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Δ-Matroids and Metroids

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1. INTRODUCTION

At about the same time in the mid-eighties the first named author and the other two authors independently introduced certain generalizations of matroids, the *Δ-matroids* (see [B1/2]) which were invented to analyse certain variants of the greedy algorithm and properties of the Euler tours of 4-regular graphs, and the *metroids* (see [DH1/2]) which were invented to analyse the combinatorial relationship between the vanishing and non-vanishing of discriminants of finite subsets of "metric" vector spaces (that is, vector spaces equipped with a sesquilinear form, cf. [Bb]) in analogy to the way matroid theory analyses the combinatorial relationship between the vanishing and non-vanishing of $n \times n$-subdeterminants of an $n \times k$-matrix ($k \geqslant n$). Both generalizations are expressed in terms of certain axioms concerning a collection $\mathscr{F}$ of subsets of a ground set E, called the free subsets. Later it was observed by the first named author (see [B2]) that the combinatorial properties of discriminants are also reflected by Δ-matroids. Simultaneously and still independently, the other two authors observed that for every free subset F in a metroid $(E, \mathscr{F})$ the set system $\{F' \Delta F := (F' \backslash F) \cup (F \backslash F') \mid F' \in \mathscr{F}\}$ also satisfies the metroid axioms (cf. the appendix to [DH2]), a fact which holds almost trivially by definition for Δ-matroids.

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Finally, things converged, we became aware of each other's work, and in the following note we show that the pair $(E, \mathscr{F})$ is a metroid if and only if it is a normal Δ-matroid (that is, a Δ-matroid in which the empty subset is a free subset). We also mention some simple applications and relate the various rank functions used in [B1/2] and [DH1/2].

It appears to be remarkable that while it is possible to prove that every normal Δ-matroid is a metroid right from the axioms without any further reference to Δ-matroid theory, this does not seem to be true for the converse. Indeed, to prove that every metroid is a Δ-matroid we shall have to use essentially all the fundamental results of metroid theory as established in [DH1]. In other words, one may consider the Δ-matroid description as a more global description of the set systems $\mathscr{F}$ in question which is augmented by a rather local description of essentially the same set systems, given by the metroid axioms.

2. Definitions and the Main Result

According to [B1/2] a Δ-matroid is a pair $(E, \mathscr{F})$ in which E is a finite set and $\mathscr{F}$ is a family of subsets of E, called the *free* (or *feasible*) subsets, which satisfy the following "symmetric exchange axiom":

(SEA) for $F', F'' \in \mathscr{F}$ and $x \in F' \Delta F'' := (F' \backslash F'') \cup (F'' \backslash F')$ there exists $y \in F' \Delta F''$ such that $F' \Delta \{x, y\} \in \mathscr{F}$.

Similarly, a metroid is defined in [DH1] to be a pair $(E, \mathscr{F})$ as above, in which the free subsets satisfy the following axioms:

(F0) $\mathscr{F} \neq \varnothing$;

(F1) If $F, G \in \mathscr{F}$ and $|G| < |F|$ then there exists some $u, v \in F \backslash G$ (possibly equal) with $G \cup \{u, v\} \in \mathscr{F}$;

(F2) If $v \in F \in \mathscr{F}$ and $F \backslash v \notin \mathscr{F}$ then there exists some $u \in F \backslash v$ with $F \backslash \{u, v\} \in \mathscr{F}$;

(F3) If $F \in \mathscr{F}$, $w, x, y, z \in E$, $w \neq x, y$, and $F \cup w$, $F \cup w \cup x$, $F \cup w \cup y \notin \mathscr{F}$, then $F \cup \{w, x, y, z\} \in \mathscr{F}$ if and only if $F \cup \{w, z\} \in \mathscr{F}$ and $F \cup \{x, y\} \in \mathscr{F}$;

(F4) If $F \in \mathscr{F}$ and $v, w, x, y, z \in E$ with $v \neq w$, then $F \cup v$, $F \cup w$, $F \cup v \cup w \notin \mathscr{F}$, and $F \cup \{v, x\}$, $F \cup \{v, w, y, z\} \in \mathscr{F}$ implies $F \cup \{v, w, x, y\} \in \mathscr{F}$ or $F \cup \{v, w, x, z\} \in \mathscr{F}$.

Our main result is the following

THEOREM 1. *Given a collection $\mathscr{F}$ of subsets F of a set E, the pair $(E, \mathscr{F})$ is a metroid if and only if $\varnothing \in \mathscr{F}$ and $(E, \mathscr{F})$ is a Δ-matroid.*

3. PROOF OF THEOREM 1

We first show that every Δ-matroid $(E, \mathscr{F})$ with $\varnothing \in \mathscr{F}$ is a metroid: Since $\varnothing \in \mathscr{F}$ we have $\mathscr{F} \neq \varnothing$, so $(F0)$ is definitely satisfied. Next we show:

(SEA)$\Rightarrow$(F1). We use induction on $k = |F \backslash G|$. We have $k > 0$ because $|G| < |F|$. If $k = 1$ we have $G \subseteq F$ so that (F1) holds with $\{u, v\} = F \backslash G$. Otherwise we choose $u' \in F \backslash G$, and apply (SEA) to G, F, and $u' \in G \Delta F$. We find $v' \in G \Delta F$ such that $G' = G \Delta \{u', v'\} \in \mathscr{F}$. If $v' \in F \backslash G$, then $G' = G \cup \{u', v'\}$ and (F1) holds with $\{u, v\} = \{u', v'\}$. If $v' \in G \backslash F$ we have $|G'| = |G|$, and $|F \backslash G'| = |F \backslash G| - 1$, so that (F1) holds by induction for G' and F. Thus we can find $\{u'', v''\} \subseteq F \backslash G'$ such that $G'' = G' \cup \{u'', v''\} \in \mathscr{F}$. Applying (SEA) to G'', G, $v' \in G'' \Delta G = \{u', v', u'', v''\}$, we can find $t \in \{u', v', u'', v''\}$ such that $H = G'' \Delta \{v', t\} \in \mathscr{F}$. In any case we have $G \subseteq H$ and $H \backslash G \subseteq F$. Moreover we have $|H \backslash G| \leqslant 2$ if $t \neq v'$, so that we may take $\{u, v\} = H \backslash G$. If $t = v'$, then $H = G \cup \{u', u'', v''\}$, and the union is disjoint. We apply (SEA) to G, H, and any $u \in H \backslash G$, and we find $v \in H \backslash G$ such that $G \cup \{u, v\} \in \mathscr{F}$.

(F2) is obviously equivalent with (SEA), applied for $F' := F$ and $F'' := \varnothing$.

(SEA)$\Rightarrow$(F3). Let $(E, \mathscr{F})$ be a Δ-matroid such that for some $F \in \mathscr{F}$ and some $w, x, y, z \in E$ with $w \neq x, y$ we have $G_1 = F \cup \{w\}$, $G_2 = F \cup \{w, x\}$, $G_3 = F \cup \{w, y\} \notin \mathscr{F}$. We have $w \notin F$ since otherwise $F = G_1$, an impossible equality. We have $G_4 = F \cup \{w, x, y\} \notin \mathscr{F}$ since otherwise $x \neq y$ and, hence, (G_4, F, w) would not satisfy (SEA). Put $X := F \cup \{w, x, y, z\}$, $Y := F \cup \{w, z\}$, $Z := F \cup \{x, y\}$. If we suppose that $X \in \mathscr{F}$, then $Y \in \mathscr{F}$ is forced by (SEA) applied to (F, X, w), and $Z \in \mathscr{F}$ is forced by (SEA) applied to (X, F, z), using that $G_4 \notin \mathscr{F}$; so we have established the left–right implication. If we suppose that $Y \in \mathscr{F}$ and $Z \in \mathscr{F}$, then (SEA) applied to (Z, Y, w) forces $X \in \mathscr{F}$, using again $G_4 \notin \mathscr{F}$; so the reverse implication holds, too.

(SEA)$\Rightarrow$(F4). Let us suppose for a contradiction that a Δ-matroid $(E, \mathscr{F})$ does not satisfy (F4). Then we can find $F \subseteq E$ and $W = \{v, w, x, y, z\} \subseteq E$ such that $v \neq w$, $F_0 = F$, $F_1 = F \cup \{v, x\}$, $F_2 = F \cup \{v, w, y, z\} \in \mathscr{F}$, and $G_1 = F \cup \{v\}$, $G_2 = F \cup \{w\}$, $G_3 = F \cup \{v, w\}$, $G_4 = F \cup \{v, w, x, y\}$, $G_5 = F \cup \{v, w, x, z\} \notin \mathscr{F}$.

Note first that $G_1, G_2 \neq F \in \mathscr{F}$ implies $v, w \notin F$, that $G_1, G_3 \neq F_1 \in \mathscr{F}$ implies $x \notin F \cup \{v, w\}$, and that $G_4, G_5 \neq F_2 \in \mathscr{F}$ implies $x \neq z, y$.

Next observe that $G_1, G_2, G_3 \notin \mathscr{F}$ implies $F \cup \{v, w, t\} \notin \mathscr{F}$ for all $t \in E$ since otherwise $t \notin F \cup \{v, w\}$ and (SEA), applied to $(F \cup \{v, w, t\}, F, t)$, would contradict $G_1, G_2, G_3 \notin \mathscr{F}$.

In particular, $F_2 \in \mathscr{F}$ implies $y \neq z$ as well as $y, z \notin F \cup \{v, w\}$. Hence $F \cap W = \emptyset$ and $|W| = 5$, so (SEA), applied to (F_1, F_2, w), leads to the final contradiction $G_3 \in \mathscr{F}$ or $F \cup \{v, w, x\} \in \mathscr{F}$ or $G_4 \in \mathscr{F}$ or $G_5 \in \mathscr{F}$.

To prove the converse, assume that $(E, \mathscr{F})$ is a metroid, i.e., a non-empty set system satisfying (F1) through (F4). Obviously, (F2) implies $\emptyset \in \mathscr{F}$. To prove that (SEA) holds, too, assume $F', F'' \in \mathscr{F}$ and $x \in F' \Delta F''$. We shall consider separately the cases $x \in F' \backslash F''$ and $x \in F'' \backslash F'$.

Assuming $x \in F' \backslash F''$, we consider the set $Z := \{z \in F' \mid F' \backslash \{x, z\} \in \mathscr{F}\}$, which is non-empty by (F2). If there is some $z \in Z$ with $z \notin F''$, we may take $y := z$ in (SEA) above to obtain the desired result. Otherwise, we have $Z \subseteq F''$ and, in particular, $x \notin Z$ that is, $Z \subseteq F' \backslash \{x\}$. Now $Z \subseteq F'' \in \mathscr{F}$ means that Z is a subset of $F' \backslash \{x\}$ which is *virtually free with respect to* F'' (F''*-free* in the language of [DH1]). Since the virtually free subsets of a metroid form a matroid [DH1] we can augment Z to a subset $Z' \subseteq F' \backslash x$ which is of maximum cardinality among all F''-free subsets in $F' \backslash x$. Then by the definition of the term F''-free, we have $Z' = (F' \backslash x) \cap G$ for some free subset $G \subseteq (F' \backslash x) \cup F''$. Moreover, by Theorem 1.7(iv) of [DH1] the metroidal rank satisfies $\rho(Z') = \rho(F' \backslash x)$, so there exists some $H \in \mathscr{F}$ such that $H \subseteq Z'$ with $|H| = \rho(F' \backslash x) := \max(|F| \mid F \in \mathscr{F}$ and $F \subseteq F' \backslash x)$. But since $F' \backslash x \notin \mathscr{F}$ by assumption, we have $\rho(F' \backslash x) = |F'| - 2$ by (F2). Together with $x \notin Z \subseteq Z'$ this implies either $H = Z' = F' \backslash \{x, y\} \in \mathscr{F}$ for some $y \in F' \backslash x$ in contradiction to $y \in Z := \{z \in F' \mid F' \backslash \{x, z\} \in \mathscr{F}\} \subseteq Z'$, or else $Z' = F' \backslash x$ in which case Z' has *defect index one with respect to* F''. It follows from Theorem 2.6 of [DH1] that there exists some $y \in F''$ with $Z' \cup y \in \mathscr{F}$, and this y is necessarily in $F'' \backslash F' \subseteq F' \Delta F''$ and satisfies

$$F' \Delta \{x, y\} = (F' \backslash x) \cup y = Z' \cup y \in \mathscr{F}.$$

In the case that $x \in F'' \backslash F'$ one argues as follows. First, if $F' \cup x \in \mathscr{F}$ one just puts $y := x$ in (SEA) above. Otherwise one consider the set

$$Z := \{z \in F' \mid (F' \cup x) \backslash z \in \mathscr{F}\}.$$

If there exists $z \in Z$ with $z \notin F''$, one puts $y := z$. Otherwise $Z \cup x \subseteq F''$ and $Z \cup x \subseteq F' \cup x$, and hence, again by Theorem 1.7(iv), there exists $Z' \subseteq F' \cup x$ and $G, H \in \mathscr{F}$ with $Z \cup x \subseteq H \subseteq Z' = (F' \cup x) \cap G$, $G \subseteq F' \cup F''$ and $|H| = \rho(F' \cup x) = |F'|$. Hence $|F'| = |H| \leqslant |Z'| \leqslant |F' \cup x| = |F'| + 1$ and therefore either $H = Z' = (F' \cup x) \backslash y \in \mathscr{F}$ for some $y \in F' \backslash Z$ in contradiction to the definition of Z, or else $Z' = F' \cup x \subseteq G \subseteq F' \cup F''$, in which case it follows as above from Theorem 2.6 of [DH1] that there exists some $y \in F'' \setminus (F' \cup x)$ with $F' \cup \{x, y\} \in \mathscr{F}$.

4. Some Consequences

Next, we wish to point out that our result has some interesting consequences concerning arbitrary Δ-matroids. To state these consequences, we introduce the following notations: For a Δ-matroid $(E, \mathscr{F})$ and some fixed $F \in \mathscr{F}$ we define

$$\rho_F\colon 2^E \to N_0\colon X \mapsto \max(|F \Delta G| \mid G \in \mathscr{F} \text{ and } F \Delta G \subseteq X);$$

$$\Phi_F\colon 2^E \times 2^E \to N_0\colon (X, Y) \mapsto \Phi_F(X \mid Y)$$

$$:= \max(|X \cap (F \Delta G)| \mid G \in \mathscr{F} \text{ and } F \Delta G \subseteq X \cup Y);$$

and

$$\delta_F\colon 2^E \times 2^E \to N_0\colon (X, Y) \mapsto \delta_F(X \mid Y) := \Phi_F(X \mid Y) - \rho_F(X).$$

Then the main result (Theorem 1.7) in [DH1] implies

Theorem 2. (i) *For fixed* $Y \subseteq E$ *the map* $\Phi_F(\cdot \mid Y)$: $2^E \to N_0$: $X \mapsto \Phi_F(X \mid Y)$ *is a matroidal rank function on* E;

(ii) *for fixed* $X \subseteq E$, *the map* $\delta_F(X \mid \cdot)$: $2^E \to N_0$: $Y \mapsto \delta_F(X \mid Y)$ *is a matroidal rank function on* E;

(iii) *for any* $X \subseteq E$ *and any* $F_1 \in \mathscr{F}$, *there exist* $F_2, F_3 \in \mathscr{F}$ *with*

$$X \cap (F_1 \Delta F) \subseteq X \cap (F_3 \Delta F), \quad F_2 \Delta F \subseteq X \cap (F_3 \Delta F), \quad \text{and} \quad |F_2 \Delta F| = \rho_F(X).$$

5. Rank Functions

Finally, we want to discuss briefly the relation between the function $\Phi(X, Y) := \Phi_\varnothing(X, Y)$, defined as in Section 4 in the case $\varnothing \in \mathscr{F}$, and the birank function ρ, defined in [B2] for $P, Q \subseteq E$ with $P \cap Q = \varnothing$ by

$$\rho(P, Q) = \max(|P \cap F| + |Q \cap \bar{F}| : F \in \mathscr{F}).$$

For example, if $\mathscr{F}$ is the independent-set of a matroid M with rank function r, the preceding formulas give $r(P) = \rho(P, \varnothing) = \Phi(P, E)$.

It is proved in [B2] that $(E, \mathscr{F})$ is a Δ-matroid if and only if ρ, called the *birank function*, satisfies the following axioms for $P, Q, P', Q' \subseteq E$, $P \cap Q = P' \cap Q' = \emptyset$:

(i) $0 < \rho(P, Q) \leqslant |P| + |Q|$;

(ii) $\rho(P, Q) \leqslant \rho(P', Q')$ if $P \subseteq P'$ and $Q \subseteq Q'$;

(iii) $\rho(P, Q) + \rho(P', Q') \geqslant \rho(P \cup P', Q \cup Q') + \rho(P \cap P', Q \cap Q')$ if $(P \cup P') \cap (Q \cup Q') = \emptyset$;

(iv) $(\rho(P + x, Q) - \rho(P, Q)) + (\rho(P, Q + x) - \rho(P, Q)) \geqslant 1$ if $X \in V \setminus (P \cup Q)$.

Similar formulas, concerning $\Phi_{\emptyset}$ and $\delta_{\emptyset}$, have been established in [DH1, Theorem 1.7].

Our last result is

THEOREM 3. *Let* $(E, \mathscr{F})$ *be a metroid. Then for* $P, Q \subseteq E$ *with* $P \cap Q = \emptyset$, *one has*

$$\rho(P, Q) = |Q| + \Phi(P, \bar{Q}).$$

Proof. Using the above definitions we easily verify that

$$|Q| + \Phi(P, \bar{Q}) = \max(|P \cap F| + |Q \cap \bar{F}| : F \in \mathscr{F}, F \cap Q = \emptyset).$$

Thus $|Q| + \Phi(P, \bar{Q}) \leqslant \rho(P, Q)$. To prove the reverse inequality we consider some $F \in \mathscr{F}$ which maximizes (lexicographically) the ordered pair

$$\pi(P, Q, F) = (|P \cap F| + |Q \cap \bar{F}|, |Q \cap \bar{F}|).$$

It is enough to observe that for such an $F \in \mathscr{F}$ we have necessarily $F \cap Q = \emptyset$ since otherwise we may apply (SEA) to $(F, \emptyset, x)$ for some $x \in F \cap Q$ to find some $y \in F = F \Delta \emptyset$ with $F' := F \Delta \{x, y\} = F \setminus \{x, y\} \in \mathscr{F}$ in contradiction to $|P \cap F'| \geqslant |P \cap F| - 1$ and $|Q \cap \bar{F}'| \geqslant |Q \cap \bar{F}| + 1$. ∎

The functions ρ and Φ can be easily computed in the following case. Let $A = (A_{ij} : i, j \in E)$ be a matrix such that for any two vectors $x = (x_i)_{i \in E}$ and $y = (y_i)_{i \in E}$ one has

$$x^t \cdot A \cdot y = \sum_{i,j} x_i A_{ij} y_j = 0$$

if and only if

$$y^t \cdot A \cdot x = \sum_{i,j} y_i A_{ij} x_j = 0.$$

For $P, Q \subseteq E$ we consider the submatrix $A[P, Q] = (A_{ij} : i \in P, j \in Q)$ and we make the convention that $A[\varnothing;\varnothing]$ is a non-singular matrix. It has been shown in [B2] and [DH1] that $(E, \{F \subseteq E \mid A[F, F] \text{ non-singular}\})$ is a Δ-matroid or a metroid, respectively. Then the birank function ρ is given by the formula

$$\rho(P, Q) = |Q| + \operatorname{rank} A[P, \bar{Q}], \qquad P, Q \subseteq E, P \cap Q = \varnothing,$$

while, in accordance with (3.3), the function Φ just coincides with the rank function:

$$\Phi(P, Q) = \operatorname{rank} A[P, Q] \qquad \text{if} \quad P \subseteq Q$$

and, therefore, in general

$$\Phi(P, Q) = \operatorname{rank} A[P, P \cup Q].$$

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