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Note

Diverse Homogeneous Sets

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A set $H \subseteq \omega$ is said to be diverse with respect to a partition Π of ω if at least two pieces of Π have an infinite intersection with H. A family of partitions of ω has the Ramsey property if, whenever $[\omega]^2$ is two-colored, some monochromatic set is diverse with respect to at least one partition in the family. We show that no countable collection of even infinite partitions of ω has the Ramsey property, but there always exists a collection of $\aleph_1$ finite partitions of ω with the Ramsey property.

1. INTRODUCTION

Let the set $[\omega]^2$ of two-element subsets of the set ω of natural numbers be colored with two colors. According to Ramsey's theorem [6], there is an infinite $H \subseteq \omega$ that is monochromatic in the sense that all of $[H]^2$ has a single color. We are interested in strengthening Ramsey's theorem to obtain monochromatic sets that are not only infinite but rather widely spread out in the following sense.

DEFINITION 1. A set $H \subseteq \omega$ is *diverse with respect to a partition Π of ω* if at least two pieces of Π have infinite intersection with H. H is *diverse with*

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respect to a family of partitions if it is diverse with respect to at least one partition in the family.

DEFINITION 2. A family of partitions of ω has the *Ramsey property* if, whenever $[\omega]^2$ is two-colored, some monochromatic set is diverse with respect to the family.

Note that, if a family of partitions is enlarged, then more sets become diverse and the Ramsey property becomes more likely. At one extreme, if a family consists of one partition Π, then the Ramsey property trivially fails; just color the pairs $\{x, y\}$ according to whether x and y are in the same piece of Π. At the other extreme, the family of all partitions of ω has the Ramsey property, because every infinite subset of ω is diverse with respect to some partition. The following two theorems (proved in Sections 2 and 3, respectively) specify the minimum possible cardinality of a family of partitions with the Ramsey property.

THEOREM 1. *No countable family of partitions of ω has the Ramsey property.*

THEOREM 2. *There is a family of $\aleph_1$ partitions of ω having the Ramsey property.*

It should also be noted that Theorems 1 and 2 are as strong as possible in terms of the number of pieces in the partitions considered. That is, the negative result (Theorem 1) applies to partitions of ω into even infinitely many pieces, while the positive result (Theorem 2) requires only partitions of ω into two pieces. (Here "as strong as possible," "even," and "only" refer to the easily proved fact that, if Π is a partition of ω into finitely many pieces, then there are two partitions Π' and Π'' of ω into infinitely many pieces such that every set diverse for Π is also diverse for at least one of Π' and Π''.)

2. PROOF OF THE NEGATIVE RESULT (THEOREM 1)

Suppose that $\mathbb{P} = \{\Pi_0, \Pi_1, ...\}$ is a family of partitions of ω. We want to produce a two-coloring of $[\omega]^2$ so that monochromatic sets fail to be diverse with respect to $\mathbb{P}$. We consider first the case where each of the partitions $\Pi_0, \Pi_1, \Pi_2, ...$ in the family $\mathbb{P}$ has just two pieces, say $\Pi_i = \{A_i, B_i\}$.

Associate to each $x \in \omega$ the infinite sequence $s(x)$ of zeros and ones whose ith term is zero if $x \in A_i$ and one if $x \in B_i$. Now, if $x, y \in \omega$ with $x < y$, color the pair $\{x, y\}$ red (resp. green) if $s(x)$ lexicographically precedes or equals (resp. follows) $s(y)$. We shall show that no set diverse with respect to $\mathbb{P}$ is monochromatic.

Let $D \subseteq \omega$ be diverse with respect to $\mathbb{P}$ and choose the least $n \in \omega$ so that $D \cap A_n$ and $D \cap B_n$ are both infinite. By deleting only finitely many $x \in D$, we obtain a set $D' \subseteq D$ so that for each $i < n$ we have $D' \subseteq A_i$ or $D' \subseteq B_i$. Now fix $x \in D' \cap A_n$ and choose $y > x$ so that $y \in D' \cap B_n$. (This is possible since $D' \cap B_n$ is infinite.) Then clearly $\{x, y\}$ is colored red since $s(x)$ lexicographically precedes $s(y)$. Similarly, if we fix $x \in D' \cap B_n$ and choose $y > x$ so that $y \in D' \cap A_n$, then $\{x, y\}$ is colored green since $s(x)$ lexicographically follows $s(y)$. Thus D is not monochromatic.

In general, if Π_i has $k(i)$ pieces (where $k(i) \leqslant \aleph_0$), then we replace Π_i by the $k(i)$ partitions $\{A, B\}$ obtainable by taking A to be one piece of Π_i and B to be the union of all the other pieces of Π_i. Clearly, if H is diverse with respect to Π_i, then it is also diverse with respect to one of these two-piece partitions. Thus, if the family $\mathbb{P} = \{\Pi_i : i \in \omega\}$ had the Ramsey property, so would the (countable) family of all the associated two-piece partitions. This would contradict the special case of the theorem proved above. So $\mathbb{P}$ does not have the Ramsey property. ∎

3. Proof of the Positive Result (Theorem 2)

We shall produce a family $\mathbb{P} = \{\Pi_\alpha : \alpha < \aleph_1\}$ of partitions of ω (into two pieces) and then show that $\mathbb{P}$ has the Ramsey property. So let $\{A_\alpha : \alpha < \aleph_1\}$ be a family of $\aleph_1$ independent subsets of ω. This means that if $\alpha_1, \alpha_2, ..., \alpha_k$ are distinct from $\beta_1, \beta_2, ..., \beta_l$, then

$$A_{\alpha_1} \cap \cdots \cap A_{\alpha_k} \cap (\omega - A_{\beta_1}) \cap \cdots \cap (\omega - A_{\beta_l}) \tag{1}$$

is non-empty, and it follows easily that all sets of the form (1) are infinite. The existence of $\aleph_1$ (and in fact $2^{\aleph_0}$) independent subsets of ω is a well-known result of [3] (given a combinatorial proof and generalized in [4]; a more accessible reference is [5, Lemma 24.8]). Now, for each $\alpha < \aleph_1$, let $\Pi_\alpha = \{A_\alpha, \omega - A_\alpha\}$; we shall show that the Ramsey property holds for the family $\mathbb{P} = \{\Pi_\alpha : \alpha < \aleph_1\}$. To do this, we first construct a suitable ultrafilter on ω and then follow a standard technique for deducing Ramsey's theorem using ultrafilters.

Temporarily call a subset X of ω *small* if, for every finite $F \subseteq \aleph_1$, there exist distinct $\alpha_1, \alpha_2, ..., \alpha_k, \beta_1, \beta_2, ..., \beta_l \in \aleph_1 - F$ such that

$$A_{\alpha_1} \cap \cdots \cap A_{\alpha_k} \cap (\omega - A_{\beta_1}) \cap \cdots \cap (\omega - A_{\beta_l}) \cap X \text{ is finite.} \tag{2}$$

In other words, X is not small if and only if all but finitely many of the A_α's remain independent when we restrict attention to X.

Suppose X and Y are small; we claim that $X \cup Y$ is also small. Indeed, let a finite $F \subseteq \aleph_1$ be given, and find α's and β's in $\aleph_1 - F$ to satisfy (2). Then use the fact that Y is small to find distinct $\gamma_1, ..., \gamma_p, \delta_1, ..., \delta_q \in \aleph_1 - (F \cup \{\alpha_1, ..., \alpha_k, \beta_1, ..., \beta_l\})$ such that

$$A_{\gamma_1} \cap \cdots \cap A_{\gamma_p} \cap (\omega - A_{\delta_1}) \cap \cdots \cap (\omega - A_{\delta_q}) \cap Y \text{ is finite.} \tag{3}$$

It follows from (2) and (3) that

$$A_{\alpha_1} \cap \cdots \cap A_{\alpha_k} \cap A_{\gamma_1} \cap \cdots \cap A_{\gamma_p} \cap (\omega - A_{\beta_1}) \cap \cdots \\ \cap (\omega - A_{\beta_l}) \cap (\omega - A_{\delta_1}) \cap \cdots \cap (\omega - A_{\delta_q}) \cap (X \cup Y)$$

is finite, which establishes that $X \cup Y$ is small.

Since ω is obviously not small, we see that the small sets constitute a proper ideal of subsets of ω. So there is an ultrafilter $\mathcal{U}$ on ω that contains no small sets. Since finite sets are clearly small, $\mathcal{U}$ is non-principal.

Now let $[\omega]^2$ be two-colored. We shall find a monochromatic set $H \subseteq \omega$ that is diverse with respect to some Π_α.

First, for each $x \in \omega$, find a set $C(x) \in \mathcal{U}$ such that all the pairs $\{x, y\}$ with $x < y$ and $y \in C(x)$ have the same color $c(x)$. Then find a set $D \in \mathcal{U}$ on which the function c is constant. Such $C(x)$ and D exist because $\mathcal{U}$ is an ultrafilter and the number of colors is finite. Being in $\mathcal{U}$, the sets $C(x)$ and D and all intersections of finitely many of them are not small. For each of the countably many sets X just mentioned, find a finite $F \subseteq \aleph_1$ serving as a counterexample to "X is small." As countably many finite sets cannot cover $\aleph_1$, fix an α belonging to none of these sets. Then, for each of the countably many sets X in question, both $X \cap A_\alpha$ and $X - A_\alpha$ are infinite.

Inductively choose an increasing sequence $x_0 < x_1 < x_2 < \cdots$ of natural numbers as follows. First, x_0 is any element of $D \cap A_\alpha$. If n is even and non-zero, then x_n is any element greater than x_{n-1} in

$$D \cap C(x_0) \cap \cdots \cap C(x_{n-1}) \cap A_\alpha.$$

If n is odd, then x_n is any element greater than x_{n-1} in

$$D \cap C(x_0) \cap \cdots \cap C(x_{n-1}) - A_\alpha.$$

Thus, if $k < n$, then $x_n \in C(x_k)$, so $\{x_k, x_n\}$ has color $c(x_k)$; furthermore, as $x_k \in D$, this color is independent of k (as well as n). So $\{x_n \colon n \in \omega\}$ is monochromatic. Furthermore, A_α contains x_n for all even n but for no odd n, so this monochromatic set is diverse for Π_α. ∎

4. Additional Remarks

1. Minor modifications of the proof of Theorem 2 establish that the partitions Π_α have the following stronger property. Whenever $[\omega]^r$, for some finite r, is colored with finitely many colors, there is a monochromatic set that is diverse with respect to each of infinitely many Π_α's. In fact, given a coloring, we can specify countably many of the Π_α's such that, for any countably many other Π_β's, some monochromatic set is diverse with respect to each of the Π_β's.

Results like these also hold for stronger partition theorems than Ramsey's, for example Silver's theorem [7]. The family of partitions used in the proof of Theorem 2 has the property that, whenever the set $[\omega]^\omega$ of infinite subsets of ω is partitioned into an analytic piece and a co-analytic piece, then there is a homogeneous set that is diverse with respect to this family. This can be proved by choosing an ultrafilter $\mathscr{U}$ containing no small sets (as in the proof of Theorem 2) and applying Theorem 4(a) of [1] to it.

2. It is easy to modify the proof of Theorem 2 to produce $\aleph_1$ partitions Π_α of ω into infinitely many pieces such that, for any two-coloring of $[\omega]^2$, there is a homogeneous set having infinite intersections with all the pieces of some Π_α. (This modification and those in Remark 1 can be combined.)

3. The considerations of the present paper were inspired by the following question of Zwicker. Call a subset X of the full binary tree ${}^{<\omega}2$ *dense* if every $s \in {}^{<\omega}2$ has an extension in X. Suppose $\mathscr{F}$ is a family of bijections $f\colon \omega \to {}^{<\omega}2$. Call $\mathscr{F}$ *Ramsey* if, whenever $[\omega]^2$ is two-colored, there is a monochromatic set $X \subseteq \omega$ such that $f(X)$ is dense for at least one $f \in \mathscr{F}$. What is the minimal size of a Ramsey family $\mathscr{F}$ as described above? Theorem 1 and (a slight generalization of) Theorem 2 show the answer to be $\aleph_1$.

4. One could also consider analogs of our results for larger or smaller cardinalities. Here are two questions, which are typical of many others that might be asked:

Let X be a set of cardinality $(2^{\aleph_0})^+$. How many partitions Π_α of X do we need, to ensure that, whenever $[X]^2$ is two-colored, some homogeneous set meets both pieces of some Π_α in uncountable sets? (Recall that, by a well-known theorem from [2], every two-coloring of $[X]^2$ has an uncountable homogeneous set.)

Fix a small positive real number ρ. For each integer $n \geqslant 2$, let $f(n)$ be the smallest number of partitions Π_α of an n-element set X_n into two pieces such that every two-coloring of $[X_n]^2$ has a homogeneous set that meets both pieces of some Π_α in sets of size $\geqslant \rho \log n$. What is the asymptotic behavior of f? We can show that, when $\rho < \frac{1}{4}$, there are constants c_1 and c_2 (depending on ρ but not on n) such that $\log \log n - c_1 \leqslant f(n) \leqslant c_2 \log n$, where the logarithms are to the base 2.

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