Mathl. Comput. Modelling Vol. 16, No. 1, pp. 1–8, 1992
Printed in Great Britain.

0895-7177/92 $5.00 + 0.00

ONE-DIMENSIONAL CUTTING STOCK PROBLEMS AND SOLUTION PROCEDURES

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Abstract—This paper provides an introduction to one-dimensional cutting stock problems and solution procedures. The first problem considered requires that both trim loss and pattern changes be controlled. Both linear programming and sequential heuristic procedures are discussed along with the ways they can be used jointly to generate the best possible solutions to this type of problem. Two other important classes of one-dimensional problems are discussed along with ways in which they can be solved.

INTRODUCTION

Common industrial problems, such as determining how to slit production rolls of paper, cut up steel bars, assign products to rail cars, containers or trailers, balance assembly lines and cycle products on a machine, appear on the surface to be quite different, but are, in fact, very closely related. They are all examples of a class of problems known as cutting and packing problems. A simple example of this type of problem is the trim loss minimization problem which occurs in the paper industry. In this problem, known quantities of various width rolls of the same diameter are to be slit from stock rolls of some standard width and diameter. The objective in this case is to find slitting patterns and their associated usage levels which produce the requirements for ordered rolls, at the least possible cost for scrap and other controllable factors. The basic restriction in this problem is that the sum of the roll widths slit from each stock roll must not exceed the usable width of the stock roll. The other problems listed above have exactly the same type of restrictions. The only difference is that the focus in these problems may be on the utilization of space or time, rather than material.

A cutting or packing problem is a one dimensional problem if this basic restriction can be stated in the form:

$$\sum_i W_i A_i \leq UW, \tag{1}$$

where, in this roll example,

- W_i is the size of requirement i (roll width i),
- A_i is the integer number of times requirement i appears in the pattern, and
- UW is the capacity (usable width of the stock roll).

Examples of higher dimensional cutting and packing problems for which restriction (1) above is not sufficient to describe a feasible pattern are given below:

- cutting rectangular pieces of glass from larger stock rectangles,
- cutting irregular shapes from a steel plate or bolt of cloth,
- cutting boards from a log,
- packing a container based on the dimensions of the items to be packed rather than their weight.

This paper will focus on one-dimensional cutting stock problems. The techniques described for cutting stock problems will work equally well for the packing problems described above, except

Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX for the assembly line balancing problem. In that situation, the presence of complex precedence relationships may make it necessary to use other approaches to solve the problem.

The next section contains a formulation of a common one-dimensional cutting stock problem along with a discussion of the major solution alternatives for solving this problem. The following section discusses some common variants of this problem along with solution approaches. The paper concludes with an identification of an important problem on which additional research is needed.

INITIAL PROBLEM DEFINITION

An initial statement of a one-dimensional cutting stock problem is given below in terms of slitting ordered rolls of some material such as paper from a single stock size. It is assumed that the order requirements are for R_i rolls of width $W_i, i = 1, \dots, n$, to be cut from stock rolls of usable width UW. All orders are for the same diameter.

$$\text{Min } C_1 \sum_j T_j X_j + C_2 \sum_j \delta(X_j), \tag{2}$$

$$\text{s.t. } RL_i \leq \sum_j A_{ij} X_j \leq RU_i, \quad \text{for all } i, \tag{3}$$

$$X_j \geq 0, \text{ integer}, \quad \text{where} \tag{4}$$

A_{ij} is the number of rolls of width W_i to be slit from each stock roll that is processed using pattern j (s.t.: subject to). In order for the elements A_{ij}, $i = 1, \dots, n$ to constitute a feasible cutting pattern, the following restrictions must be satisfied:

$$\sum_i A_{ij} W_i \leq UW, \tag{5}$$

$$A_{ij} \geq 0, \quad \text{integer}. \tag{6}$$

- X_j is the number of stock rolls to be processed according to pattern j;
- T_j is the number of units of trim loss incurred by pattern j,

$$T_j = UW - \sum_i A_{ij} W_i;$$

- C_1 is the dollar value of trim loss per unit;
- C_2 is the cost of changing patterns in dollars;
- $\delta(X_j)$ is 1 for $X_j > 0$ and 0 otherwise; and
- RL_i, RU_i are the lower and upper bounds on the order requirement, R_i, for customer order i reflecting the general industry practice of allowing overruns within specific limits. Depending on the situation R_i may be equal to either RL_i or RU_i or both.

Note that the objective in this example is not simply to minimize trim loss. In virtually all industrial applications, it is necessary to consider other factors in addition to trim loss. In this example, a cost is associated with a pattern changes and, therefore, controlling the number of patterns used to satisfy the order requirements is an important consideration. In other applications, some other factor might be important. For example, when assigning discrete products to containers by weight, to minimize the number of containers used, it might be desirable to have like products shipped in the same container to facilitate material handling at both the origin and destination of the shipment.

Because optimal solutions to this problem can be found only for values of n smaller than those typically found in practice, heuristic procedures represent the only feasible approach to solving this type of problem. Two types of heuristic procedures have been widely used to solve one-dimensional cutting stock problems. One approach uses the solution to a linear programming (LP) problem as its starting point. The LP solution is then massaged in some way to provide a solution to the problem. The second approach is to generate cutting patterns sequentially to satisfy some portion of the remaining requirements. The sequential heuristic (SH) procedure terminates when all ordered requirements (3) are satisfied.

LP SOLUTIONS

Almost all LP based procedures for solving cutting stock problems can be traced back to the seminal work of Gilmore and Gomory [1,2]. They described how the next pattern to enter the basis could be found by solving an associated knapsack problem. Problems of minimizing trim losses could then be solved by linear programming techniques without first generating every feasible slitting pattern.

A large number of slitter patterns may exist when narrow widths are to be slit from a wide stock roll. Pierce [3] showed that, in such situations, the number of slitting patterns can easily run into the millions. However, only a small fraction of all possible slitting patterns need to be considered in finding the minimum trim loss solution. The delayed pattern generation technique developed by Gilmore and Gomory made it possible to solve trim loss minimization problems in much less time than would be required to generate the slitting patterns to prepare the input for a general-purpose linear programming algorithm.

Referring to (2)–(4), if the cost of changing patterns is dropped from the objective function, and the integer requirement on pattern usage is relaxed, the linear programming problem to be solved can be stated as

$$\text{Min} \sum_j X_j, \tag{7}$$

$$\sum_j A_{ij} X_j \geq R L_i, \quad \text{for all } i, \tag{8}$$

$$X_j \geq 0. \tag{9}$$

Note that minimizing the number of production rolls rather than trim loss forces the upper bound on each order width to be automatically satisfied because (8) can be restated as an equality.

If U_i is the dual price associated with constraint i in (8), the reduced cost for any nonbasic pattern $A = (A_i, \ldots, A_n)$ is $1 - \sum_i U_i A_i$. Therefore, the LP problem (7)–(9), can be solved without first generating all possible cutting patterns. The next pattern to enter the basis, if one exists, can be found by solving a knapsack problem:

$$Z = \max \sum_i U_i A_i, \tag{10}$$

$$\sum_i W_i A_i \leq UW, \tag{11}$$

$$A_i \geq 0, \quad \text{integer.} \tag{12}$$

If $Z \leq 1$, the current solution is optimal. If $Z > 1$, then $A = (A_1, \ldots, A_n)$ can be used to improve the solution.

This LP solution can then be massaged in a number of ways to obtain integer values for X_j and completion of each order. The primary disadvantage of using LP to solve the problem in (2)–(4) is that the number of nonzero cutting patterns in the LP solution will be close to n, the number of sizes ordered. This may be acceptable only if controlling trim loss is very difficult and LP is the only way to find a low trim loss solution.

SEQUENTIAL HEURISTIC SOLUTIONS

With this approach, a solution is constructed one pattern at a time, until all the order requirements (3) are satisfied. The first documented SH procedure capable of finding better solutions than those found manually by human schedulers was described by Haessler [4]. The key to success with this type of procedure is to make intelligent choices as to which pattern will be selected next. The pattern selected should have low trim loss, high usage and leave a set of requirements for future patterns that does not guarantee trouble because the remaining sizes do not combine well.

The following procedure is capable of making effective pattern choices in a variety of situations:

1. Compute descriptors of the order requirements yet to be scheduled. Typical descriptors would be the number of stock rolls still to be slit and the average number of ordered rolls to be cut from each stock roll.
2. Set goals for the next pattern to be entered into the solution. Goals should be established for trim loss, pattern usage and number of rolls in the pattern.
3. Search exhaustively for a pattern that meets those goals.
4. If a pattern is found, add this pattern to the solution at the maximum possible level without exceeding R_i, for all i, reduce the order requirements and return to 1.
5. If no pattern is found, reduce the goals for the next pattern and return to 3.

The pattern usage goal is a lower bound on the level at which the pattern will enter the solution if it is chosen; for example, if some width has an unmet requirement of 10 rolls and the pattern usage goal is 4, that width may not appear more than twice in a pattern that satisfies the goal. If no pattern is satisfactory, then some goal—most commonly pattern usage—must be relaxed, allowing more patterns to be considered. If the pattern usage goal is changed to 3 in the above example, then the width for which 10 rolls are required can appear in the pattern three times. Termination can be guaranteed by eventually reducing the goals to permit a pattern of lowest trim loss at a usage level of one.

The primary advantage of the sequential approach is its ability to limit pattern changes by searching for high usage patterns, and to eliminate rounding problems by working only with integer values. It may, however, result in greatly increased trim loss because of what might be called ending conditions. For example, if care is not taken as each pattern is accepted and the requirements reduced, the widths remaining at some point in the process may not have an acceptable trim loss solution. Such would be the case if only 36-inch rolls are left to be slit from 100-inch stock rolls.

JOINT SOLUTION PROCEDURES

In order to find the best possible solution to a class of one-dimensional cutting problem such as the one defined in (2)–(4), the two basic solution approaches can be combined in a number of ways as described below.

1. The SH procedure can be used with the LP solution to obtain an integer solution which satisfies all order requirements in the following way. The LP solution could first be rounded down, and then nonzero usage patterns could be increased in unit increments so long as production of any size does not exceed RU_i. Any orders for which production falls below RL_i can be completed by using the SH procedure to generate new patterns.
2. The SH procedure is used to generate a solution which is saved, and also used as an initial basis in the LP procedure. Additional LP iterations are made only to the extent the trim loss is reduced. The better of the SH and LP solutions is selected according to the criterion specified in (2).
3. The problem is first solved as an LP problem in order to obtain optimal dual prices.

   These dual prices are used as an additional test before accepting a pattern in a SH procedure to ensure that the pattern does not contain a disproportionate share of the low relative dual price sizes. For patterns in the optimal or alternate optimal trim loss solutions,

$$Z = \sum_i A_i U_i = 1.$$

If the value of Z is less than .97 or .98, for a pattern accepted in a SH procedure, the total trim loss of the SH solution may be increased significantly. Although this test makes it possible to avoid making some mistakes when selecting a pattern using a SH approach, it is not foolproof because the SH may use the pattern at too high a level.

As the SH nears the completion of the requirements and the pattern selection decision becomes more difficult, patterns for residual requirements are generated using LP. If the residual LP solution does not meet some target value which is based on the original LP solution to the entire problem, the sequentially generated patterns are dropped in reverse order of generation and the expanded residual problem is resolved using LP. This dropping of sequentially generated patterns continues until either a satisfactory solution is obtained or all the patterns are dropped, at which point the LP solution with the best possible trim loss is generated.

The advantage of this last approach is that it truly integrates the strengths of the SH procedure to consider factors such as slitter changes and the LP procedure to minimize trim loss into a single procedure. This procedure is capable of giving either a pure SH or LP solution if that is what is best. Most importantly, however, is its ability to generate solutions which are part SH and part LP and, therefore, likely to be better than either the pure LP or SH solutions. Haessler [5] has used it effectively to solve difficult, from a trim loss standpoint, problems of the type defined in (2)–(4).

OTHER ONE-DIMENSIONAL PROBLEMS

To this point, the only problem discussed has involved controlling the number of patterns used along with the trim loss. Although this is an important class of one-dimensional problems, it is by no means the only important class. Two other important classes will be considered here. The first deals with multiple stock sizes and the second deals with pattern restrictions imposed by the product or process.

Multiple Stock Sizes

A general statement of this problem, for the same order requirements used earlier, is as follows. In this case, we permit the possibility that different stock sizes may be available at different locations, and therefore freight cost also influences the choice of which stock size will be used.

$$\text{Min} \sum_k \Big(C_{1k} \sum_j T_{jk} X_{jk} + \sum_i \sum_j C_{3ki} A_{ijk} X_{jk}\Big),$$

$$\text{s.t. } RL_i \le \sum_k \sum_j A_{ijk} X_{jk} \le RU_i, \quad \text{for all } i,$$

$$\sum_j X_{jk} \le M_k, \quad \text{for all } k,$$

$$X_{jk} \ge 0, \quad \text{integer},$$

where

- A_{ijk} is the number of rolls for order i to be cut from stock width k using pattern j. In order for A_{ijk} to be a feasible cutting pattern, for $i = 1, \ldots, n$, the following condition must be satisfied

$$\sum_i A_{ijk} W_i \le UW_k$$

$$A_{ijk} \ge 0, \quad \text{integer}.$$

- X_{jk} is the number of stock of width k to be processed according to pattern j.
- T_{jk} is the trim loss incurred by using pattern j with stock width k.

$$T_{jk} = UW_k - \sum_i A_{ijk} W_i.$$

- C_{1k} is the dollar value of trim loss per unit for stock width k.
- C_{3ki} is the cost of shipping one roll for order i which is produced from stock width k. It is assumed that the stock width defines the production location. If all the production options are at the same location, this value can be set to 0.
- M_k is the maximum number of rolls of stock width k which can be used.

Real world problems of this sort usually require the power of LP to deal with the complex tradeoffs which can arise.

The LP relaxation of this problem, developed by Beged Dov [6] is

$$\min \sum_{jk} C_{jk} X_{jk},$$

$$\text{s.t. } \sum_{jk} A_{ijk} X_{jk} \geq R L_i, \quad \text{for all } i,$$

$$X_{jk} \geq 0.$$

For the most general case with varying costs for trim loss and freight, C_{jk} can be represented as follows

$$C_{jk} = C_k + C_{pk}\Big(\sum_i A_{ijk} W_i\Big) + \sum_i C_{3ki} A_{ijk}.$$

where

- C_k is the cost excluding material of making one stock roll of width k.
- C_{pk} is the material cost per inch of that portion of production roll k actually used.
- C_{3ki} is the cost of shipping one roll for order i from the location where stock width k is made.

In this situation, the shadow prices must be adjusted before generating the cutting pattern that enters the solution next, if one exists. The following associated problem must be solved for each stock width.

$$Z_k = \max \sum_i (U_i - C_{pk} W_i - C_{3ki})\, A_{ijk} - C_k,$$

$$\text{s.t. } \sum_i W_i A_{ijk} \leq U W_k,$$

$$A_{ijk} \geq 0, \quad \text{integer}.$$

The pattern for which Z_k has the largest value greater than zero enters the solution.

Pattern Restrictions

For this discussion refer back to the problem defined in (2)–(4). The definition of an acceptable pattern will be extended to consider restrictions beyond

$$\sum_i A_{ij} W_i \leq UW.$$

1. Limited slitter capacity

   If C is the maximum number of rolls which can be slit from each stock width on the primary slitter, then each pattern must satisfy the additional restriction

$$\sum_i A_{ij} \leq C.$$

   If the rolls are too narrow, then it may be necessary to generate master rolls to be slit at a secondary operation. In this case, the objective would be to minimize the number of such rolls. In a SH procedure, this can be accomplished by initially combining the narrowest rolls into patterns that contain master rolls for secondary slitting, until the remaining rolls are wide enough to be slit at the primary operation. In a LP procedure this can be handled using two stock sizes where the second stock size has a higher cost to reflect the secondary slitting, and a lower usable width to reflect the edge trim required at a secondary slitter.

2. Two-stage processing

   In some applications, such as the production of film or coated grade of paper, the stock rolls must be slit, processed through a finishing operation which is not capable of processing the full stock width roll, and then slit to the width required by the final customer. Haessler [7] discussed a procedure for solving this problem in the film industry.

   In this case, each pattern $(A_1, A_2, \ldots, A_n)$ must be capable of being partitioned to meet the requirements of the finishing operation. Each component A_i of the pattern can be thought of as being the requirements R_i of a trim problem with a limited number of stock sizes of usable width, FW, where FW is the maximum width roll which the finishing equipment can process.
3. Quality variations across the width of the stock size

   Sweeney and Haessler [8] developed a procedure for solving the one-dimensional trim problem in the case where the order requirements may be satisfied by lower quality grades which may occur during the production process. If there are no quality variations across the width of the stock roll, this first quality material can be used to satisfy any order for that grade. A two phase procedure is used. In the first phase all non perfect stock rolls are considered. Each roll is assigned a value based on the nature of the quality variations. Patterns are generated for each stock roll, beginning with those with the severest quality variations. Each pattern is given a value based on the shadow prices obtained from solving an LP problem for the order requirements and stock rolls without any quality defects. If the value of the pattern exceeds the value of the stock roll, the pattern is accepted for that one stock roll. Any order requirements not met from stock rolls with quality variations are slit from first quality stock rolls based on an LP solution to this residual problem found in phase two.
4. Order contiguity

   In cutting stock problems where the time required to satisfy all the requirements is larger, it may be necessary to try to produce all of each order within some limited time period. This may be due to a need to ship some order by a certain time, or it may be due to a general need to be shipping something all the time because of limited dock or work in-process storage space. This problem is extremely important in mills which produce large volumes of commodity grades of paper, such as kraft liner and medium for corrugated boxes. The issue may be defined as limiting the number of orders that have been partially produced at any point in time. The problem is further complicated if some orders require multiple railcars or truck trailers to ship, and the issue is defined as limiting the number of partially filled vehicles at any point in time.

   There are a number of ways to approach this problem. Although using one might work reasonably well in a given situation, none really provides a satisfactory general purpose method of dealing with this problem. The most powerful of these approaches involve pattern restrictions based on predetermined or dynamic sequencing of orders.

   A predetermined sequencing would involve partitioning either orders, or full railcar or trailer portions of orders into subgroups. In an LP approach, the only patterns generated initially would be those containing ordered sizes assigned to the same subgroup. Once an optimal LP solution for each subgroup is obtained, the procedure stops if a judgment is made that the trim loss is low enough. If the trim loss is judged to be too high, additional patterns can be generated using ordered sizes from adjacent subgroups. This expansion of ordered widths which can be combined in the same pattern continues until a satisfactory trim loss solution is obtained. Clearly, the effectiveness of this type of procedure depends on how well the subgroups have been formed and the production sequence in which they have been arranged.

   A more dynamic approach would involve selecting subgroups during the solution process based on trim and shipment considerations. An LP solution for all orders regardless of contiguity would indicate overall attainable trim loss and optimal dual prices. Once a starting order is established, an integrated LP and SH procedure could simultaneously generate patterns and start new orders as needed to provide low trim loss patterns which complete orders in a systematic fashion.

CONCLUSION

This paper has considered a variety of one-dimensional cutting stock problems and the ways in which they can be solved. Of the problems identified, the most difficult one is the order continuity problem. If the number of orders is more than a few, the number of sequences in which the orders can be arranged for completion becomes very large. Research is needed to find ways to sequence orders for production or to partition groups of orders for joint production, so that the requirements of the material handling and shipping departments can be considered when generating solutions to large cutting stock problems. An even more complex variation of this problem exists when the same grade is being run simultaneously on two or more machines. In this case, product flow off each machine must be coordinated to ensure efficient handling and timely shipment of each order.

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