Intermediate- and extreme-sum processes

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Let $X_1, X_2, \ldots, X_n$ be the order statistics of $n$ independent random variables with a common distribution function $F$ and let $k_n$ be positive numbers such that $k_n \to \infty$ and $k_n / n \to 0$ as $n \to \infty$. With suitable centering and norming, we investigate the weak convergence of the intermediate-sum process $\sum_{t=0}^{[\infty]} X_{n+1-i,n}$, $a \leq t \leq b$, where $0 < a < b < \infty$, and the weak convergence of the extreme-sum process $\sum_{i=1}^{[k_n]} X_{n+1-i,n}$, $0 \leq t < b$. Convergence is with respect to the supremum norm and can take place along a subsequence of the positive integers $\{n\}$.

order statistics * intermediate-sum processes * extreme-sum processes * weak convergence * extreme-value domain of attraction

1. Introduction and statement of results

Let $X, X_1, X_2, \ldots$, be a sequence of independent non-degenerate random variables with a common distribution function $F(x) = P(X \leq x)$, $x \in \mathbb{R}$, and for each integer $n \geq 1$ let $X_{1,n} \leq \cdots \leq X_{n,n}$ denote the order statistics based on the sample $X_1, \ldots, X_n$. Let $\{k_n\}$ be a sequence of positive numbers such that

$$k_n \to \infty \text{ and } k_n / n \to 0 \quad \text{as } n \to \infty,$$

and consider the sum process

$$I_n(a, t) = I_n(a, t; k_n) = \sum_{i=1}^{[tk_n]} X_{n+1-i,n}, \quad a \leq t \leq b,$$

of intermediate order statistics, where $0 < a < b$, and the sum process

$$E_n(t) = E_n(t; k_n) = \begin{cases} \sum_{i=1}^{[tk_n]} X_{n+1-i,n}, & 1/k_n \leq t \leq b, \\ 0, & 0 \leq t < 1/k_n, \end{cases}$$

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of extreme order statistics, where \([x]\) is the smallest integer not smaller than \(x\) and an empty sum is understood as zero.

The asymptotic distribution of the intermediate sum \(I_n(a, b)\) for fixed \(0 < a < b\) has been thoroughly investigated in [3]. We found necessary and sufficient conditions for the existence of constants \(A_n > 0\) and \(C_n \in \mathbb{R}\) such that \(A_n^{-1}(I_n(a, b) - C_n)\) converges in distribution along subsequences of the positive integers \(\{n\}\) to non-degenerate limits and completely described the possible subsequential limiting distributions. Exactly the same programme has been carried out previously in [2] for the extreme sums \(E_n(1)\). The aim of the present paper is to investigate the weak convergence of the suitably centered and normalized processes \(I_n(a, t)\) and \(E_n(\cdot)\) in the supremum norm on \([a, b]\) and \([0, b]\), respectively.

Consider the inverse or quantile function of \(F\) defined as

\[
Q(s) = \inf\{x: F(x) \geq s\}, \quad 0 < s \leq 1,
\]

and introduce the associated left-continuous non-decreasing function

\[
H(s) = -Q((1 - s)^-), \quad 0 \leq s < 1.
\]

Consider the centering functions

\[
\mu_n(a, t) = \mu_n(a, t; k_n) = -n \int_{1/k_n/n}^{[sk_n]/n} H(s) \, ds, \quad 0 \leq a < t.
\]

We say that a sequence \(\{\xi_n(t): a \leq t \leq b\}_{n=1}^\infty\) of stochastic processes has a distributionally equivalent version \(\{\eta_n(t): a \leq t \leq b\}_{n=1}^\infty\) if the distributional equality \(\xi_n(t): a \leq t \leq b\} = \eta_n(t): a \leq t \leq b\}\) holds for each \(n \geq 1\), that is, all finite-dimensional distributions of \(\xi_n(\cdot)\) and \(\eta_n(\cdot)\) are the same on \([a, b]\) for each \(n \geq 1\).

**Theorem 1.** Let \(\{k_n\}\) be a sequence as in (1.1) and fix \(0 < a_0 < b_0 < \infty\). Suppose that there exist a subsequence \(\{n'\}\) of the positive integers and positive numbers \(B_n\) along it such that for a function \(\varphi\) continuous on \([a_0, b_0]\), necessarily non-negative, non-decreasing and satisfying \(\varphi(a_0) = 0\), we have

\[
\varphi_n(a_0; x) := \int_{a_0}^{x} dH\left(\frac{sk_n}{n}\right) / B_n \to \varphi(x) \text{ at each } x \in [a_0, b_0] \text{ as } n' \to \infty.
\]

Then on a suitable probability space, for any choice of \(a_0 < a < b < b_0\), there exists a sequence \(\{\tilde{I}_n(a, t): a \leq t \leq b\}_{n=1}^\infty\) of distributionally equivalent versions of the sequence \(\{I_n(a, t): a \leq t \leq b\}_{n=1}^\infty\) and a standard Wiener process \(W(t), t \geq 0\), such that

\[
\sup_{a \leq t \leq b} \left| \frac{1}{\sqrt{k_n} B_n} \left( \tilde{I}_n(a, t) - \mu_n(a, t) \right) - \int_a^t W(s) \, d\varphi(s) \right| \to 0 \quad \text{a.s.}
\]

as \(n' \to \infty\).

We note that by Theorem 1 in [3] for convergence in distribution of the process

\[
Y_n(a, t) := \{I_n(a, t) - \mu_n(a, t)\} / \sqrt{k_n B_n},
\]

as \(n' \to \infty\).
at a fixed point \( a \leq t \leq b \) with \( a_0 < a < b < b_0 \) along a subsequence of \( \{n\} \) we can always choose

\[
B_n = \Delta_n(a_0, b_0) := \max(H(b_0k_n/n) - H(a_0k_n/n), 1) > 0.
\]

Then, with this choice of \( B_n \), for the non-decreasing, left-continuous functions \( \varphi_n(a_0, x) \) we have \( 0 \leq \varphi_n(a_0; x) \leq 1 \) on \([a_0, b_0]\). Hence by a Helly selection one can always find a subsequence \( \{n'\} \subset \{n\} \) and a non-negative, non-decreasing, left-continuous function \( \varphi \) on \((a_0, b_0)\) such that \( \varphi_n(a_0; x) \) converges to \( \varphi(x) \) as \( n' \to \infty \) at any continuity point \( x \) of \( \varphi \). Theorem 1 in [3] shows that one can hope for weak convergence of \( I_n(\cdot) \) in the supremum norm on \([a, b]\) only in the case when \( \varphi \) is continuous on some interval containing \([a, b]\). This is the underlying reason for condition (1.6).

Now we turn to the weak-convergence problem of the extreme-sum process \( E_n(t) = \sum_{i=1}^{n} (0, t) \). Despite the problem is now the behavior in the vicinity of zero, we still need a reference point \( a_0 > 0 \) as in (1.6), which can in principle be chosen to be \( b_0 \).

**Theorem 2.** Let \( \{k_n\} \) be a sequence as in (1.1) and fix \( 0 < a_0 < b_0 < \infty \). Suppose there exist a subsequence \( \{n'\} \) of the positive integers and positive numbers \( B_{n'} \) such that for a function \( \varphi \) continuous on \((0, b_0)\), necessarily non-decreasing and satisfying \( \varphi(a_0) = 0 \), we have

\[
\varphi_n(a_0; x) = \int_{a_0}^{x} dH\left(\frac{sk_{n'}}{n'}\right)/B_{n'} \to \varphi(x) \quad \text{at each } x \in (0, b_0]
\] (1.9)

as \( n' \to \infty \) and

\[
\lim_{a \to 0} \limsup_{n' \to \infty} \int_{0}^{a} \sqrt{x} \ d\varphi_n(a_0, x) = 0 \quad \text{and} \quad \lim_{a \to 0} \int_{0}^{a} \sqrt{x} \ d\varphi(x) = 0.
\] (1.10)

Then on a suitable probability space, for any choice of \( 0 < b < b_0 \), there exist a sequence \( \{\tilde{E}_n(t) : 0 < t \leq b\}_{n-1}^{\infty} \) of distributionally equivalent versions of the sequence \( \{E_n(t) : 0 < t \leq b\}_{n-1}^{\infty} \) and a standard Wiener process \( W(t), t \geq 0 \), such that

\[
\sup_{0 < t < b} \left| \frac{1}{\sqrt{n'}} \tilde{E}_n(t) - \mu_n(0, t) \right| - \int_{0}^{t} W(s) \ d\varphi(s) \to_r 0
\]
as \( n' \to \infty \).

We note that it is easy to see using integration by parts that if (1.9) holds and there exists a constant \( \beta > -\frac{1}{2} \) such that \( \varphi_n(a_0, x) < x^\beta \) for all \( n' \) large enough and all \( x > 0 \) small enough, then condition (1.10) is also satisfied.

Now we formulate a corollary to Theorems 1 and 2 under the classical condition of extreme value theory. We say that \( F \) is in the domain of attraction of an extreme value distribution if \( (X_{n,n} - c)/a_n \) converges in distribution to a non-degenerate random variable \( Y \), where \( a_n > 0 \) and \( c_n \in \mathbb{R} \) are some constants. As pointed out in
with earlier references, this happens if and only if there exists a constant $\gamma \in \mathbb{R}$ such that

$$\lim_{t \to 0} \frac{H(sx) - H(sy)}{H(su) - H(sv)} = \begin{cases} \frac{(x^{-\gamma} - y^{-\gamma})/\left(u^{-\gamma} - v^{-\gamma}\right)}{\log(x/y) / \log(u/v)}, & \text{if } \gamma \neq 0, \\ \log(x/y) / \log(u/v), & \text{if } \gamma = 0, \end{cases}$$

(1.11)

for all distinct $0 < x, y, u, v < \infty$. In this case we write $F \in \Lambda_\gamma$, where, with suitable choices of $a_n$ and $c_n$,

$$\Lambda_\gamma(y) = P\{Y \leq y\} = \begin{cases} \exp(-y^{1/\gamma}), & \text{if } \gamma > 0, \\ \exp(-(-y)^{1/\gamma}), & \text{if } \gamma < 0, \\ \exp(-\exp(-y)), & \text{if } \gamma = 0, \end{cases}$$

For any $\gamma \in \mathbb{R}$, set

$$u_\gamma = \begin{cases} 1, & \text{if } \gamma > 0, \\ e, & \text{if } \gamma = 0, \end{cases} \quad \text{and} \quad v_\gamma = \begin{cases} (1 + \gamma)^{-1/\gamma}, & \text{if } \gamma > 0, \\ 1, & \text{if } \gamma = 0, \end{cases}$$

so that $u_\gamma^{-\gamma} - v_\gamma^{-\gamma} = -\gamma$ if $\gamma \neq 0$ and $\log(u_0/v_0) = 1$. For a sequence $\{k_n\}$ satisfying (1.1) we define

$$\Delta_n(\gamma) = H(u, k_n/n) - H(v, k_n/n).$$

Choosing the reference point of Theorem 2 as $a_0 = 1$, introduce now

$$\varphi_n(x) = \varphi_n(1, x) = \int_1^x dH(\frac{sk_n}{n})/\Delta_n(\gamma)$$

$$= \left\{ H\left(\frac{xk_n}{n}\right) - H\left(\frac{k_n}{n}\right) \right\} / \Delta_n(\gamma),$$

(1.12)

which is well-defined for $0 < x < n/k_n$. Then, if $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$, we obtain from (1.11) that

$$\varphi_n(x) = \varphi_\gamma(x) := \begin{cases} (1 - x^{-\gamma})/\gamma, & \text{if } \gamma \neq 0, \\ \log x, & \text{if } \gamma = 0, \end{cases}$$

(1.13)

as $n \to \infty$, that is, we have (1.9) with the continuous function $\varphi = \varphi_\gamma$ along the whole $\{n\}$ and for any $b_0 > 0$, or, what is the same, (1.6) for any $0 < a_0 < b_0$. Hence the first statement of Corollary 1 below follows from Theorem 1 and the second statement will follow from Theorem 2 after proving (1.10) for the present $\varphi_n$ and $\varphi = \varphi_\gamma$.

**Corollary 1.** If $F \in \Lambda_\gamma$ for some $\gamma \in \mathbb{R}$ and $\{k_n\}$ satisfies (1.1), then on a suitable probability space, for any choice of $b > 0$, there exist a sequence $\{\tilde{E}_n(t); 0 \leq t \leq b\}_{n=1}^\infty$ of distributionally equivalent versions of the sequence $\{E_n(t); 0 \leq t \leq b\}_{n=1}^\infty$ and a standard Wiener process $W(t)$, $t \geq 0$, such that for the sequence $\{\tilde{I}_n(a, t) = \tilde{E}_n(t) -$
\[ \tilde{E}_n(a) \triangleq \sup_{a < t < b} \left\{ \frac{1}{\sqrt{k_n} \Delta_n(\gamma)} \left( \tilde{E}_n(t) - n \int_{[tk_n]^{-\gamma}/n}^{[tk_n]^{\gamma}/n} Q(1-s) \, ds \right) \right\} \]

almost surely as \( n \to \infty \). Furthermore, if \( \gamma < \frac{1}{2} \), then for any \( b > 0 \),

\[ \sup_{0 < t < b} \left| \frac{1}{\sqrt{k_n} \Delta_n(\gamma)} \left( \tilde{E}_n(t) - n \int_{0}^{[tk_n]^{\gamma}/n} Q(1-s) \, ds \right) \right| \to 0 \]

as \( n \to \infty \).

Notice that if \( F \in \Lambda_\gamma \) and \( \gamma \geq \frac{1}{2} \), then we have (1.13) but condition (1.10) is not satisfied by the limiting function \( \varphi =\varphi_\gamma \). In this case, according to Corollary 2 in [2], the appropriately centered extreme sums \( E_n(t) - \mu_n(0+, t) \) require a norming sequence \( A_n > 0 \) to converge in distribution (denoted by \( \Rightarrow \)) to a non-degenerate random variable \( V \) that is heavier than the one needed by the centered intermediate sums. Namely, it follows from Corollary 2 in [2] that if \( F \in \Lambda_\gamma \) for some \( \gamma \geq \frac{1}{2} \), then there is a sequence \( A_n = A_n(k_n) > 0 \), completely specified in [2], such that

\[
\{ I_n(a, b) - \mu_n(a, b) \}/A_n \to_P 0 \quad \text{for all } 0 < a < b
\]

and

\[
\{ E_n(t) - \mu_n(0+, t) \}/A_n \to_P V \quad \text{for all } t > 0,
\]

where if \( \gamma = \frac{1}{2} \), then \( V \) is the same standard normal random variable for all \( t > 0 \), and if \( \gamma > \frac{1}{2} \), then \( V \) is the same stable random variable with index \( 1/\gamma \) for all \( t > 0 \).

Our next corollary discloses a curious Darling-Erdős type behavior for the extreme-sum process. Whenever \( F \in \Lambda_\gamma \) for some \( \gamma < \frac{1}{2} \) and \( \{ k_n \} \) satisfies (1.1) write

\[
e_n(t) := \sigma_t \gamma^{-1/2} \frac{E_n(t) - n \int_{[tk_n]^{-\gamma}/n}^{[tk_n]^{\gamma}/n} Q(1-s) \, ds}{\sqrt{k_n} \Delta_n(\gamma)},
\]

where

\[
\sigma_\gamma = (\frac{1}{2}(1-\gamma)(1-2\gamma))^{1/2},
\]

and for \( T > 0 \) and \( \gamma < \frac{1}{2} \) set

\[
A(T) = (2 \log \max(T, e))^{1/2} \tag{1.14}
\]

and

\[
B_\gamma(T) = A(T) + (\log(\sqrt{\lambda_\gamma}/2\pi))^{1/2}/A(T) \tag{1.15}
\]
where
\[ \lambda_\gamma = \frac{1}{4}(1 - 2\gamma). \] (1.16)

This behavior will be a consequence of the weak convergence of a time-changed variant of \( e_n(\cdot) \) to the stochastic process
\[ V_\gamma(x) := \sigma_\gamma e^{(1/2-\gamma)x} \int_0^x W(u)u^{-1-\gamma} \, du, \quad 0 \leq x < \infty. \]

It is readily verified that \( V_\gamma(\cdot) \) is a sample-continuous mean zero stationary Gaussian process with covariance function given, for \( x \geq 0 \) and \( h \geq 0 \), by
\[ rensely. \]

**Corollary 2.** Assume that \( F \in \Lambda_\gamma \) with \( \gamma < \frac{1}{2} \) and let \( \{k_n\} \) be a sequence satisfying (1.1). Then for any fixed \( 0 < c < 1 \) the sequence \( \{e_n(e^{-x}): 0 \leq x \leq \log(1/c)\} \) of processes converges weakly in the Skorohod space \( D[0, \log(1/c)] \) to the process \( \{V_\gamma(x): 0 \leq x \leq \log(1/c)\} \). Furthermore,
\[ \sup_{e_n(t) - B_\gamma(\log(1/c))} = \exp(-e^{-x}) \]
for all \( x \in \mathbb{R} \).

Finally we would like to connect Theorem 1 to the classical theory of domains of partial attraction for the whole sums \( X_1 + \cdots + X_n \) and thereby show that Theorem 1 is not empty for any choice of \( 0 < a < b \) and non-negative, non-decreasing continuous function \( \varphi \) on \([a, b]\). In particular, we claim the following: Let \( 0 < a < b < \infty \) be arbitrary and let \( \varphi \) be any non-negative, non-decreasing, continuous function on \([a, b]\). Then there exist a distribution function \( F \), a subsequence \( \{n'\} = \{n'_j\}_{j=1}^\infty \), and a sequence \( k'_j - k'_j \) satisfying \( k'_j \to \infty \) and \( k'_j/n'_j \to 0 \) as \( j \to \infty \) such that for the versions \( \tilde{I}_{n'_j} \) of the intermediate sums \( I_{n'_j} \) pertaining to \( F \) in Theorem 1 we have (1.7) as \( n' \to \infty \). In fact, there is a universal \( F \) that does the job for all \( 0 < a < b \) and all functions \( \varphi \) on \([a, b]\) with the described properties.

Indeed, let \( 0 < a < b \) be arbitrary and \( \varphi \) be any continuous, non-negative, non-decreasing function on \([a, b]\). Choose \( 0 < a_0 < a < b < b_0 < \infty \) and extend the definition of \( \varphi \) so that the extended \( \varphi \) is continuous and non-decreasing on \([a_0, b_0]\) and \( \varphi(a_0) = 0 \). Now define
\[ \psi(s) = \begin{cases} \varphi(a_0) - \varphi(b_0), & 0 < s \leq a_0, \\ \varphi(s) - \varphi(b_0), & a_0 < s \leq b_0, \\ 0, & s > b_0. \end{cases} \]
and introduce \( R(x) = -\inf\{s > 0: \psi(s) \geq -x\}, \ x > 0 \). Consider the spectrally right-sided infinitely divisible distribution without a normal component, the right Lévy measure of which is \( R(x), \ x > 0 \). Then by the classical theorem of Khinchin [4, p. 184] there is an \( F \) in the domain of partial attraction of this infinitely divisible law. Using Theorems 3 and 5 in [1], this means, in particular, that there exists a subsequence \( \{n_j\}_{j=1}^\infty \) of the positive integers such that if \( Q \) denotes the quantile function belonging to \( F \) then we have

\[
-Q((1-s/n_j)^-) / B_j' \to \psi(s), \ s > 0,
\]

as \( j \to \infty \), where \( B_j' > 0 \) are some constants. Now define \( n_j^i = \lfloor n_j^{3/2} \rfloor \) and \( k_j^i = \lfloor n_j^{3/2} / n_j \rfloor, \ j = 1, 2, \ldots \). Then \( k_j^i \to \infty \) and \( k_j^i / n_j^i \to 0 \) as \( j \to \infty \), and it follows from (1.17) that

\[
\left\{ H\left( \frac{s k_j^i}{n_j^i} \right) - H\left( \frac{a_0 k_j^i}{n_j^i} \right) \right\} / B_j' \to \psi(s) - \psi(a_0) - \varphi(s)
\]

for each \( a_0 \leq s \leq b_0 \). This means that condition (1.6) is satisfied and hence by Theorem 1 we obtain (1.7) along \( \{n_j^i\} = \{n_j\}_{j=1}^\infty \), with \( B_j \) replaced by the present \( B_j' \). A universal \( F \) is obtained by using the distribution function \( F \) of any of the universal laws of Doeblin [4, p. 189].

In order to give a flavor of the content of Theorems 1 and 2, we close this section by an illustrative example. Set

\[
Q(1-s) = \{\beta + \sin(\log s)\} s^{-\gamma}, \ 0 < s \leq 1,
\]

where \( \gamma > 0 \) and \( \beta > (1 + \gamma) / \gamma \). Differentiation shows that \( Q \) is an actual quantile function. For \( j = 1, 2, \ldots \), set

\[
n_j^i = \lfloor \exp(4\pi j) \rfloor \quad \text{and} \quad k_j^i = n_j^i \exp(-2\pi j),
\]

so that \( k_j^i / n_j^i = \exp(-2\pi j), \ j = 1, 2, \ldots \). Also let

\[
B_j' = B_n_j = \exp(2\pi\gamma j), \ j = 1, 2, \ldots,
\]

and choose \( a_0 > 0 \) arbitrarily. Then for any \( a_0 \leq x < \infty \) and all \( j \) large enough,

\[
\varphi_{n_j^i}(a_0; x) = \left\{ Q\left( 1 - x \frac{k_j^i}{n_j^i} \right) - Q\left( 1 - a_0 \frac{k_j^i}{n_j^i} \right) \right\} / B_j'
\]

\[
= (\beta + \sin(\log a_0)) a_0^{-\gamma} - (\beta + \sin(\log x)) x^{-\gamma}.
\]

We see that Theorem 1 applies along \( \{n_j^i\} = \{n_j\}_{j=1}^\infty \) for all \( \gamma > 0 \) and all \( a_0 < a < b < \infty \) and, moreover, it is easily checked that Theorem 2 is also applicable when \( 0 < \gamma < \frac{1}{2} \). Notice that by (1.11) the distribution function corresponding to such a \( Q \) is not in the domain of attraction of \( \Lambda_y \) for any \( \gamma \).

2. Proofs

Let \( U_1, U_2, \ldots, \) be independent random variables uniformly distributed on \( (0, 1) \) with corresponding order statistics \( U_{1,n} \leq \cdots \leq U_{n,n} \). Consider the uniform empirical
and quantile processes \( \alpha_n(t) = \sqrt{n}(G_n(t) - t) \) and \( \beta_n(t) = \sqrt{n}(t - U_n(t)), \) \( 0 \leq t \leq 1, \)
where \( G_n(t) = n^{-1}\#\{1 \leq k \leq n: U_k \leq t\}, 0 \leq t \leq 1, \) and \( U_n(t) = \inf\{0 \leq s \leq 1: G_n(s) \geq t\}, \) \( 0 \leq t \leq 1, \) \( U_n(0) = U_{1,n} \) so that \( U_n(t) = U_{k,n} \) if \( (k-1)/n < t \leq k/n, \) \( k = 1, \ldots, n. \)
The tail empirical and quantile processes pertaining to the given sequence \( \{k_n\} \)

satisfying \( (1.1) \) are defined as

\[
w_n(s) = (n/k_n)^{1/2} \alpha_n(sk_n/n)
\]

and

\[
v_n(s) = (n/k_n)^{1/2} \beta_n(sk_n/n), \quad 0 \leq s \leq n/k_n.
\]

As pointed out in [6], \( w_n(\cdot) \) converges weakly in the Skorohod space \( D[0, T] \), for any \( T > 0, \) to a standard Wiener process. Then by a Skorohod construction and the left-continuous version of Lemma 1 of Vervaat [8] (both also in [7]) we see that the sequences \( \{w_n(\cdot)\}_{n=1}^{\infty} \) and \( \{v_n(\cdot)\}_{n=1}^{\infty} \) have distributionally equivalent versions

\[
\{\hat{w}_n(\cdot)\}_{n=1}^{\infty} \quad \text{and} \quad \{\hat{v}_n(\cdot)\}_{n=1}^{\infty}
\]
on a rich enough probability space that carries a standard Wiener process \( W \) such that

\[
\sup_{0 \leq s \leq T} |\hat{w}_n(s) - W(s)| \to 0 \quad \text{and} \quad \sup_{0 \leq s \leq T} |\hat{v}_n(s) - W(s)| \to 0 \quad \text{a.s.} \quad (2.1)
\]
as \( n \to \infty. \)

In order to obtain the distributionally equivalent copies \( \hat{I}_n \) of Theorem 1, we first note that from (1.4),

\[
(X_{1,n}, \ldots, X_{n,n}) = \mathcal{G}(-H(U_{1,n}), \ldots, -H(U_{n,n})) \quad \text{for each} \quad n \geq 1.
\]

Define \( B_n > 0 \) arbitrarily for an \( n \) which is not a member of \( \{n'\} \) in (1.6). Then, using the notations in (1.2) and (1.5), starting out from the equality

\[
\{I_n(a, t) - \mu_n(a, t): a \leq t \leq b\}
\]

\[
= \mathcal{G}\left\{-\int_{U_n(ak_n/n)}^{U_n(ik_k/n)} nH(u) \, dG_n(u) + \int_{[ak_k]/n}^{[ik_k]/n} H(u) \, du: a \leq t \leq b\right\}, \quad n \geq 1,
\]

and then integrating by parts, for the processes \( Y_n(a, t) \) in (1.8) and for

\[
Y_n^*(a, t) := M_n^*(a, t) - R_n^*(a) + R_n^*(t),
\]

where

\[
M_n^*(a, t) = \frac{n}{\sqrt{k_n} B_n} \int_{[ak_k]/n}^{[ik_k]/n} (G_n(u) - u) \, dH(u)
\]

and

\[
R_n^*(t) = \frac{n}{\sqrt{k_n} B_n} \int_{[tk_k]/n}^{U_n(ik_k/n)} (G_n(u) - u) \, dH(u)
\]

we obtain

\[
\{Y_n(a, t): a \leq t \leq b\}
\]

\[
= \mathcal{G} \left\{Y_n^*(a, t) = M_n^*(a, t) - R_n^*(a) + R_n^*(t): a \leq t \leq b\right\}, \quad n \geq 1. \quad (2.2)
\]
Substituting now $u = sk_n/n$ and transferring to the probability space of the versions $\tilde{w}_n$ and $\tilde{v}_n$ in (2.1), we get

$$\{ Y_n(a, t): a \leq t \leq b \} = \{ \tilde{Y}_n(a, t) := \tilde{M}_n(a, t) - \tilde{R}_n(a) + \tilde{R}_n(t): a \leq t \leq b \}, \quad n \geq 1. \quad (2.3)$$

where, with the obvious extension of the definition of $\varphi_n$ for an arbitrary $n$,

$$\tilde{M}_n(a, t) = \int_{\lfloor ak_n \rfloor / k_n}^{\lfloor tk_n \rfloor / k_n} \tilde{w}_n(s) \, d\varphi_n(a_0; s)$$

and

$$\tilde{R}_n(t) = \int_{\lfloor tk_n \rfloor / k_n}^{(1 - \sqrt{k_n})/k_n} \left\{ \tilde{w}_n(s) + s\sqrt{k_n} - \frac{\lfloor tk_n \rfloor}{\sqrt{k_n}} \right\} \frac{dH(sk_n/n)}{B_n}$$

$$= \int_0^{\frac{1}{\sqrt{k_n}}} \left\{ \tilde{w}_n\left( \frac{\lfloor tk_n \rfloor}{k_n} + \frac{x}{\sqrt{k_n}} \right) + x \right\} \frac{dH(\lfloor tk_n \rfloor / n + x\sqrt{k_n}/n)}{B_n}.$$

**Proof of Theorem 1.** Relations (1.8) and (2.3) show the existence of versions $\{ I_n(a, t): a \leq t \leq b \}$ of $\{ I_n(a, t): a \leq t \leq b \}$ as claimed in the statement in the theorem once we prove that

$$\sup_{a \leq t \leq b} |\tilde{R}_n(t)| \to 0 \quad \text{a.s. as } n' \to \infty \quad (2.4)$$

and

$$\sup_{a \leq t \leq b} \left| \tilde{M}_n(a, t) - \int_a^t W(s) \, d\varphi_n(s) \right| \to 0 \quad \text{a.s. as } n' \to \infty. \quad (2.5)$$

By (2.1) and the fact that $|W(\cdot)|$ is bounded on any finite interval with probability 1, there exists an almost surely finite random variable $K > 0$ such that for all $n'$ large enough,

$$\sup_{a \leq t \leq b} |\tilde{R}_n(t)|$$

$$\leq K \sup_{a \leq t \leq b} \int_{-K}^K \frac{dH(\lfloor tk_n \rfloor / n' + x\sqrt{k_n}/n')}{B_n}$$

$$= K \sup_{a \leq t \leq b} \frac{H(\lfloor tk_n \rfloor / n' + K\sqrt{k_n}/n') - H(\lfloor tk_n \rfloor / n' - K\sqrt{k_n}/n')}{B_n}$$

$$= K \sup_{a \leq t \leq b} \left\{ \varphi_n\left( a_0, \frac{\lfloor tk_n \rfloor}{k_n} + \frac{K}{\sqrt{k_n}} \right) - \varphi_n\left( a_0, \frac{\lfloor tk_n \rfloor}{k_n} - \frac{K}{\sqrt{k_n}} \right) \right\}.$$

almost surely. Since $\varphi_n(a_0, \cdot)$ is non-decreasing, by condition (1.6) and the continuity of $\varphi$ this bound goes to zero as $n' \to \infty$ and hence we have (2.4).
To prove (2.5), first notice that (2.1) and (1.6) easily imply that
\[ \sup_{a < t < b} \left| \tilde{M}_n(a, t) - \int_{[a, t]} W(s) \, d\varphi_n(a_0, s) \right| \to 0 \quad \text{a.s. as } n' \to \infty. \]

Next, for all \( n' \) large enough,
\[ \sup_{a < t < b} \left| \int_{[a, t]} W(s) \, d\varphi_n(a_0, s) - \int_{[a, t]} W(s) \, d\varphi_n(a_0, s) \right| \leq 2 \sup_{a_0 < s < b_0} |W(s)| \sup_{a < t < b} \left| \varphi_n\left( a_0, \frac{tk_n}{k_{n'}} \right) - \varphi_n(a_0, t) \right|, \]
and this bound goes to zero again by (1.6) and the continuity of \( \varphi \) as \( n' \to \infty \). Finally, (2.5) and hence the theorem will follow from these relations if we show that
\[ \sup_{a < t < b} \left| \int_{a}^{t} W(s) \, d\varphi_n(a_0, s) - \int_{a}^{t} W(s) \, d\varphi(s) \right| \to 0 \quad \text{a.s.} \tag{2.6} \]
as \( n' \to \infty \).

To verify this, notice that with probability 1 for any given \( \varepsilon > 0 \) there exists a (random) partition \( a = t_0 < t_1 < \cdots < t_m < t_{m+1} = b \) such that
\[ \left| \int_{t_{i-1}}^{t_i} W(s) \, d\varphi_n(a_0, s) - \int_{t_{i-1}}^{t_i} W(s) \, d\varphi(s) \right| < \frac{1}{2} \varepsilon \]
and
\[ \int_{t_i}^{t_{i+1}} |W(s)| \, d\varphi_n(a_0, s) + \int_{t_i}^{t_{i+1}} |W(s)| \, d\varphi(s) \leq \varepsilon \]
for all \( i = 0, \ldots, m \) and \( n' \) large enough, where \( m \) does not depend on \( n' \). Thus for any \( t_i \leq t \leq t_{i+1} \) and \( i = 0, \ldots, m \),
\[ \left| \int_{a}^{t} W(s) \, d\varphi_n(a_0, s) - \int_{a}^{t} W(s) \, d\varphi(s) \right| < \varepsilon \]
almost surely for all \( n' \) large enough, proving (2.6). \( \Box \)

**Proof of Theorem 2.** First we note that it is easily checked that (1.10) implies the finiteness of \( \mu_n(0, b) \) for all \( n \) large enough, so that the representation (2.2) holds true for \( a = 0 \). Hence, again defining \( B_n > 0 \) arbitrarily for an \( n \in \{n'\} \),
\[ \left\{ \frac{E_n(t) - \mu_n(0, t)}{\sqrt{k_n B_n}} : 0 \leq t \leq b \right\} \]
\[ = \{ M_n^*(0, a) + M_n^*(a, t) - R_n^*(0) + R_n^*(t) : 0 \leq a \leq t \leq b \} \]
for each \( n \geq 1 \). Furthermore, it follows from the derivation of (2.2) and (2.3) that for each \( n \geq 1 \),
\[ \{(M_n^*(0, a), M_n^*(a, t), R_n^*(0), R_n^*(t)) : 0 \leq a \leq t \leq b \} \]
\[ = \{(\tilde{M}_n(0, a), \tilde{M}_n(a, t), \tilde{R}_n(0), \tilde{R}_n(t)) : 0 \leq a \leq t \leq b \}. \]
Hence, in view of the fact that now we have (2.4) and (2.5) for any fixed $0 < a < b < b_0$, it suffices to prove

\[
\lim_{a \to 0} \lim_{n' \to \infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq a} \left| M_n^a(0, t) \right| > \varepsilon \right\} = 0, \tag{2.7}
\]

\[
\lim_{a \to 0} \mathbb{P} \left\{ \sup_{0 \leq t \leq a} \int_0^t W(s) \, d\varphi(s) > \varepsilon \right\} = 0, \tag{2.8}
\]

and

\[
\lim_{a \to 0} \lim_{n' \to \infty} \mathbb{P} \left\{ \sup_{0 \leq t \leq a} |R_n^a(t)| > \varepsilon \right\} = 0, \tag{2.9}
\]

where $\varepsilon > 0$ is arbitrary.

We have

\[
E \left( \sup_{0 \leq t \leq a} \left| M_n^a(0, t) \right| \right) \leq \frac{1}{\sqrt{k_n} \cdot B_n} \int_0^{[ak_n]/n'} \sqrt{n'} \left| G_n(u) - u \right| \, dH(u)
\]

\[
\leq \frac{\sqrt{n'}}{\sqrt{k_n} \cdot B_n} \int_0^{[ak_n]/k_n} \sqrt{u} \, dH(u)
\]

\[
= \int_0^{[ak_n]/k_n} \sqrt{s} \, d\varphi_n(a_0, s)
\]

\[
\leq \int_0^{2a} \sqrt{s} \, d\varphi_n(a_0, s),
\]

where the last inequality holds for all $n'$ large enough. Hence, using condition (1.10), by the Markov inequality we obtain (2.7).

Also,

\[
E \left( \sup_{0 \leq t \leq a} \left| W(s) \, d\varphi(s) \right| \right) \leq E \int_0^a |W(s)| \, d\varphi(s) \leq \int_0^a \sqrt{s} \, d\varphi(s),
\]

and hence condition (1.10) and the Markov inequality again imply (2.8).

Finally, using the fact that for any $a > 0$,

\[
\frac{n}{k_n} U_n \left( \frac{ak_n}{n} \right) \rightarrow_d a \quad \text{as } n \to \infty,
\]

which follows for example from (2.1), we obtain that for each $a > 0$,

\[
\sup_{0 \leq t \leq a} |R_n^a(t)| \leq \frac{1}{\sqrt{k_n} \cdot B_n} \int_0^{U_n((ak_n)/n')} \sqrt{n'} \left| G_n(u) - u \right| \, dH(u)
\]

\[
= \frac{1}{\sqrt{k_n}} \int_0^{n'U_n((ak_n)/n')/k_n} \sqrt{n'} \left| G_n \left( \frac{sk_n}{n'} \right) - \frac{sk_n}{n'} \right| \, d\varphi_n(a_0, s)
\]

\[
\leq \frac{1}{\sqrt{k_n}} \int_0^{2a} \sqrt{n'} \left| G_n \left( \frac{sk_n}{n'} \right) - \frac{sk_n}{n'} \right| \, d\varphi_n(a_0, s) + o_p(1)
\]
as $n' \to \infty$. But the expectation of the first term here is not greater than

$$\int_0^{2a} \sqrt{s} \, d\varphi_n(a_0, s),$$

and hence we obtain (2.9) as above. □

**Proof of Corollary 1.** Since

$$\int_0^a \sqrt{s} \, d\varphi_n(s) = \int_0^a s^{-\gamma - 1/2} \, ds \to 0 \quad \text{as } a \downarrow 0$$

whenever $\gamma < \frac{1}{2}$, we only have to prove that

$$\lim_{a \to 0} \lim_{n \to \infty} \sup_{s} \int_0^a \sqrt{s} \, d\varphi_n(s) = 0 \quad \text{if } \gamma < \frac{1}{2},$$

(2.10)

where $\varphi_n$ is given in (1.12).

Fix $0 < a < 1$. Then we have

$$\int_0^a \sqrt{s} \, d\varphi_n(s) = \sum_{i=0}^{\infty} \int_{a/2^i}^{a/2^{i+1}} \sqrt{s} \, d\varphi_n(s) \leq \sqrt{a} \sum_{i=0}^{\infty} r_i(n, a),$$

where, using (1.11),

$$r_i(n, a) = 2^{-i/2} \{ \varphi_n(a/2^i) - \varphi_n(a/2^{i+1}) \}$$

$$= 2^{-i/2} \frac{H(2^{-(i+1)}ak_n/n) - H(2^{-i}ak_n/n)}{H(u,k_n/n) - H(v,k_n/n)}$$

$$= 2^{-i/2} \frac{H(2^{-i}ak_n/n) - H(ak_n/n)}{H(u,k_n/n) - H(v,k_n/n)}$$

$$\times \prod_{m=1}^{i} \frac{H(2^{-(m+1)}ak_n/n) - H(2^{-m}ak_n/n)}{H(2^{-m}ak_n/n) - H(2^{-(m-1)}ak_n/n)}$$

$$\leq 2^{-i/2} \left\{ a^{-\gamma} \left[ (\frac{1}{2})^{-\gamma} - 1 \right] + a^{1/2} \right\} (2^\gamma + a)^i$$

$$= \left\{ a^{-\gamma} \left[ \frac{2^\gamma - 1}{|\gamma|} + a^{1/2} \right] \right\} (2^\gamma - 1/2 + a2^{-1/2})^i$$

for all $n$ large enough, where we use the convention that $|((\frac{1}{2})^{-\gamma} - 1)/\gamma| = \log 2$ if $\gamma = 0$. Hence for all $n$ large enough and all $a > 0$ small enough,

$$\int_0^a \sqrt{s} \, d\varphi_n(s) \leq \left( a + a^{1/2 - \gamma} \frac{|2^\gamma - 1|}{|\gamma|} \right) (1 - \gamma^{1/2})^{-1} a2^{-1/2} - 1.$$

Since this bound goes to zero as $a \downarrow 0$, (2.10) follows. □

**Proof of Corollary 2.** The first statement follows directly from Corollary 1.
Calculation shows that for the covariance function $r_{\gamma}(\cdot)$ of the process $V_{\gamma}(\cdot)$ we have

$$r_{\gamma}(h) = 1 - \frac{1}{2} \lambda_{\gamma} h^2 + o(h^2) \quad \text{as } h \to \infty,$$

where $\lambda_{\gamma}$ is as in (1.16). Applying now Theorem 8.2.6 in Leadbetter, Lindgren and Rootzén [5], we get that for all $x \in \mathbb{R}$,

$$\lim_{T \to \infty} P \left\{ A(T) \left( \sup_{0 \leq t < T} V_{\gamma}(t) - B_{\gamma}(T) \right) \leq x \right\} = \exp(-e^{-x}),$$

where $A(T)$ and $B_{\gamma}(T)$ are given in (1.15) and (1.16). This and the first statement now easily imply the second statement after a time change. □

References


