$

$$= \lim_{\varepsilon \to 0} \varepsilon^2 \max_{x_0 \in [D_0]} \ln \text{Prob}\,\{\tau^{\varepsilon}_{x_0} \le T\}$$

$$= \max_{x_0 \in [D_0]} \lim_{\varepsilon \to 0} \varepsilon^2 \ln \text{Prob}\,\{\tau^{\varepsilon}_{x_0} \le T\}$$

$$= \max_{x_0 \in [D_0]} \left[- \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0) \right]$$

$$= - \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2}(y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0).$$

Proof of Lemma 2.1. Since $X(t) = Q\hat{X}(t)Q^T$,

$$(\hat{y} - e^{\hat{A}t}\hat{x}_0)^T \hat{X}^{-1}(t)(\hat{y} - e^{\hat{A}t}\hat{x}_0)$$

$$= (Q^{-1}y - Q^{-1}e^{At}QQ^{-1}x_0)^T Q^T X^{-1}(t) \times Q(Q^{-1}y - Q^{-1}e^{At}QQ^{-1}x_0)$$

$$= (y - e^{At}x_0)^T X^{-1}(t)(y - e^{At}x_0).$$

Therefore,

$$\hat{\varphi}(\hat{D}_0) = \min_{\hat{x}_0 \in [\hat{D}_0]} \min_{\substack{\hat{y} \in \partial \hat{D} \\ 0 \le t \le T}} \tfrac{1}{2}(\hat{y} - e^{\hat{A}t}\hat{x}_0)^T \hat{X}^{-1}(t)(\hat{y} - e^{\hat{A}t}\hat{x}_0)$$

$$= \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2}(y - e^{At}x_0)^T X^{-1}(t) \times (y - e^{At}x_0) = \varphi(D_0).$$

APPENDIX 2: PROOFS FOR SECTION 3

Proof of Theorem 3.1. Observe that

$$\varphi(D_0) \ge \min_{x_0 \in [D_0]} \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2}\lambda_{\min}(X^{-1}(t))\,\|y - e^{At}x_0\|^2$$

$$\ge \frac{\min\limits_{x_0 \in [D_0]} \min\limits_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{At}x_0\|^2}{2\lambda_{\max}(X(T))}.$$

Since A is Hurwitz and there exists $M > 0$ such that

$$A^T M + MA < 0.$$

$D_0 = \{x_0 \in \mathbf{R}^n : x_0^T M x_0 \le R_0^2\}$ is an invariant set for (2.1) with $\varepsilon = 0$. Since

$$\Gamma \triangleq \min_{y \in \partial D} \|y\| - \frac{R_0}{\sqrt{\lambda_{\min}(M)}} > 0,$$

this invariant set is contained in D. Therefore,

$$\varphi(D_0) \ge \frac{\min\limits_{x_0 \in [D_0]} \min\limits_{\substack{y \in \partial D \\ 0 \le t \le T}} \|y - e^{At}x_0\|^2}{2\lambda_{\max}(X(T))}$$

$$\ge \frac{\left(\min\limits_{y \in \partial D} \|y\| - \dfrac{R_0}{\sqrt{\lambda_{\min}(M)}}\right)^2}{2\lambda_{\max}(X(T))} \triangleq \Phi_l. \qquad \text{(A2.1)}$$

(A2.1) and (2.7) imply that (2.1) is (D, T)-stable with probability $1 - e^{-[(\alpha - \delta)/\varepsilon^2]}$ where

$$0 < \alpha < \Phi_1.$$

Proof of Theorem 3.2. Let $B = \{x \in \mathbf{R}^n : \|x\| \le R\}$ and $R = \max_{y \in \partial D} \|y\|$. Then

$$\varphi(D_0) \le \min_{\substack{y \in \partial D \\ 0 \le t \le T}} \tfrac{1}{2} y^T X^{-1}(t) y$$

$$\le \min_{\substack{y \in \partial B \\ 0 \le t \le T}} \tfrac{1}{2} y^T X^{-1}(t) y$$

$$= \min_{0 \le t \le T} \tfrac{1}{2}\lambda_{\min}(X^{-1}(t))R^2$$

$$= \frac{R^2}{2\lambda_{\max}(X(T))} \triangleq \Phi_2, \qquad \text{(A2.2)}$$

(A2.2) and (2.7) imply that (2.1) is not (D, T)-stable with probability $1 - e^{-[(\alpha + \delta)/\varepsilon^2]}$, where $\alpha > \Phi_2$.

APPENDIX 3: PROOFS FOR SECTION 4

Proof of Theorem 4.1. *Sufficiency*. If $\text{Im}\, C \subseteq \text{Im}\, B$, the disturbability of (A, C) guarantees the controllability of (A, B). Then, as it was shown in Meerkov and Runolfsson (1988), there exist a sequence $\{K_\alpha\}$ such that $(A + BK_\alpha)$ is Hurwitz for all $\alpha \in [1, \infty)$, a set D_0 which is invariant set for $\dot{x} = (A + BK_\alpha)x$ for all α and, in addition,

$$\lim_{\alpha \to \infty} X_\infty(K_\alpha) = 0,$$

where

$$(A + BK_\alpha)X_\infty(K_\alpha) + X_\infty(K_\alpha)(A + BK_\alpha)^T + CC^T = 0.$$

Since $X(T, K_\alpha) \le X_\infty(K_\alpha)$, $\forall T \ge 0$

$$\lim_{\alpha \to \infty} X(T, K_\alpha) = 0, \quad \forall T \ge 0,$$

where

$$\dot{X}(t, K_\alpha) = (A + BK_\alpha)X(t, K_\alpha) + X(K_\alpha)(A + BK_\alpha)^T + CC^T, \; X(0, K_\alpha) = 0.$$

Therefore, by using Theorem 2.1,

$$\lim_{\alpha \to \infty} \varphi(D_0, K_\alpha) = \infty,$$

which implies that (1.1) is srp-controllable.

Necessity. Assume that (1.1) is srp-controllable. Then $\sup_K \varphi(D_0, K) = \infty$ which implies that $\inf_K \lambda_{\max}(X(T, K)) = 0$. In particular, this is true for $T = \infty$. Finally, it was shown in Meerkov and Runolfsson (1988) that $\inf_K \lambda_{\max}(X_\infty(K)) = 0$ implies $\text{Im}\, C \subseteq \text{Im}\, B$. Q.E.D.

Proof of Theorem 4.2. It follows from (A2.2) and the positive definiteness of $X(T, \hat{K})$ that

$$\hat{\varphi}(\hat{D}_0, \hat{K}) \le \frac{R^2}{2\lambda_{\max}(X(T, \hat{K}))} \le \frac{nR^2}{2\,\text{Tr}\, X(T, \hat{K})}, \qquad \text{(A3.1)}$$

where $R^2 = \max_{y \in \partial \hat{D}} \|y\|^2$. Let $\hat{\mathcal{K}}[0, T]$ be the set of time-varying feedback gains $\hat{K}(t)$, $t \in [0, T]$, and $\hat{\mathcal{K}}$ be the set of constant feedback gains $\hat{K}$. Then clearly

$$\inf_{\hat{K} \in \hat{\mathcal{K}}[0, T]} \text{Tr}\, X(T, \hat{K}) \le \inf_{\hat{K} \in \hat{\mathcal{K}}} \text{Tr}\, X(T, \hat{K}). \qquad \text{(A3.2)}$$

Combining (A3.1) and (A3.2) gives

$$\sup_{\hat{K}\in\hat{\mathcal{K}}} \hat{\varphi}(D_0, \hat{K}) \le \frac{nR^2}{2 \inf\limits_{\hat{K}\in\hat{\mathcal{K}}[0,T]} \operatorname{Tr} X(T, \hat{K})}. \tag{A3.3}$$

Finally, it was shown in O'Malley and Jameson (1975) that

$$\inf_{\hat{K}\in\hat{\mathcal{K}}[0,T]} \operatorname{Tr} X(T, \hat{K}) = \operatorname{Tr} \hat{C}_2^T P_{22}^*(0)\hat{C}_2, \tag{A3.4}$$

where $P_{22}^*(t)$ is the positive definite solution of (4.5). Substituting (A3.4) into (A3.3) gives the upper bound.

To obtain the lower bound note that as in (A2.1)

$$\hat{\varphi}(\hat{D}_0, \hat{K}) \ge \frac{\min\limits_{x_0\in[\hat{D}_0]} \min\limits_{\substack{y\in\partial\hat{D}\\ 0\le t\le T}} \|y - e^{(\hat{A}+\hat{B}\hat{K})t}x_0\|^2}{2\lambda_{\max}(X(T, \hat{K}))}. \tag{A3.5}$$

Moreover, since

$$\lambda_{\max}(X(T, \hat{K})) \le \lambda_{\max}(X(\infty, \hat{K})) \le \operatorname{Tr} X(\infty, \hat{K}), \tag{A3.6}$$

it follows that

$$\hat{\varphi}(\hat{D}_0, \hat{K}) \ge \frac{\min\limits_{x_0\in[\hat{D}_0]} \min\limits_{\substack{y\in\partial\hat{D}\\ 0\le t\le T}} \|y - e^{(\hat{A}+\hat{B}\hat{K})t}x_0\|^2}{2 \operatorname{Tr} X(\infty, \hat{K})}. \tag{A3.7}$$

Let

$$P_\rho = \begin{bmatrix} P_{\rho 11} & P_{\rho 12} \\ P_{\rho 12}^T & P_{\rho 22} \end{bmatrix},$$

be the positive definite solution of (4.4). It is shown in Kwakernaak and Sivan (1972) that

$$\inf_{\hat{K}\in\hat{\mathcal{K}}} \operatorname{Tr} X(\infty, \hat{K}) = \lim_{\rho\to 0} \operatorname{Tr} X(\infty, \hat{K}_\rho) = \lim_{\rho\to 0} \operatorname{Tr} P_\rho \hat{C}\hat{C}^T = \operatorname{Tr} P_0 \hat{C}\hat{C}^T. \tag{A3.8}$$

Furthermore, the limit

$$\lim_{\rho\to 0} \frac{1}{\rho} P_\rho \hat{B}\hat{B}^T P_\rho = \lim_{\rho\to 0} \frac{1}{\rho} \begin{bmatrix} P_{11\rho}^2 & P_{11\rho}P_{12\rho} \\ P_{12\rho}^T P_{11\rho} & P_{12\rho}^T P_{12\rho} \end{bmatrix}, \tag{A3.9}$$

exists, and thus,

$$P_0 = \lim_{\rho\to 0} P_\rho = \lim_{\rho\to 0} \begin{bmatrix} P_{11\rho} & P_{12\rho} \\ P_{12\rho}^T & P_{22\rho} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & P_{22} \end{bmatrix}. \tag{A3.10}$$

A straightforward calculation shows that P_{22} satisfies (4.6). Finally, by (4.5) and (4.6), the proof of Theorem 3.1 and (A3.5)

$$\begin{aligned}
\sup_{\hat{K}\in\hat{\mathcal{K}}} \hat{\varphi}(\hat{D}_0, \hat{K}) &\ge \sup_{\hat{K}\in\hat{\mathcal{K}}} \frac{\min\limits_{x_0\in[\hat{D}_0]} \min\limits_{\substack{y\in\partial\hat{D}\\ 0\le t\le T}} \|y - e^{(\hat{A}+\hat{B}\hat{K})t}x_0\|^2}{2 \operatorname{Tr} X(\infty, \hat{K})} \\
&\ge \lim_{\rho\to 0} \frac{\left(R - \frac{R_0}{\sqrt{\lambda_{\min}(M)}}\right)^2}{2 \operatorname{Tr} X(\infty, \hat{K}_\rho)} \\
&= \frac{\left(R - \frac{R_0}{\sqrt{\lambda_{\min}(M)}}\right)^2}{2 \lim\limits_{\rho\to 0} \operatorname{Tr} P_\rho \hat{C}\hat{C}^T} \\
&= \frac{\left(R - \frac{R_0}{\sqrt{\lambda_{\min}(M)}}\right)^2}{2 \operatorname{Tr} \hat{C}_2^T P_{22} \hat{C}_2}.
\end{aligned} \tag{A3.11}$$

The proof is complete. Q.E.D.