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Existence of Particle-like Solutions of the Einstein–Yang/Mills Equations

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In the paper in Ref. [2], we struggled to prove the existence of a bounded, smooth solution to the Einstein–Yang/Mills equations with $SU(2)$ gauge group. Bartnik and McKinnon in [1] derived these equations, and obtained numerical evidence for the existence of such solutions. The equations reduce to a system of two ordinary differential equations for the unknown functions $A(r)$ and $w(r)$ in the region $r>0$ (cf. [1, 2]),

$$rA' + (1 + 2w'^2)A = 1 - \frac{(1-w^2)}{r^2}, \tag{1}$$

$$r^2Aw'' + \left[r(1-A) - \frac{(1-w^2)^2}{r} \right] w' + w(1-w^2) = 0, \tag{2}$$

with initial conditions

$$A(0) = 1, \qquad w(0) = 1, \qquad w'(0) = 0. \tag{3}$$

The solutions of (1)–(3) are parametrized by $\lambda = -w''(0)$. Furthermore, for any compact λ-interval, there is an $R > 0$ such that the one-parameter family of smooth solutions $(A(r, \lambda), w(r, \lambda))$ is defined for $r \leqslant R$, and the solution depends continuously on λ. The problem is to show that for some λ

$$\lim_{r \to \infty} (w(r, \lambda), w'(r, \lambda)) = (-1, 0). \tag{4}$$

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One difficulty in dealing with these equations is that they are highly nonlinear, and they become singular at $\bar{r}$ if $A(\bar{r})=0$. The purpose of this note is to show how the methods which we have recently developed in [3], (where we prove the existence of infinitely many λ for which (4) holds), allow us to simplify considerably the proof of the result in [2].

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In [2], we have shown that for λ near 0, the solution of (1)–(3) satisfies the following:

$$\text{there is an "exit-time" } r_e(\lambda) \text{ such that}$$

$$\begin{array}{lll} \text{(i)} & w(r_e(\lambda), \lambda) = -1 & \\ \text{(ii)} & w(r, \lambda) < 1 \quad \text{if} \quad 0 < r < r_e(\lambda) & \quad (5) \\ \text{(iii)} & w'(r, \lambda) < 0, A(r, \lambda) > 0 \quad \text{for} \quad 0 < r \leqslant r_e(\lambda). & \end{array}$$

Moreover, if $\lambda \geqslant 2$, we proved that the λ-orbit "crashes" in the sense that $A(\bar{r}, \lambda)=0$ for some finite $\bar{r}$ (depending on λ), and $0<w(\bar{r}, \lambda)<1$.

Now define $\bar{\lambda}$ to be the supremum of those λ which satisfy (5); clearly $\bar{\lambda}<2$. In [2] we proved that the $\bar{\lambda}$-orbit is a connecting orbit; i.e., satisfies (4). This was done by eliminating all alternative behavior for this orbit. Namely, if $\Gamma=\{(w, w', A, r): w^2 \leqslant 1, w' \leqslant 0\}$, then by an easy transversality argument, the $\bar{\lambda}$-orbit cannot exit Γ through $w=-1$, $w'<0$, nor can it exit Γ through $w=0$, $w^2<1$, for otherwise orbits with smaller λ would also exit Γ in the same manner. We also showed in [2] that the $\bar{\lambda}$-orbit cannot stay in Γ for all $r>0$ without satisfying (4). Hence we only had to rule out crashing for the $\bar{\lambda}$-orbit.

In order to rule out crash, we considered several cases. Thus assume that $A(\bar{r}, \bar{\lambda})=0$, and $A(r, \bar{\lambda})>0$ for $r<\bar{r}$, and let $\bar{w}=w(\bar{r}, \bar{\lambda}) \equiv \lim_{r \nearrow \bar{r}} w(r, \bar{\lambda})$; the three cases for the $\bar{\lambda}$-orbit are $\bar{w}>0$, $\bar{w}=0$, and $\bar{w}<0$. The first case, $\bar{w}>0$ was ruled out by [2, Proposition 5.8]. The two other cases were quite difficult, and involved a complicated complex-plane argument.

In this paper we show how to avoid these difficulties via the methods which we have developed in [3]. The idea is to find a point P in $\mathbb{R}^4$, where the $\bar{\lambda}$-orbit would be if it did not crash, $(P=\lim_{\lambda \nearrow \bar{\lambda}}(w(\tilde{r}, \lambda), w'(\tilde{r}, \lambda), A(\tilde{r}, \lambda), \tilde{r}))$, and then work backwards in r; i.e., we show that the orbit through P for $r<\tilde{r}$ arrives at the "starting point" $(w, w', A, r)=(1, 0, 1, 0)$.

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We shall show now that the $\bar{\lambda}$-orbit does not crash. For this we note that we have proved in [3], that there are numbers $\tau > 0$, $R_1 > 0$, and $w_1, -1 < w_1 < 0$, satisfying for all λ, $0 \leqslant \lambda \leqslant 2$, the following:

$$\text{if either } r \geqslant R_1 \text{ or } -1 \leqslant w(r, \lambda) \leqslant w_1, \text{ then}$$
$$-\tau \leqslant w'(r, \lambda) < 0, \text{ and } A(r, \lambda) > 0. \tag{6}$$

Next, choose w_2 such that $-1 < w_2 < w_1$, and for $\lambda < \bar{\lambda}$, define $r_{w_2}(\lambda)$ by $w(r_{w_2}(\lambda), \lambda) = w_2$. We now consider two cases:

(i) there is a sequence $\lambda_n \nearrow \bar{\lambda}$ for which the corresponding "times" $\{r_{w_2}(\lambda_n)\}$ are bounded, or

(ii) no such sequence as in (i) exists; i.e., $\underline{\lim}\ r_{w_2}(\lambda_n) = \infty$.

Suppose first that we are in case (i). We consider the points P_n in $\mathbb{R}^4$ defined by

$$P_n = (w_2, w'(r_{w_2}(\lambda_n), \lambda_n), A(r_{w_2}(\lambda_n), \lambda_n), r_{w_2}(\lambda_n)).$$

Because we are in case (i), there exists a $B > 0$ such that $R \leqslant r_{w_2}(\lambda_n) \leqslant B$, where $[0, R]$ is the interval of local existence discussed in Section 1 above. Using (6), we have $-\tau < w'(r_{w_2}(\lambda_n), \lambda_n) < 0$, and from [3, Proposition 3.7], there exists an $\alpha > 0$ such that $1 \geqslant A(r_{w_2}(\lambda_n), \lambda_n) \geqslant \alpha$. It follows that the sequence $\{P_n\}$ has a limit point $P = (w_2, \tilde{w}', \tilde{A}, \tilde{r})$, where $-\tau \leqslant \tilde{w}' \leqslant 0$, $\alpha \leqslant \tilde{A} \leqslant 1$, and $R \leqslant \tilde{r} \leqslant B$.

Now consider the backwards orbit from P; i.e., the solution $(w(r), w'(r), A(r), r)$ of (1)–(2), with $(w(\tilde{r}), w'(\tilde{r}), A(\tilde{r}), \tilde{r})) = (w_2, \tilde{w}', \tilde{A}, \tilde{r})$, defined for $0 < r < \tilde{r}$. We claim that this orbit cannot crash in the region $\mathscr{R} = \{(w, w') : -1 \leqslant w \leqslant 0,\ w' \leqslant 0\}$, and that it meets the line $w = 0$ at a point where $w' < 0$. In fact, if there were a crash in $\mathscr{R}$ at some $r_1 < \tilde{r}$, then defining $v = Aw'$, we would have $v(r_1) = 0$, and $-1 < w(r_1) \leqslant 0$, $w'(r_1) \leqslant 0$ (see [2, Proposition 3.3]). Since $v' = -2w'^2 v/r - (1 - w^2)\, w/r^2$, the mean-value theorem yields the contradiction $0 > (r_1 - \tilde{r})\, v'(\xi) = v(r_1) - v(\tilde{r}) = -v(\tilde{r}) \geqslant 0$. Therefore this backward orbit cannot crash in $\mathscr{R}$. It cannot cross the line $w' = 0$ at a point where $w < 0$, since at such points, $w'' > 0$ (as follows from (2)), nor can it go to the point $(0, 0)$ in finite r. We next show that the backward orbit through P can not stay in $\mathscr{R}$ for all $r > 0$. Assume the contrary. We have by definition, $\lim w(r, \lambda_n) = \tilde{w}$, so by continuous dependence of the solution on parameters, $\lim w(r, \lambda_n) = w(r)$ for $r < \tilde{r}$, as long as $w(r)$ does not crash. Choose $r' > 0$ such that $w(r', \lambda) > \frac{1}{2}$ for all λ, $0 \leqslant \lambda \leqslant 2$; (recall $w(0, \lambda) \equiv 1$). Then $\frac{1}{2} \leqslant \lim w = w(r')$, and this is a contradiction. Hence the orbit leaves $\mathscr{R}$ for some $r > r'$. Therefore the backwards orbit from P reaches the line $w=0$ at some $r_0<\bar{r}$, where $w'(r_0)<0$ and $A(r_0)>0$.

Now since $A(r_0)>0$, the solution $(w(r), w'(r), A(r), r)$ through the point $(0, w'(r_0), A(r_0), r_0)$ can be continued backwards in r, to a point $Q=(w(r_\varepsilon), w'(r_\varepsilon), A(r_\varepsilon), r_\varepsilon)$, where $w(r_\varepsilon)>0$, $w'(r_\varepsilon)<0$, and $A(r_\varepsilon)>0$, for some $r_\varepsilon<r_0$, r_ε near r_0. Thus we have traced the point P backwards into the region $\mathscr{S}=\{(w, w'): w>0, w'<0\}$. Now from [2, Proposition 5.14], the $\bar{\lambda}$-orbit (starting at $w=1$, $w'=0$, $A=1$, $r=0$) cannot crash in $\mathscr{S}$ in forward time. Since

$$w(r_\varepsilon, \bar{\lambda})=\lim w(r_\varepsilon, \lambda_n)=w(r_\varepsilon)$$

it follows that the $\bar{\lambda}$-orbit reaches Q without crashing and joins up with the backwards orbit from P. Thus the $\bar{\lambda}$-orbit arrives at P without crashing in forward time. In view of (6), this orbit cannot crash for $r>\bar{r}$. This completes the proof in case (i).

Suppose now that (ii) holds. Then we can find a sequence $\lambda_n \nearrow \bar{\lambda}$ such that $r_{w_2}(\lambda_n)>R_1+1$; i.e., $w(R+1, \lambda_n)>w_2>-1$. Define points P_n in $\mathbb{R}^4$ by

$$P_n=(w(R_1+1, \lambda_n), w'(R_1+1, \lambda_n), A(R_1+1, \lambda_n), R_1+1).$$

We have $1\geqslant w(R_1+1, \lambda_n)\geqslant w_2$, and from (6), $-\tau<w'(R+1, \lambda_n)\leqslant 0$. Furthermore, from [3, Proposition 3.9], there is an $\alpha>0$ such that $\alpha\leqslant A(R_1+1, \lambda_n)\leqslant 1$. Thus $\{P_n\}$ has a limit point $P=(\tilde{w}, \tilde{w}', \tilde{A}, R_1+1)$, where $1\geqslant\tilde{w}\geqslant w_2>-1$, $-\tau\leqslant\tilde{w}'\leqslant 0$, and $\alpha\leqslant\tilde{A}\leqslant 1$. The special case $P=(0, 0, \tilde{A}, R_2+1)$ is ruled out in [3]. Now if $\tilde{w}\geqslant 0$, then the same argument as given above in case (i) will work to show that the $\bar{\lambda}$-orbit does not crash. We may thus assume that $\tilde{w}<0$. If $\tilde{w}'=0$, then an easy transversality argument would show that for large n, the λ_n-orbits would cross the line $w'=0$ at points near $\tilde{w}$, and this is impossible. Thus we may assume that $\tilde{w}'<0$, and now the same argument as given in case (i) applies to show that the $\bar{\lambda}$-orbit cannot crash.

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